Andrei B. Alexandra
Mathematics                                                                                          1
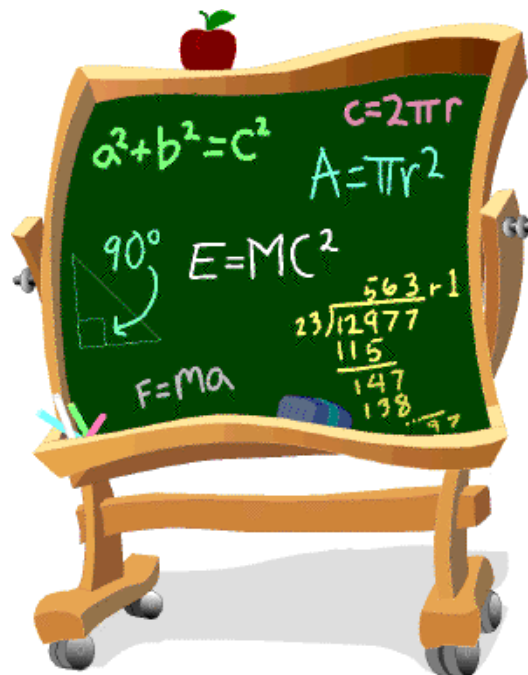
# Zeros of Cubic Functions

I am going to investigate the zeros of a cubic function. Zeros of functions are in other words roots of functions. A cubic function might have a root, two roots or three roots. An eas y way to find the roots of a function is by using the Remainder Theorem, which states that $a$ is a root of $f(x)$ if and only if $f(a) = 0$. I will make a very god use of this Theorem troughout the essay.

My mission is to find the equation of the tangent lines to the average of two of the three roots, by taking the roots two at a time. Then to find where the tangent line intersect the curve again in order to be able to state a conjecture concerning the roots of the cubic function and the tangent line at the average value of these roots.

Let us consider the cubic function $f\ x\ = 2x^3 + 6x^2 - 4.5x - 13.5$ and take a look at the graph of it:
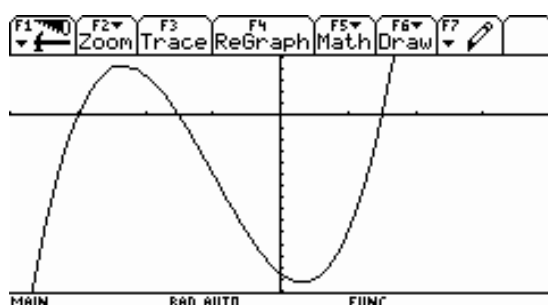


\* Window values:

```
xmin=-4.
xmax=4.
xscl=1.
ymin=-15.
ymax=5.
yscl=1.
xres=2.
```

First I will state the roots of the function as follow
- -3.0
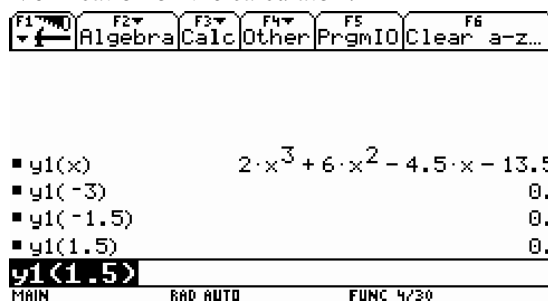- -1.5
- 1.5

And then prove this using the Remainder Theorem:
$x = \alpha$ is a root of $f(x) \Leftrightarrow f(\alpha) = 0$
Substituting x with the values of the roots:
$f\ x\ = 2x^3 + 6x^2 - 4.5x - 13.5$

- $f(-3) = 2(-3)^3 + 6(-3)^2 - 4.5(-3) - 13.5$
  $f(-3) = -54 + 54 + 13.5 - 13.5$
  $f(-3) = 0$

- $f(-1.5) = 2(-1.5)^3 + 6(-1.5)^2 - 4.5(-1.5) - 13.5$
  $f(-1.5) = -6.75 + 13.5 + 6.75 - 13.5$
  $f(-1.5) = 0$

- $f(1.5) = 2(1.5)^3 + 6(1.5)^2 - 4.5(1.5) - 13.5$
  $f(1.5) = 6.75 + 13.5 - 6.75 - 13.5$
  $f(1.5) = 0$

Verification on the calculator :

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The next step will be to find the equation of the tangent line of two of the three roots. The first thing to do when finding the equation of a tangent line is to find the slope (gradient). Hence to find the derivative of the function $\Rightarrow f(x) = 2x^3 + 6x^2 - 4.5x - 13.5$

and $f'(x) = 6x^2 + 12x - 4.5$

Second is to find the average of two of the roots using the formula $\dfrac{(a+b)}{2}$ , ($a$ and $b$ being the roots) and then substitute x with the value of the average in the function as following:

- Roots -3.0 and -1.5

Average $= \dfrac{(-3.0 - 1.5)}{2} = -2.25$

$f(-2.25) = 2(-2.25)^3 + 6(-2.25)^2 - 4.5(-2.25) - 13.5$
$f(-2.25) = 4.21875 \approx 4.22$
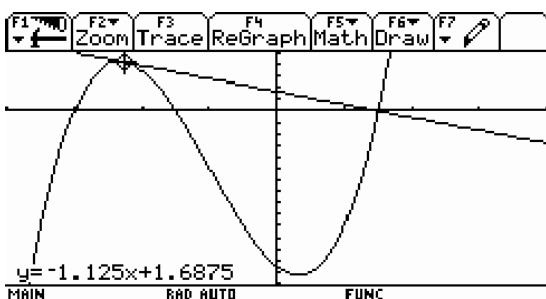
This implies that the slope at (-2.25, 4.22) will be:
$f'(-2.25) = 6(-2.25)^2 + 12(-2.25) - 4.5 = -1.125$

I will use the formula of the equation of a line $y - y_1 = m(x - x_1)$ and rewriting it to find $y = m(x - x_1) + y_1$ which is the equation of the tangent. * $x_1$ is the average of two of the roots respectively $\dfrac{(a+b)}{2}$ ; $y_1$ is $f\left(\dfrac{a+b}{2}\right)$ and $m$ is $f'\left(\dfrac{a+b}{2}\right)$.

In this case $x_1 = -2.25$, $y_1 = -4.22$ and $m = -1.125 \Rightarrow y = -1.125x + 1.7$



The same steps I used for the average of the other roots:

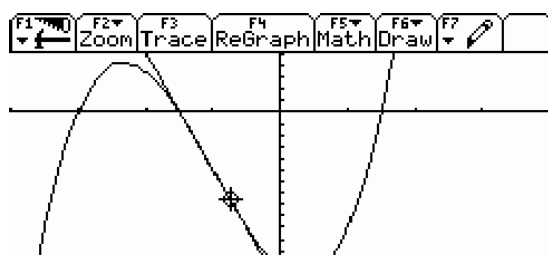- Roots -3.0 and 1.5

Average $= \dfrac{(-3.0 + 1.5)}{2} = -0.75$

$f(-0.75) = 2(-0.75)^3 + 6(-0.75)^2 - 4.5(-0.75) - 13.5$
$f(-0.75) = -7.59375 \approx -7.6$

Slope at (-0.75, -7.6) will be: $f'(-0.75) = 6(-0.75)^2 + 12(-0.75) - 4.5 = -10.125$

**Equation of the tangent** where $x_1 = -0.75$, $y_1 = -7.6$ and $m = -10.125 \Rightarrow$
$y = -10.125x - 15.2$

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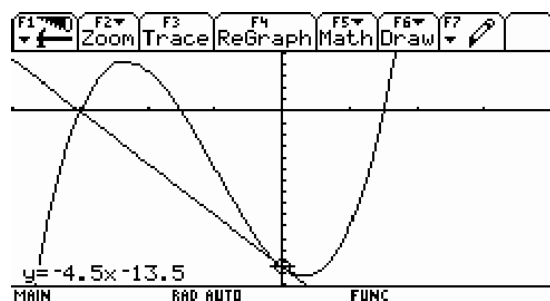Mathematics                                                                                                    4

- Roots -1.5 and 1.5

Average = $\dfrac{(-1.5+1.5)}{2} = 0$

$f(0) = 2(0)^3 + 6(0)^2 - 4.5(0) - 13.5$

$f(0) = -13.5$

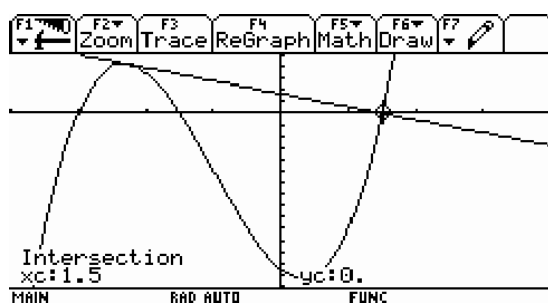Slope at (0, -13.5) will be: $f'(0) = 6(0)^2 + 12(0) - 4.5 = -4.5$

**Equation of the tangent** where $x_1 = 0$, $y_1 = -13.5$ and $m = -4.5$ $\Rightarrow$ $y = -4.5x - 13.5$



The next thing to do is to find out where the tangents lines will intersect with the curve again. For that to occur the equation of the tangent lines should be equal with 0 ( $y = 0$ ).
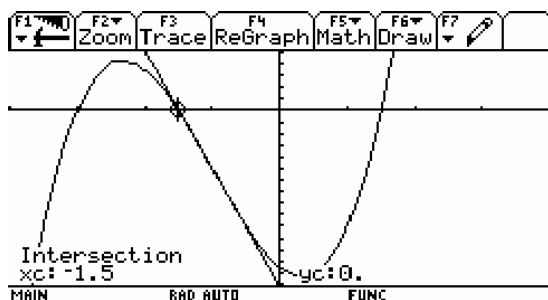
- $y = -1.125x + 1.7$

  $-1.125x + 1.7 = 0 \Rightarrow x = 1.5 \Rightarrow$ **The intersection will be at** (1.5, 0)0



- $y = -10.125x - 15.2$

  $-10.125x - 15.2 = 0 \Rightarrow x = -1.5 \Rightarrow$ **The intersection will be at** (-1.5, 0)
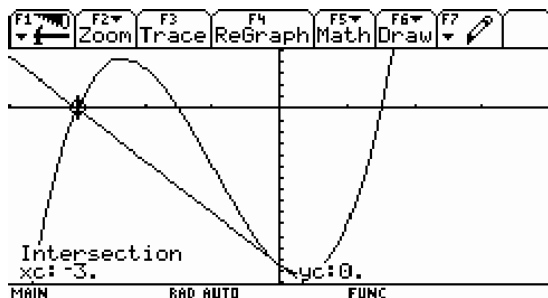
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- $y = -4.5x - 13.5$
  $-4.5x - 13.5 = 0 \Rightarrow x = -3.0 \Rightarrow$ **The intersection will be at** (-3.0, 0)



As I observe in the graph and in my calculations the tangent line at the average of two of the roots will intersect the graph at the other root. An observation which I can base it on my conjecture: **The equation of a tangent at the average of any two roots will intersect the y-axis at the third root.**

The next step will be to test my conjecture in another similar cubic function using the same steps as previous.

Let us consider the function: $f(x) = 2x^3 - 3x^2 - 3x + 2$

The graph of the function:
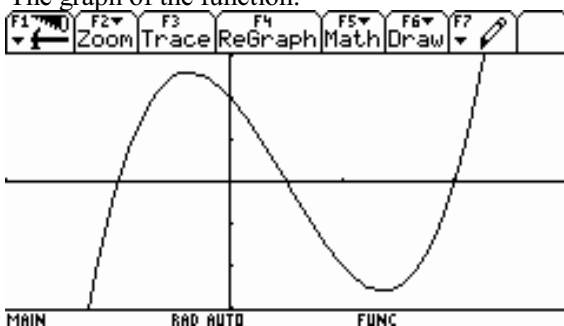


*Window values:

xmin=-2.
xmax=3.
xscl=1.
ymin=-3.
ymax=3.
yscl=1.
xres=2.

* The average: $\dfrac{(a+b)}{2}$

* The equation of tangent: $y - y_1 = m(x - x_1) \Rightarrow y = m(x - x_1) + y_1$

* The slope ($f'(x)$) of the function

* $f'(x) = 6x^2 - 6x - 3$

The roots of the function are:

- -1
- $\dfrac{1}{2}$
- 2

Proof using the Remainder Theorem:

$f(x) = 2x^3 - 3x^2 - 3x + 2$

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- $f(-1) = 2(-1)^3 - 3(-1)^2 - 3(-1) + 2$

  $f(-1) = -2 - 3 + 3 + 2$

  $f(-1) = 0$

- $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) + 2$
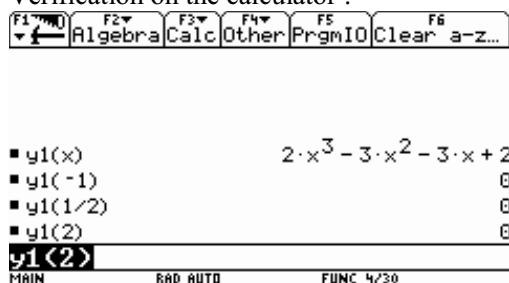
  $f\left(\frac{1}{2}\right) = 0.25 - 0.75 - 1.5 + 2$

  $f\left(\frac{1}{2}\right) = 0$

- $f(2) = 2 \cdot 2^3 - 3 \cdot 2^2 - 3 \cdot 2 + 2$

  $f(2) = 16 - 12 - 6 + 2$

  $f(2) = 0$

Verification on the calculator :

| F1 | F2 | F3 | F4 | F5 | F6 |
|---|---|---|---|---|---|
| | Algebra | Calc | Other | PrgmIO | Clear a-z... |

```
■ y1(x)                    2·x³ – 3·x² – 3·x + 2
■ y1(-1)                                       0
■ y1(1/2)                                      0
■ y1(2)                                        0
y1(2)
MAIN        RAD AUTO        FUNC 4/30
```

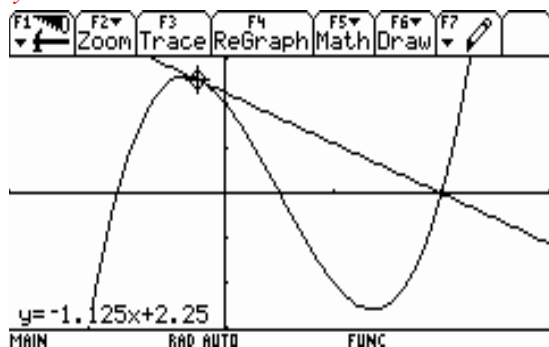- Roots -1 and $\frac{1}{2}$

Average $= \dfrac{-1 + \dfrac{1}{2}}{2} = -0.25$

$f(-0.25) = 2 \cdot (-0.25)^3 - 3 \cdot (-0.25)^2 - 3 \cdot (-0.25) + 2$

$f(-0.25) = 2.53125 \approx 2.53$

Slope at (-0.25, 2.53) will be: $f'(-0.25) = 6(-0.25)^2 - 6(-0.25) - 3 = -1.125$

**Equation of the tangent** where $x_1 = -0.25$, $y_1 = 2.53$ and $m = -1.125$ $\Rightarrow$

$y = -1.125x + 2.25$

```
F1    F2    F3    F4      F5    F6  F7
     Zoom Trace ReGraph Math Draw

y=-1.125x+2.25
MAIN        RAD AUTO        FUNC
```

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Intersection occurs when $y = 0 \Rightarrow -1.125x + 2.25 = 0 \Rightarrow x = 2 \Rightarrow$ **the intersection will be at**
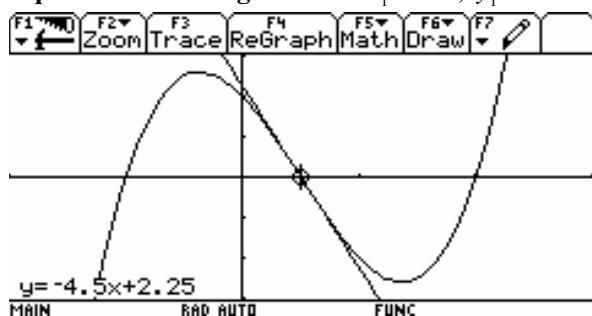**(2, 0)**

- Roots -1 and 2

$\text{Average} = \dfrac{-1+2}{2} = 0.5$

$f(0.5) = 2 \cdot (0.5)^3 - 3 \cdot (0.5)^2 - 3 \cdot (0.5) + 2$

$f(0.5) = 0$

Slope at (0.5, 0) will be: $f'(0.5) = 6(0.5)^2 - 6(0.5) - 3 = -4.5$

**Equation of the tangent** where $x_1 = 0.5$, $y_1 = 0$ and $m = -4.5 \Rightarrow$ $y = -4.5x + 2.25$



Intersection occurs when $y = 0 \Rightarrow -4.5x + 2.25 = 0 \Rightarrow x = 0.5 = \dfrac{1}{2} \Rightarrow$ **the intersection will be**
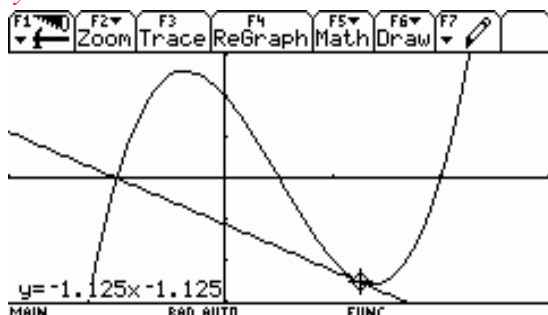**at** (0.5, 0)

- Roots 2 and $\dfrac{1}{2}$

$\text{Average} = \dfrac{2 + \dfrac{1}{2}}{2} = 1.25$

$f(1.25) = 2 \cdot (1.25)^3 - 3 \cdot (1.25)^2 - 3 \cdot (1.25) + 2$

$f(1.25) = -2.53125 \approx -2.53$

Slope at (1.25, -2.53) will be: $f'(1.25) = 6(1.25)^2 - 6(1.25) - 3 = -1.125$

**Equation of the tangent** where $x_1 = 1.25$, $y_1 = -2.53$ and $m = -1.125 \Rightarrow$
$y = -1.125x - 1.125$



Intersection occurs when $y = 0 \Rightarrow -1.125x - 1.125 = 0 \Rightarrow x = -1 \Rightarrow$ **the intersection will be**
**at** (-1, 0)

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Observing from my calculations the equation of the tangent line to the average of two of the three roots will intersect the graph at the third root.

**With this I can conclude that the conjecture stands for other functions as well.**

Now to prove my conjecture I will use the same steps and formulas but this time I will not use values but variables to have a general formula.

$f(x) = k(x-a)(x-b)(x-c)$ where $a$, $b$ and $c$ are the roots and k is a constant that changes the position of the curve. To simplify my calculation I will assume that $k = 1 \Rightarrow$
$$f(x) = (x-a)(x-b)(x-c)$$
This could be written also as:
$$f(x) = (x-a)(x^2 - bx - cx + bc)$$
*this might be useful when finding the derivative of the function in an easier way $\Rightarrow$
$$f'(x) = (x-a)(2x - b - c) + 1(x^2 - bx - cx + bc)$$
$$f'(x) = 2x^2 - bx - cx - 2ax + ab + ac + x^2 - bx - cx + bc$$
$$f'(x) = 3x^2 - 2ax - 2bx - 2cx + ab + ac + bc$$
$$f'(x) = 3x^2 - 2x(a+b+c) + a(b+c) + bc$$

The average of two of the roots will be $\dfrac{a+b}{2}$.

$$f\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)\left(\frac{a+b}{2} - c\right)$$
$$f\left(\frac{a+b}{2}\right) = \frac{(a+b-2a)(a+b-2b)(a+b-2c)}{8}$$
$$f\left(\frac{a+b}{2}\right) = \frac{(b-a)(a-b)(a+b-2c)}{8}$$
$$f\left(\frac{a+b}{2}\right) = \frac{-(a-b)(a-b)(a+b-2c)}{8}$$
$$f\left(\frac{a+b}{2}\right) = \frac{-(a-b)^2(a+b-2c)}{8}$$

The slope will be :
$$f'\left(\frac{a+b}{2}\right) = \frac{3(a+b)^2}{4} - \frac{2(a+b)(a+b+c)}{2} + \frac{2a(b+c) + 2bc}{2}$$
$$f'\left(\frac{a+b}{2}\right) = \frac{3(a+b)^2 - 4(a+b)(a+b+c) + 4a(b+c) + 4bc}{4}$$
$$f'\left(\frac{a+b}{2}\right) = \frac{(a+b)\ 3(a+b) - 4(a+b+c)\ + 4a(b+c) + 4bc}{4}$$
$$f'\left(\frac{a+b}{2}\right) = \frac{(a+b)(3a + 3b - 4a - 4b - 4c) + 4ab + 4ac + 4bc}{4}$$
$$f'\left(\frac{a+b}{2}\right) = \frac{(a+b)(-a - b - 4c) + 4ab + 4ac + 4bc}{4}$$

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$$f'\left(\frac{a+b}{2}\right) = \frac{-a^2 - ab - 4ac - ab - b^2 - 4bc + 4ab + 4ac + 4bc}{4}$$

$$f'\left(\frac{a+b}{2}\right) = \frac{-a^2 - ab - ab - b^2 + 4ab}{4}$$

$$f'\left(\frac{a+b}{2}\right) = \frac{-a^2 + 2ab - b^2}{4}$$

$$f'\left(\frac{a+b}{2}\right) = -\left(\frac{a-b}{2}\right)^2$$

To find the equation of the tangent I will use the previous formulas:

\* $y = m(x - x_1) + y_1$

\* $m = f'\left(\frac{a+b}{2}\right) = -\left(\frac{a-b}{2}\right)^2$

\* $x_1 = \frac{a+b}{2}$

\* $y_1 = f\left(\frac{a+b}{2}\right) = \frac{-(a-b)^2(a+b-2c)}{8}$

$$y = -\left(\frac{a-b}{2}\right)^2\left(x - \frac{(a+b)}{2}\right) - \frac{(a-b)^2(a+b-2c)}{8}$$

$$y = -\frac{(a-b)^2}{4}\left(x - \frac{(a+b)}{2}\right) - \frac{(a-b)^2}{8} \times (a+b-2c)$$

$$y = -\frac{(a-b)^2}{4}\left(x - \frac{(a+b)}{2}\right) + \frac{(a+b-2c)}{2}$$

$$y = -\frac{(a-b)^2}{4}\left(\frac{2x - a - b + a + b - 2c}{2}\right)$$

$$y = -\frac{(a-b)^2}{4}\left(\frac{2x - 2c}{2}\right)$$

$$y = -\frac{(a-b)^2}{4}\left(\frac{2(x-c)}{2}\right)$$

$$y = -\frac{(a-b)^2}{4} \mid x - c$$

As $y = 0$ when the tangent cuts the $x$-axis $\Rightarrow -\frac{(a-b)^2}{4} \mid x - c = 0 \Rightarrow x - c = 0 \Rightarrow x = c$

As x = c, which is the third i.e. the root not included in the average, we can conclude that the conjecture is true for all cubic equations. Hence, **the equation of tangent at the average of any two roots will intersect the y–axis at the third root.**
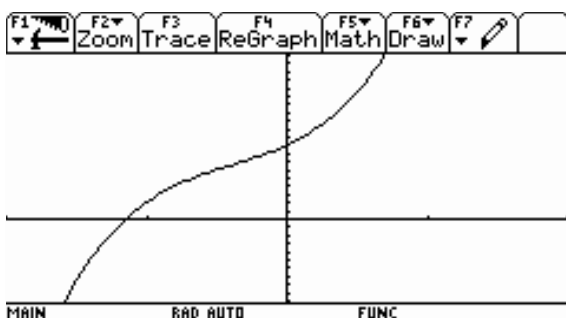
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Forward I will investigate the above properties with cubic functions which have one root or two roots.
I will first start with the cubic function which has only one root:
Let us consider the function: $f(x) = 6x^3 + 7x^2 + 8x + 9$



*Window values:
xmin=-2.
xmax=2.
xscl=1.
ymin=-10.
ymax=20.
yscl=1.
xres=2.

The root of the equation is:

- $x = -1.1456$

Proof using the remainder theorem:

- $f(x) = 6x^3 + 7x^2 + 8x + 9$

$f(-1.1456) = 6(-1.1456)^3 + 7(-1.1456)^2 + 8(-1.1456) + 9$

$f(-1.1456) = 0$

Equation of the tangent line

- Roots -1.1456 and -1.1456

$$\text{Average} = \frac{-1.1456 - 1.1456}{2} = -1.1456$$

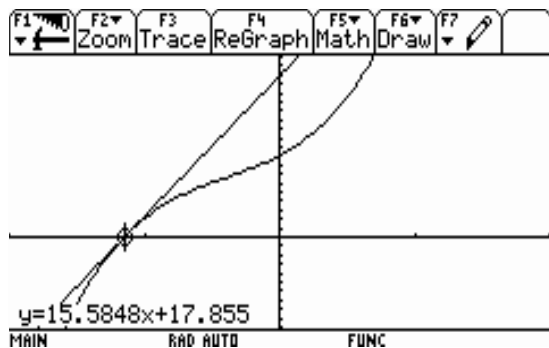$f(-1.1456) = 6(-1.1456)^3 + 7(-1.1456)^2 + 8(-1.1456) + 9$

$f(-1.1456) = 0$

$f'(x) = 18x^2 + 14x + 8$

Slope at (-1.1456, 0) will be: $f'(-1.1456) = 18(-1.1456)^2 + 14(-1.1456) + 8 = 15.6$

**Equation of the tangent** where $x_1 = -1.1456$, $y_1 = 0$ and $m = 15.6$ at $y = 0$
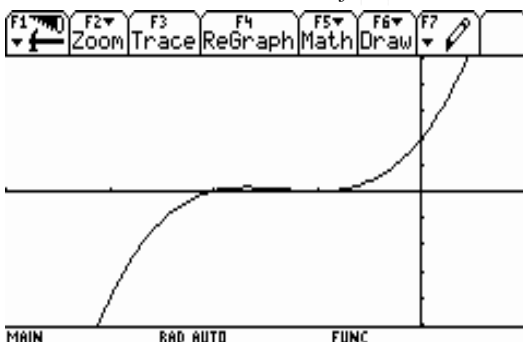
$x = -1.1456$



y=15.5848x+17.855

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**This shows that the conjecture is applicable to a cubic equation with one root, although not required.**

Now I will investigate a cubic function with two roots.
Let us consider the function: $f(x) = x^3 + 4x^2 + 5x + 2$



*Window values:

xmin=-4.
xmax=1.
xscl=1.
ymin=-5.
ymax=5.
yscl=1.
xres=2

The roots of the function are:
- $x = -2$
- $x = -1$

Proof using the remainder theorem:
$f(x) = x^3 + 4x^2 + 5x + 2$

- $f(-1) = (-1)^3 + 4(-1)^2 + 5(-1) + 2$

  $f(-1) = 0$

- $f(-2) = (-2)^3 + 4(-2)^2 + 5(-2) + 2$

  $f(-2) = 0$

Equation of the tangent line (average of two of roots):
- Roots -2 and -1

  $\text{Average} = \dfrac{-1-2}{2} = -1.5$

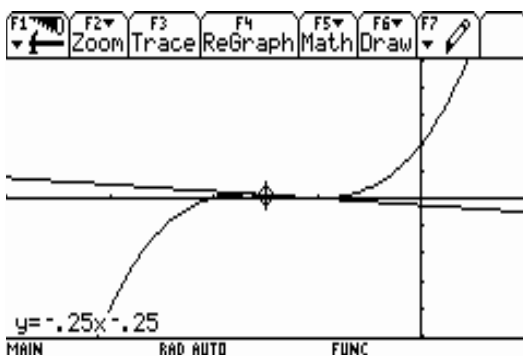  $f(-1.5) = (-1.5)^3 + 4(-1.5)^2 + 5(-1.5) + 2$

$f(-1.5) = 0.125$

$f'(x) = 3x^2 + 7x + 5$

Slope at (-1.5, 0.125) will be: $f'(-1.1456) = 3(-1.1456)^2 + 7(-1.1456) + 5 = -0.25$

**Equation of the tangent** where $x_1 = -1.5$, $y_1 = 0.125$ and $m = -0.25 \Rightarrow$ at $y = 0$, $x = -1$



**This again shows that the conjecture is applicable to a cubic equation with two roots.**

With this I will conclude that the conjecture stated is valid.