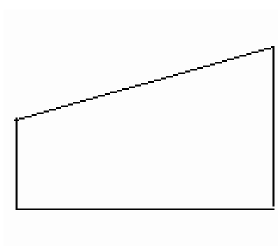
There are different ways through which can estimating the area under a curve of a certain function and one of the many methods is by the trapezium rule.

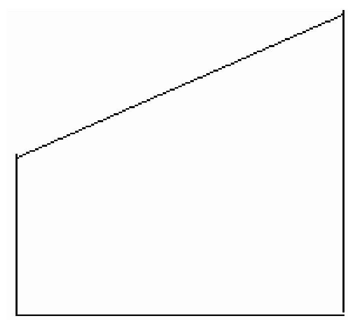
The area of a trapezium (trapezoid) is determined by;

$A = \frac{1}{2}\ a+b\ \ h$ Or in words, the average of two sides times the base, which could also be expressed as $\left(\frac{a+b}{2}\right)h$ where by if considering trapeziums formed under a curve; parallel y intervals form the sides and Δx the base, which also doubles as the height.

To determine the area under the function $g(x) = x^2 + 3$, with in the interval of $a = 0$, $b = 1$, one can estimate the area by trapeziums over subintervals by finding the Δx. $\approx$

In the first attempt approximate an area the curve of the given function above, two trapeziums will be considered.





The height (h) in this case, is the difference between the X intervals Δx

Therefore, for the two trapeziums, h = Δx is determined by;

$\Delta x = \frac{b-a}{n}$, Where by n is the number of trapeziums or intervals.

So, we know b is 1 and a is 0 comparing to the integrals of the function,

$$\int_a^b g\ x$$

$$\int_0^1 x^2 + 3$$

So now, $\Delta x = \dfrac{b-a}{n}$

$= \dfrac{1-0}{2}$

$= \dfrac{1}{2}$

Giving us 0.5, as our height (h)

NB: Δx , 0.5 is considered to be the height because the trapezoids are rotated 90^0 which makes their base the height.

To be able to calculate the area of both trapeziums, we have to consider all the lengths required hence, $(a+b)h$

We will therefore have to find the values of Y intervals, which form $(a+b)$ sides. This can show by;

$y_0 = f(a)$
$= f(0)$
$= 0^2 + 3$
$= 3 units$

$y_1 = f(a + \Delta x)$
$= f(0.5)$
$= 0.5^2 + 3$
$= 3.25 units$

$y_2 = fa + 2\Delta x)$
$= f(1) = 1^2 + 3$
$= 4 units$

This in one way or the other can be proved by;
Considering y_0, if 0 is put into the function $g(x) = (x^2 + 3)$ instead of x
Will result in 3 which is the interval of x_0 when a is 0.
And also considering intervals, 0.5 and 1 respectively instead for x, in the function $g(x) = (x^2 + 3)$, the results are 3.5 and 4 respectively, which proves my y intervals.

Having found all the required lengths, it's now possible to calculate an approximation of an area under the curve of the given function above using 2 trapeziums

Therefore, areas for the two trapeziums;

$$A_1 \approx \frac{1}{2}\left(y_0 + y_1\right)\Delta x$$
$$= \frac{1}{2}(3+3.25)0.5$$
$$= \frac{3.125}{2}$$
$$= 1.5626 a.u$$

$$A_2 \approx \frac{1}{2}\left(y_1 + y_2\right)\Delta x$$
$$= \frac{1}{2}(3.25+4)0.5$$
$$= \frac{3.625}{2}$$
$$= 1.8125 a.u$$

In order to estimate the total area under the curve of the given function, we need to consider area for both trapeziums,

This leads us to:

$$\approx A_1 + A_2$$

$$\approx 1.5625 + 1.8125$$

Gi ving; $T.A \approx 3.375 a.u$

It's good to compare the area got by estimation and area by using function to make notice of the limitations of approximation

Therefore, $A = \int_0^1 g\left(x\right) = x^2 + 3$

Giving, $= \left[\frac{x^3}{3} + 3\right]_0^1$

$$= \frac{1}{3} + 3$$
$$= \frac{1+9}{3}$$

And therefore arriving at $= \frac{10}{3}$ or $3.333...$

Comment: what can be noticed is that there is a big difference if the trapezium rule is used to estimate the area than when the function is used. There is a difference of $\approx 0.042(3\text{dp})$ which is quite a big difference.

In the case where the number of intervals is increased to 5, we still take the same procedure, to estimate the area. But in this case, we now have more intervals (n) increased to five which will vary the lengths of the trapeziums. We will therefore have to determine a new height Δx and $(a+b)$, the side, which form the y intervals.

Therefore, $\Delta x = \dfrac{b-a}{n}$, where in this case n = 5 but b and a are the same, hence $(x=0, x=1)$

This will give us, $\dfrac{1-0}{5} = 0.2$

And the y intervals,

$$y_0 = f(a) = f(0) = 0^2 + 3 = 3units$$

$$y_1 = f(a+\Delta x) = f = (0.2) = (0.2)^2 + 3 = 3.04units$$

$$y_2 = f(a+2\Delta x) = 0.4 = (0.4)^2 + 3 = 3.16units$$

$$y_3 = f(a+3\Delta x) = 0.6 = (0.6)^2 + 3 = 3.36units$$

$$y_4 = f(a+4\Delta x) = (0.8) = (0.8)^2 + 3 = 3.64units$$

$$y_5 = f(b) = 1 = 1^1 + 3 = 4units$$

The areas will now be

$$A_1 \approx \frac{1}{2}(y_0+y_1)\Delta x = \frac{1}{2}(3+3.04)0.2 = \frac{1.208}{2} = 0.604a.u$$

$$A_3 \approx \frac{1}{2}(y_2+y_3)\Delta x = \frac{1}{2}(3.16+3.36)0.2 = 0.652a.u$$

$$A_4 \approx \frac{1}{2}(y_3+y_4)\Delta x = \frac{1}{2}(3.36+3.64)0.2 = 0.7a.u$$

$$A_5 \approx \frac{1}{2}(y_4+y_5)\Delta x = \frac{1}{2}(3.64+4)0.2 = 0.764a.u$$

Therefore, total approximation of the area

$\approx A_1 + A_2 + A_3 + A_4 + A_5$

$= 0.604 + 0.62 + 0.652 + 0.7 + 0.764$

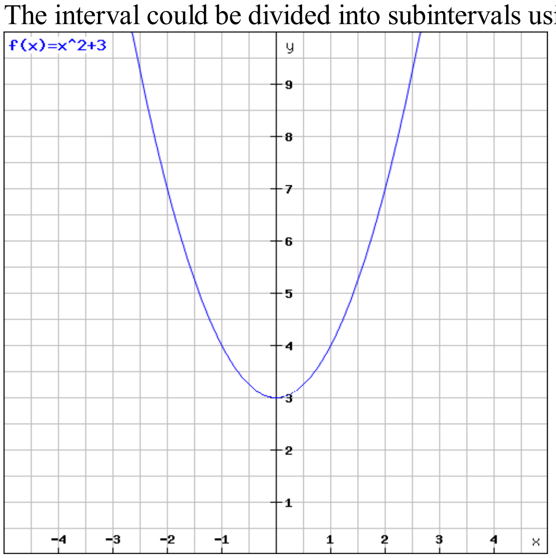
$= 3.34 a.u$

Comment; it can be observed after two approximations that the more trapeziums you use to approximate the area under the curve of the function $g(x) = x^2 + 3$, the better approximation and the nearer you come to the area derived when calculated using the integral.

With help of technology, the area under a curve approximated using the trapezoid rule requires splitting the interval $[a, b]$ into n subintervals of length based on equally spaced points $a = x_0, x_1, x_2 ... x_n = b$ with corresponding ordinates $y_o, y_1, y_2, ... y_n$.

If we consider an area of under a curve on the interval $[0,1]$

The interval could be divided into subintervals using $n = 10$



And then construct trapeziums over each sub interval,

The sum of the trapezoids constructed will probably give an approximation of an area under the curve on the interval $[a = o, b = 1]$

The new base (h) will therefore be,

$$\Delta x = \frac{b-a}{n} \quad = \frac{1-0}{10} \quad = 0.1$$

And the sides will be,

$y_0 = f(a)$
$= 3$

$y_1 = f(a + \Delta x)$
$= (0.1)^2 + 3$
$= 3.1$

$y_2 = f(a + 2\Delta x)$
$= (0.2)^2 + 3$
$= 3.04$

$y_3 = f(a + 3\Delta x)$
$= (0.3)^2 + 3$
$= 3.09$

$y_4 = (0.4)^2 + 3$
$= 3.16$

$y_5 = (0.5)^2 + 3$
$= 3.25$

$y_6 = (0.6)^2 + 3$
$= 3.36$

$y_7 = (0.7)^2 + 3$
$= 3.49 units$

$y_8 = (0.8)^2 + 3$
$= 3.64 units$

$y_9 = (0.9)^2 + 3$
$= 3.81$

$y_{10} = (1)^2 + 3$
$= 4 units$

Having found all the sides, y and Δx which also doubles as the base or height, we can now approximate the areas of each trapezium.

Therefore area will be as follows

$A_1 \approx \frac{1}{2}(y_0 + y_1)\Delta x$
$= \frac{1}{2}(3 + 3.01)0.1$
$= 0.3005$a.u

$A_2 = \frac{1}{2}(3.01 + 3.04)0.1$
$= 0.3025 a.u$

$A_3 = \frac{1}{2}(3.04 + 3.09)0.1$
$= 0.3065 a.u$

$A_4 = \frac{1}{2}(3.09 + 3.16)0.1$
$= 0.3125 a.u$

$A_5 = \frac{1}{2}(3.16 + 3.25)0.1$
$= 0.3205 a.u$

$A_6 = \frac{1}{2}(3.25 + 3.36)0.1$
$= 0.3305$

$A_7 = \frac{1}{2}(3.36 + 3.49)0.1$
$= 0.3425 a.u$

$A_8 = \frac{1}{2}(3.49 + 3.64)0.1$
$= 0.3565 a.u$

$A_9 = \frac{1}{2}(3.64 + 3.81)0.1$
$= 0.3725$

$A_{10} = \frac{1}{2}(3.81 + 4)0.1$
$= 0.3905 a.u$

Therefore, total approximation area for $n = 10$ trapeziums
$\approx A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10}$
$= (0.3005 + 0.3025 + 0.3065 + 0.3125 + 0.3205 + 0.3305 + 0.3425 + 0.3565 + 0.3725 + 0.3905) a.u$
$= 3.335 a.u$

Comment; having estimated the area using 2, 5 and 10 trapeziums respectively, it can be noticed that the more trapeziums one uses to draw a conclusion, the nearer or the better approximation one gets. **I.e**. the more trapeziums one uses to approximate an area under a curve, the nearer you come to the solution got when u use the function to determine the area.

For instance, when two trapeziums are considered, the approximation is 3.375 which makes a big difference from the area derived when using the function.

Also when 5 trapeziums are considered, the approximation is 3.34 which also gives a greater difference of 1.002

But then it can be proved that the greater number of trapeziums one considers approximating the area under a curve, the more accurate answer you arrive at. Using 10 trapeziums gives an approximation of 3.335 which nearer to the area derived when using the function which is $\frac{10}{3}$ or $3.333...$

The difference between the x intervals (Δx), which also doubles as the height (h) of the all the trapeziums under the curve of the function $g(x) = x^2 + 3$ is determined by the difference between b and a divided by the number of trapeziums (n) or the number of intervals.

Therefore, for the function with unknown number of trapeziums,

To find the Δx or height for the area $x = a = 0$ to $x = b = 1$ we use;

$$\Delta x = \frac{b-a}{n} \qquad = \frac{1-0}{n} \qquad = \frac{1}{n}$$

And we also need, the sides, which are the y intervals, therefore;

$$\begin{array}{llll} y_0 = f(a) = 0 & y_1 = f(a+\Delta x) & y_2 = f(a+2\Delta x) & y_n = f(b) = 1 \\ = 0^2 + 3 & = 3 + \left(\frac{1}{n}\right) units & = 3 + 2\left(\frac{1}{n}\right) \quad \ldots & = 1^2 + 3 \\ = 3units & & & = 4units \end{array}$$

Therefore; the area for the curve $f(x)$ will be given by summing up the areas of the n trapeziums,

Therefore,

$$a_1 \approx \left(y_0 + y_1\right)\frac{1}{2n} \qquad a_2 \approx \frac{1}{2}\left(\left(\frac{1}{n}\right)^2 + 3 + \left(\frac{2}{n}\right)^2 + 3\right)\frac{1}{n} \quad \ldots$$

$$\approx \left(3 + \left(3 + \frac{1}{n}\right)\right) \qquad \approx \frac{1}{2n}\left(6 + \left(\frac{3}{n}\right)^2\right)a.u$$

$$\approx \left(6 + \frac{1}{n}\right)a.u$$

$$\ldots$$

$$A_n = \frac{1}{2}\left(y_{n-1} + y_n\right)\Delta x$$

That leads to the statement below approximating the area under a curve with n trapeziums;

$$T.A = a_1 + a_2 \ldots a_n$$

$$\approx \frac{1}{2n}\left(3 + \left(\frac{1}{n}\right)^2 + 3\right) + \frac{1}{2n}\left(3 + \left(\frac{2}{n}\right)^2 + 3\right)\ldots\frac{1}{2n}\left(y_{n-1} + 4\right)$$

$$\approx \frac{1}{2n}\left(\left(6 + \left(\frac{1}{n}\right)^2\right) + 6 + \left(\frac{3}{n}\right)^2 \ldots \left(y_{n-1} + 4\right)\right)$$

$$\approx \left(6 + \left(\frac{1}{n}\right)^2\right)\frac{1}{2n} + \left(6 + \left(\frac{3}{n}\right)^2\right)\frac{1}{2n}\ldots\left(y_{n-1} + 4\right)\frac{1}{2n}$$

$$\approx \left(12 + \left(\frac{1}{n}\right)^2 + \left(\frac{3}{n}\right)^2 \ldots \left(y_{n-1} + 4\right)\right)a.u$$

To approach this task, just like the previous one's we need the intervals, which in this case is tricky because no numerical are involved at all but the problem can still be encountered supposing the alphabets were numerals.

He will therefore still require the base or height, which the difference between b and a divided by n or the number of trapeziums.

Therefore, $\Delta x = \dfrac{b-a}{n}$

And the y intervals, which are not known in this case, we will therefore have to consider;
$y_0, y_1, y_2, \ldots, y_{n-1}, y_n$

$y_0 = f(a)$

$$y_1 = f(a + \Delta x)$$
$$= \left(a + \frac{b-a}{n}\right) units$$

$$y_2 = f(a + 2\Delta x)$$
$$= \left(a + 2\left(\frac{b-a}{n}\right)\right) units$$

The same approach is taken to estimate the area,

Therefore, area 1

$$\approx \frac{1}{2}(y_0 + y_1)\Delta x$$

$$\approx \frac{1}{2}(y_0 + y_1)\frac{b-a}{n}$$
$$\approx \frac{b-a}{2n}\left(a + \left(a + \frac{b-a}{n}\right)\right)$$

$$\approx \frac{b-a}{2n}\left(2a + \left(\frac{b-a}{n}\right)\right) a.u$$

Area 2

$$\approx \frac{1}{2}(y_1 + y_2)\Delta x$$

$$\approx \frac{1}{2}\left(y_1 + y_2\right)\frac{b-a}{n}$$

$$\approx \frac{b-a}{2n}\left(y_1 + y_2\right)$$

$$\approx \frac{b-a}{2n}\left(a + \frac{b-a}{n}\right) + a + 2\left(\frac{b-a}{n}\right)$$

$$\approx \frac{b-a}{2n}\left(2a + \left(\frac{b-a}{n}\right) + 2\left(\frac{b-a}{n}\right)\right) a.u$$

…

$$\text{Area}_n$$

$$\approx \frac{1}{2}\left(y_{n-1} + y_n\right)\Delta x$$

$$\approx \frac{1}{2}\left(y_{n-1} + y_n\right)\frac{b-a}{n}$$

$$\approx \frac{b-a}{2n}\left(y_{n-1} + y_n\right) a.u$$

Therefore, statement estimating area under any curve $y = f(x)$, from $x = a$ to $x = b$

$$\approx A_1 + A_2 + \ldots + A_n$$

$$\approx \frac{b-a}{2n}\left(2a + \left(\frac{b-a}{n}\right)\right) + \frac{b-a}{2n}\left(2a + \left(\frac{b-a}{n}\right) + 2\left(\frac{b-a}{n}\right)\right) + \ldots + \frac{b-a}{2n}\left(y_{n-1} + y_n\right)$$

$$\approx \frac{b-a}{2n}\left(4a + \left(\frac{b-a}{n}\right)^2 + 2\left(\frac{b-a}{n}\right) + \ldots + \left(y_{n-1} + y_n\right)\right) a.u$$

To find the areas of the following functions through estimating the area of $n = 8$ intervals, we will need to take the same approach as previously.

We still require the base, which also doubles as the height and the sides $(a+b)$ to be able to estimate the areas

Therefore, the base of the function with 8 intervals with in the range of $a = 1, b = 3$ will be given by;

$$\Delta x = \frac{b-a}{n}$$

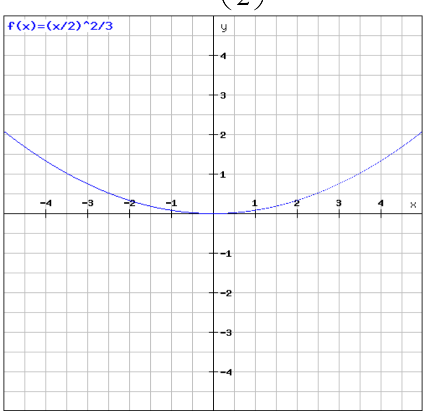
$$= \frac{3-1}{8}$$

$$= \frac{2}{8} = \frac{1}{4} = 0.25$$

This base applies to all the given functions as they are in the same range $a = 1, b = 3$ also with the same number of intervals, $n = 8$

To be able to calculate the areas, we all also require the sides $(a+b)$, which are the y intervals. But the sides will vary when used to the three function because when $a = 1$ and $b = 3$ are put in the given functions instead for x will give different results.

For the function $\int_1^3 \left(\frac{x}{2}\right)^{\frac{2}{3}} dx$,



we can split the area covered under the curve formed by the function to estimate its area.
Y intervals, which form the other sides of the formed trapeziums, are crucial in estimating the trapezium areas. They form what in the formulae is $(a+b)$

$y_1 = f\ a + \Delta x$	$y_2 = f\ a + 2\Delta x$	$y_3 = f\ a + 3\Delta x$	$y_4 = f\ a + 4\Delta x$	$y_5 = f\ a + 5\Delta x$
$= 0.63 + 0.25$	$= 0.63 + 0.5$	$= 0.63 + 0.75$	$= 0.63 + 1$	$= 0.63 + 1.25$
$= 0.88 units$	$= 1.13 units$	$= 1.38 units$	$= 1.63 units$	$= 1.88 units$

$y_6 = f\ a+6\Delta x$	$y_7 = f\ a+7\Delta x$	$y_8 = f\ a+8\Delta x$	$y_9 = f\ a+9\Delta x$
$=0.63+1.5$	$=0.63+1.75$	$=0.63+1.75$	$=0.63+1.75$
$=2.13units$	$=2.38units$	$=2.63units$	$=2.88units$

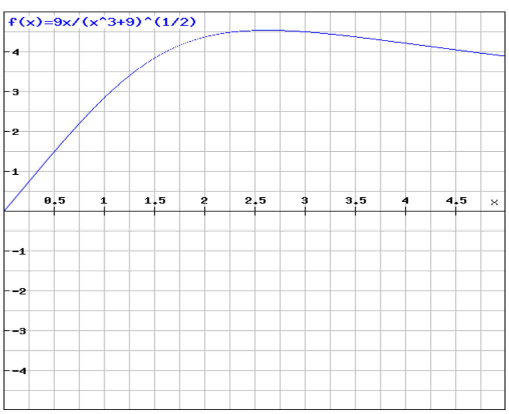
Knowing all the necessary sides Δx and the y intervals, its then possible to estimate the area of every trapezium, the results give us, by applying the sides to the formulae of finding the area of trapeziums $\frac{1}{2}\ a+b\ h$ where in our case we consider; $\frac{1}{2}\ y_1+y_2\ \Delta x+...+\frac{1}{2}\ y_{n-1}+y_n\ \Delta x$ Therefore, approximation for the areas of $n=8$ trapeziums gives the following results.

$a_1=\frac{1}{2}\ y_1+y_2\ \Delta x$	$a_3=\frac{1}{2}\ y_3+y_4\ \Delta x$	$a_5=\frac{1}{2}\ y_5+y_6\ \Delta x$	$a_7=\frac{1}{2}\ y_7+y_8\ \Delta x$
$=0.25125a.u$	$=0.37625a.u$	$=0.50125a.u$	$=0.62625a.u$
$a_2=\frac{1}{2}\ y_2+y_3\ \Delta x$	$a_4=\frac{1}{2}\ y_4+y_5\ \Delta x$	$a_6=\frac{1}{2}\ y_6+y_7\ \Delta x$	$a_8=\frac{1}{2}\ y_8+y_9\ \Delta x$
$=0.31375a.u$	$=0.43875a.u$	$=0.56375a.u$	$=0.68875a.u$

Therefore total approximation area will be; $\approx a_1+a_2+a_3+...+a_8 a.u$

This gives us the an approximation of $3.76a.u$

For the function $\int_1^3 = \frac{9x}{\sqrt{x^3+9}}dx$, knowing the height which Δx the sides will be,



The same procedure can apply to the above functions and other function as well to estimate their areas using trapezoids are the $(a+b)$ sides keep on varying depending on the structure of the curve formed, we always have to find news sides to that function but Δx supposing the functions are in the same range.

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$$y_1 = f(a+\Delta x) = 2.85 + 0.25 = 3.1 units$$

$$y_2 = f(a+2\Delta x) = 2.85 + 0.5 = 3.35 units$$

$$y_3 = 3.6 units \quad y_5 = 4.1 units$$
$$y_4 = 3.85 units \quad y_6 = 4.35 units$$

$$y_7 = 4.6 units$$
$$y_8 = 4.85 units$$
$$y_9 = 5.1 units$$

The area estimated areas are as follows;

Areas

$$a_1 \approx \frac{1}{2}(y_1 + y_2)\Delta x = \frac{1}{2}(3.1 + 3.35)0.25 = 0.80625 a.u$$

$$a_2 \approx \frac{1}{2}(y_2 + y_3)\Delta x = \frac{1}{2}(3.35 + 3.85)0.25 = 0.9 a.u$$

$$a_3 \approx \frac{1}{2}(y_3 + y_4)\Delta x = 0.93125 a.u$$

$$a_4 \approx \frac{1}{2}(y_4 + y_5)\Delta x = 0.99375 a.u$$

$$a_6 \approx \frac{1}{2}(y_6 + y_7)\Delta x = 1.11875 a.u$$

$$a_8 \approx \frac{1}{2}(y_8 + y_9)\Delta x = 1.24375 a.u$$

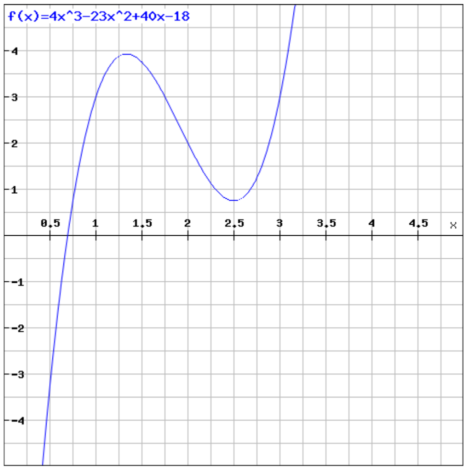
$$a_5 \approx \frac{1}{2}(y_5 + y_6)\Delta x = 1.05625 a.u$$

$$a_7 \approx \frac{1}{2}(y_7 + y_8)\Delta x = 1.18125 a.u$$

The total approximation of the areas will therefore be;

$$\approx a_1 + a_2 + a_3 + \ldots + a_8$$
$$= (0.80625 + 0.9 + 0.93125 + 0.99375 + 1.05625 + 1.11875 + 1.18125 + 1.24375)\, a.u$$
$$= 8.2 a.u$$

For the function $\int_1^3 (4x^3 - 23x^2 + 40x - 18)\, dx$,



The y intervals of the new function will then be;

$y_1 = f(a+\Delta x)$
$= 3+0.25$
$= 3.25units$

$y_2 = f(a+2\Delta x)$
$= 3+0.5$
$= 3.5units$

$y_3 = 3.75units$
$y_4 = 4units$

$y_5 = 4.25units$
$y_6 = 4.5units$
$y_7 = 4.75units$

$y_8 = 5units$
$y_9 = 5.25units$

Having known all the required sides $\Delta x \ and (a+b)$ its then possible to estimate the areas of all the eight trapeziums formed under the curve.

Therefore the areas,

$a_1 \approx \frac{1}{2}(y_1+y_2)\Delta x$
$= \frac{1}{2}(3.25+3.5)0.25$
$= 0.84375a.u$

$a_2 \approx \frac{1}{2}(y_2+y_3)\Delta x$
$= \frac{1}{2}(3.5+3.75)0.25$
$= 0.90625a.u$

$a_3 = 0.96875a.u$
$a_4 = 1.03125a.u$
$a_5 = 1.09375a.u$

$a_6 = 1.15625a.u$
$a_7 = 1.21875a.u$
$a_8 = 1.28125a.u$

Which gives a total approximation area $\approx a_1 + a_2 + a_3 + ... + a_8$

$\approx 6a.u$