

## Introduction

This portfolio will investigate the patterns and aspects of infinite surds. Technologies, graphs, and charts will be used in the process of the investigation, allowing the understanding of infinite surds to be more comprehensive. In the beginning, two examples of infinite surds will be examined, and some similarities between the two may be found. Using the knowledge gained from the previous two examples, we will try to come up with some general statements and restrictions that are true for all infinite surds. First we will start by defining a surd.

### Definition of a Surd

In order to qualify as a surd, a number has to be:

1. The root of a positive rational number
2. Unable to be expressed as a fraction
3. Can only be expressed in its exact value with the root sign
4. An irrational number

For example,  $\sqrt{2}$  is a surd, but  $\sqrt{4}$  is not since its exact value can be expressed as a fraction and integer.

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$$

This is an example of an infinite surd, where identical surds are being added under the previous root repeatedly.

We can also turn this into a sequence where  $a_1 = \sqrt{1 + \sqrt{1}}$ ,  $a_2 = \sqrt{1 + \sqrt{1 + \sqrt{1}}}$ ,

$$a_3 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}$$
, and  $a_\infty = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$

To find the relationship between  $a_n$  and  $a_{n+1}$ , we can first find a formula for  $a_{n+1}$  in terms of  $a_n$ , the substitution method along with the algebraic process can be used to find the formula.

### Formula for $a_{n+1}$ In Terms of $a_n$

Step1

Let  $n=1$  which makes  $a_n=a_1$

Step2

$$a_1 = \sqrt{1 + \sqrt{1}}$$

$$a_2 = \sqrt{1 + \sqrt{1 + \sqrt{1}}}$$

From this, we can see that the expressions within the dotted boxes are equivalent

Step3

Now we can substitute  $a_1$  as part of  $a_2$  knowing that  $a_1$  is equivalent to part of  $a_2$

$$a_2 = \sqrt{1 + a_1}$$

By substituting  $a_1$  into the  $a_2$  equation, the above equation resulted.

Step4

Since  $n=1$  and  $a_{n+1}=a_2$ ,

$$a_{n+1} = \sqrt{1 + a_n}$$

### Decimal Values of the First Ten Terms of the Sequence

Below are the values of the first ten terms of the sequence accurate to the 9<sup>th</sup> place after the decimal point.

$$a_1 = \sqrt{1 + \sqrt{1}} = 1.414213562 \dots$$

$$a_2 = \sqrt{1 + \sqrt{1 + \sqrt{1}}} = 1.53773974 \dots$$

$$a_3 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}} = 1.585182 \dots$$

$$a_4 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}} = 1.6184754 \dots$$

$$a_5 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}} = 1.662207 \dots$$

$$a_6 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}}} = 1.67429 \dots$$

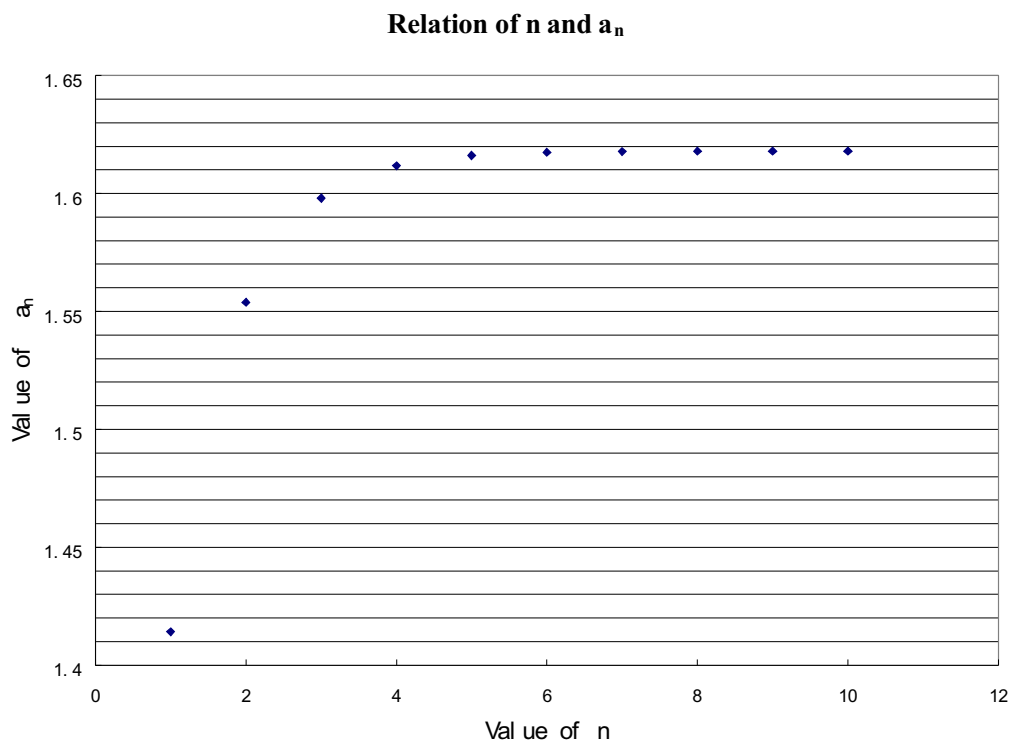
$$a_7 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}}}} = 1.678291 \dots$$

$$a_8 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}}}}} = 1.679731 \dots$$

$$a_9 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}}}}}} = 1.68062 \dots$$

$$a_{10} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}}}}}}}} = 1.618033989\dots$$

A graph can be plotted using the data from the previous page.



As the value of  $n$  increases, the value of  $a_n$  seems to increase less. The chart below describes the difference between  $a_n$  and  $a_{n+1}$  as the value of  $n$  increases.

**Difference of  $a_n$  and  $a_{n+1}$**

$a_n - a_{n+1}$	Decimal value rounded to the exact billionth
$a_1 - a_2$	-0.139560412
$a_2 - a_3$	-0.044279208
$a_3 - a_4$	-0.013794572
$a_4 - a_5$	-0.004273453
$a_5 - a_6$	-0.001321592
$a_6 - a_7$	-0.000408492
$a_7 - a_8$	-0.000126240
$a_8 - a_9$	-0.000039011
$a_9 - a_{10}$	-0.000012055

According to the chart, as the value of  $n$  increases, the difference between  $a_n$  and  $a_{n+1}$  decreases. Based on this, we can predict that as the value of  $n$  becomes very large,  $a_n - a_{n+1} \rightarrow 0$ , and we can come up with an expression that represents  $a_n - a_{n+1}$  as  $n$  approaches infinity.

**Equation for  $a_n - a_{n+1}$  When  $n \rightarrow \infty$**

$$\lim_{n \rightarrow \infty} (a_n - a_{n+1}) = 0$$

Since  $a_n = a_{n+1}$ , when  $n = \infty$ , their difference would be 0. So this means that the sequence is a convergent sequence, which is a sequence that approaches a limit.

When  $n = \infty$ , the value of  $n$  and  $n+1$  is identical, since infinity is the largest number possible. No matter how much infinity is increased by or decreased by, its value will always stay the same.  
 $\therefore \infty + x = \infty - x = \infty$

## Solving the Sequence

Now we can solve for the sequence  $a_{\infty} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$

### Step 1

Let  $a_{\infty} = x$ , making  $a_{\infty+1} = x$

Then substitute  $x$  into the expression  $a_{n+1} = \sqrt{1 + a_n}$  resulting in  $x = \sqrt{1 + x}$

### Step 2

To simplify this equation in order to solve for  $x$ , we first square both sides, then allow one side of the equation to become "0". This way, a quadratic equation will result

$$(x)^2 = (\sqrt{1 + x})^2$$

$$x^2 = 1 + x$$

$$x^2 - x - 1 = 1 + x - 1 - x$$

$$x^2 - x - 1 = 0$$

### Step 3

Use the quadratic formula  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$  to solve for " $x$ ", the exact value of the infinite

surds. In this case, only the positive root of " $x$ " will be taken since only positive numbers have real roots and the process of solving for the root of a negative number will result in an unreal solution.

$$x = \frac{-(-1) + \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{1 + \sqrt{1 + 4}}{2}$$

$$x = \frac{1 + \sqrt{5}}{2}$$

The exact value of the infinite surd  $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$  is  $\frac{1 + \sqrt{5}}{2}$ . When expressed as a decimal

rounded to the nearest billionth, the value of the infinite surd is 1.618033989.

To prove that the previous methods used are true for all infinite surds, we can apply them in a different situation.

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 \dots}}}}}$$

In this case, the sequence would be expressed as  $a_1 = \sqrt{2 + \sqrt{2}}$ ,  $a_2 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$ ,

$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$
, etc.

Now we will try to find a formula for  $a_{n+1}$  to show the relationship between  $a_n$  and  $a_{n+1}$ .

### Formula for $a_{n+1}$ In Terms of $a_n$

Step 1

Let  $n=1$ , making  $a_n=a_1$

Step 2

$$a_1 = \sqrt{2 + \sqrt{2}}$$

$$a_2 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$$

Once again,  $a_1$  is identical to part of  $a_2$ , so we can substitute  $a_1$  into  $a_2$ .

Step 3

The following equation resulted by substitution

$$a_2 = \sqrt{2 + a_1}$$

Since  $n=1$  and  $a_{n+1}=a_2$

$$a_{n+1} = \sqrt{2 + a_n}$$

### Decimal Values of the First Ten Terms of the Sequence

Below are the values of the first ten terms of the sequence accurate to the 9<sup>th</sup> place after the decimal point.

$$a_1 = \sqrt{2 + \sqrt{2}} = 1.84779065 \dots$$

$$a_2 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} = 1.96157056 \dots$$

$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} = 1.98491653 \dots$$

$$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} = 1.995092 \dots$$

$$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}} = 1.998767 \dots$$

$$a_6 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}} = 1.999404 \dots$$

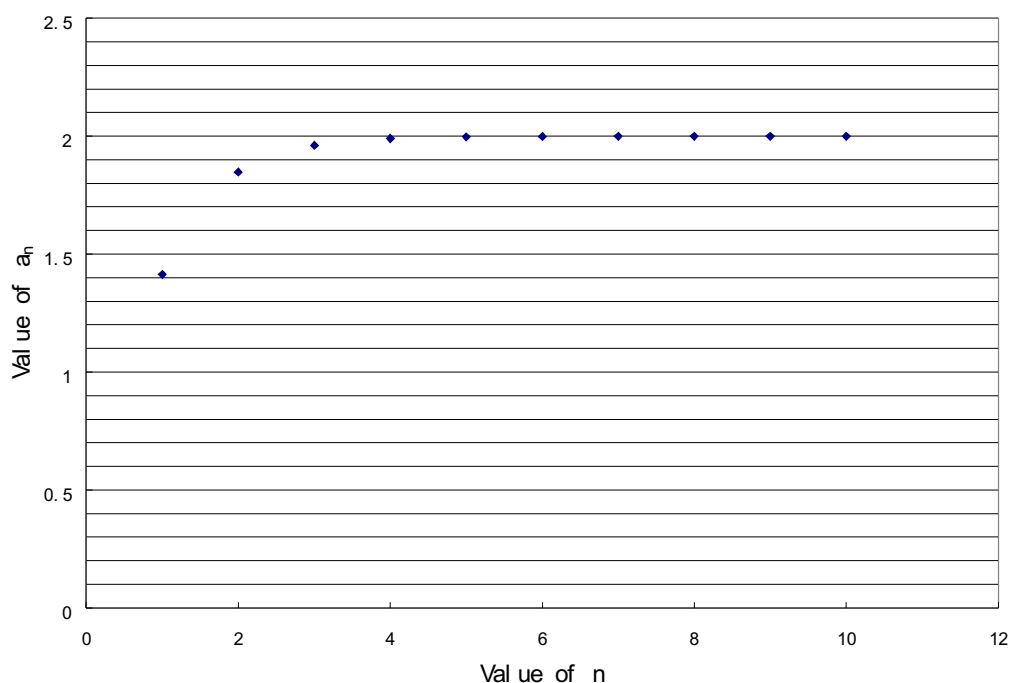
$$a_7 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}}} = 1.9996251 \dots$$

$$a_8 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}}}} = 1.9999088 \dots$$

$$a_9 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}}}}} = 1.9999767 \dots$$

$$a_{10} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}}}}}} = 1.999999412 \dots$$

### Relation of $a_n$ and $a_{n+1}$



Once again, as the value of  $n$  increases, the value of  $a_n$  seems to increase less. This is similar to what has happened with the previous example. The chart below describes the difference between  $a_n$  and  $a_{n+1}$  as the value of  $n$  increases.

### Difference of $a_n$ and $a_{n+1}$

$a_n - a_{n+1}$	Decimal value rounded to the exact billionth
$a_1 - a_2$	-0.1138114958
$a_2 - a_3$	-0.028798892
$a_3 - a_4$	-0.007221459
$a_4 - a_5$	-0.001806725
$a_5 - a_6$	-0.000451767
$a_6 - a_7$	-0.000112947
$a_7 - a_8$	-0.000028237
$a_8 - a_9$	-0.000007059
$a_9 - a_{10}$	-0.000001765

Compare to the previous example, the difference between  $a_n$  and  $a_{n+1}$  is less in this case. However, the same pattern occurred. As the value of  $n$  increases, the difference between  $a_n$  and  $a_{n+1}$  decreases. This means that when the value of  $n$  is very large  $a_n - a_{n+1} \rightarrow 0$ . The expression  $\lim_{n \rightarrow \infty} (a_n - a_{n+1}) = 0$ , used to describe the difference between  $a_n$  and  $a_{n+1}$  when  $n \rightarrow \infty$  for the last example, is also true for in

this case, proving that  $a_\infty = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}}$  is also a convergent sequence.

### Solving the Sequence

Now we can solve for the sequence  $a_{\infty} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}}$  the same way as we solved for the previous example

#### Step 1

Let  $a_{\infty} = x$ , making  $a_{\infty+1} = x$

Then substitute x into the expression  $a_{n+1} = \sqrt{2 + a_n}$  resulting in  $x = \sqrt{2 + x}$

#### Step 2

Square each side, then allow one side of the equation to become 0, a quadratic equation resulted.

$$(x)^2 = (\sqrt{2 + x})^2$$

$$x^2 = 2 + x$$

$$x^2 - x - 2 = 2 + x - 2 - x$$

$$x^2 - x - 2 = 0$$

#### Step 3

The quadratic formula  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$  will once again be used to solve for x, and only the positive root will be taken.

$$x = \frac{-(-1) + \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)}$$

$$x = \frac{1 + \sqrt{1 + 8}}{2}$$

$$x = \frac{1 + \sqrt{9}}{2}$$

$$x = 4/2$$

$$x = 2$$

The exact value of the infinite surd  $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}}$  is 2.

## General Infinite Surd

After looking at two examples of the infinite surds, we are ready to come up with some general statements with the general infinite surd below.

$$\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \dots}}}}}$$

### General Equation for $a_{n+1}$ in Terms of $a_n$

Step 1

$$a_1 = \sqrt{k + \sqrt{k}}$$

$$a_2 = \sqrt{k + \sqrt{k + \sqrt{k}}}$$

$$a_2 = \sqrt{k + a_1}$$

Since  $n=1$  and  $a_{n+1}=a_2$ , the equation below resulted.

$$a_{n+1} = \sqrt{k + a_n}$$

Since these are equivalent terms  
 $a_n$  can be substituted into  $a_2$

### Exact Value in Terms of k

#### Step 1

Let  $n = \infty$ , making  $a_{n+1} = a_{\infty+1} = a_{\infty}$

#### Step 2

Substitute  $a_{\infty}$  into the equation  $a_{n+1} = \sqrt{k + a_n}$

#### Step 3

Express this as a quadratic expression by square both sides then allow one side to equal to 0

$$a_{\infty} = \sqrt{k + a_{\infty}}$$

$$(a_{\infty})^2 = (\sqrt{k + a_{\infty}})^2$$

$$a_{\infty}^2 = k + a_{\infty}$$

$$a_{\infty}^2 - a_{\infty} - k = 0$$

#### Step 4

To solve for the equation, the quadratic formula  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$  is needed. Substitute the numbers

into the formula then take only the positive root.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) + \sqrt{(-1)^2 - 4(1)(-k)}}{2(1)}$$

$$x = \frac{1 + \sqrt{1 + 4k}}{2}$$

The expression  $\frac{1 + \sqrt{1 + 4k}}{2}$  represents the exact value of the general infinite surd.

### Restrictions of the expression

Before finding the values of  $k$  that will make the expression an integer, the restrictions has to be discussed. In this expression, the square root of the discriminate,  $1 + 4k$ , is an element of any odd positive integers, 0 or  $\infty$ .

This restriction is stated due to the fact that an odd integer divided by 2 will result in a decimal instead of an integer. Since 1 add any even integer square root of  $1 + 4k$  will result in an odd number,  $\sqrt{1 + 4k}$  has to be an odd positive integer, "0", or " $\infty$ ". The reason why that the discriminate has to be positive is because the square root of any negative number will result in an unreal solution. Examples are provided below to prove the restrictions.

#### Example #1

Allow  $\sqrt{1 + 4k}$  in the expression  $\frac{1 + \sqrt{1 + 4k}}{2}$  to equal to 10, an even number. This will result in  $\frac{1 + 10}{2}$ , which has an exact value of 5.5. This is *not* an integer.

#### Example #2

Allow  $\sqrt{1 + 4k}$  in the expression  $\frac{1 + \sqrt{1 + 4k}}{2}$  to equal to -1, a negative number. This will result in  $\frac{1 + \sqrt{-3}}{2}$ , which has an unreal solution.

#### Example #3

Allow  $\sqrt{1 + 4k}$  in the expression  $\frac{1 + \sqrt{1 + 4k}}{2}$  to equal to 13, an odd number. This will result in  $\frac{1 + 13}{2}$ , which has an exact value of 7. This is an integer.

### Some Values of k That Make the Expression an Integer

By allowing the expression  $\frac{1 + \sqrt{1 + 4k}}{2}$  to equal to some positive integers, and then solve for “k”, we are able to find some values of “k” that makes the expression an integer.

Value of k	Value of the expression
0	1
2	2
6	3
12	4
20	5
30	6
42	7
56	8
72	9
90	10

A general pattern can be found from the chart that represents relation between the value of “k” and the value of the expression.

### Relation between Value of “k” and Value of the Expression

The formula  $k = x^2 - x$  represents the relations between “k” and the value of the expression. To prove this, we can substitute 6 as “x”, the value of the expression, which will give us,  $k = (6)^2 - 6$ , and  $k = 30$ . This result is the same as the value in the chart. We can also try to put the number 9 into x’s place, resulting in  $k = (9)^2 - 9$ , which will give us  $k = 72$ . This is also the same as the result in the chart. We can factor the formula  $k = x^2 - x$  and this will result in  $k = x(x - 1)$ . By looking at the formula, we can tell that x multiply a number that is one lower than itself will equal to k. This means that the product of two consecutive numbers equals to k. The restriction for this is that x has to be an integer. This is because only the product of two integers will be an integer.

### Proving the General Statement

Now we can try to substitute some values into  $k$  to see if the general statement  $k = x(x-1)$  works and if the restrictions are true. Then we will calculate the value of  $a_{\infty}$  using the expression  $\frac{1 + \sqrt{1 + 4k}}{2}$ , to see if the value of  $k$  found will make the expression an integer.

Let  $x = 0$

$$k = 0(0-1)$$

$$k = 0$$

$$\frac{1 + \sqrt{1 + 4(0)}}{2} = \frac{1 + \sqrt{1}}{2} = \frac{2}{2} = 1$$

Let  $x = 20$

$$k = 20(20-1)$$

$$k = 380$$

$$\frac{1 + \sqrt{1 + 4(380)}}{2} = \frac{1 + \sqrt{1521}}{2} = \frac{40}{2} = 20$$

Let  $x = -31$

$$k = (-31)(-31-1)$$

$$k = 992$$

$$\frac{1 + \sqrt{1 + 4(992)}}{2} = \frac{1 + \sqrt{3969}}{2} = \frac{64}{2} = 32$$

We have tested 0, and both a positive and a negative integer, and the value of  $a_{\infty}$  for all of them are integers.

Here are some values that do not satisfy the restrictions. We will now test if the restrictions are true.

Let  $x = 0.5$

$$k = 0.5(0.5-1)$$

$$k = -0.25 \quad \frac{1 + \sqrt{1 + 4(-0.25)}}{2} = \frac{1 + \sqrt{0}}{2} = \frac{1}{2} = 0.5$$

Let  $x = -8.9$

$$k = -8.9(-8.9-1)$$

$$k = 88.11 \quad \frac{1 + \sqrt{1 + 4(88.11)}}{2} = \frac{1 + \sqrt{352.44}}{2} = \frac{19.8}{2} = 9.9$$

Both the positive and the negative decimals are tested, and the results of  $a_{\infty}$  for both are not integers

### How Big Can $k$ and $a_{\infty}$ Be?

Right now, we know that for the expression  $\frac{1 + \sqrt{1 + 4k}}{2}$  to be an integer, “k” has to be the product of

any two consecutive integers, and we also know that  $\frac{1 + \sqrt{1 + 4k}}{2} \geq 1$ . Now, we will consider a special

case for  $k$ . When  $k = \infty$ , the value of the expression should also be infinity. This is shown by the

calculations below.  $\frac{1 + \sqrt{1 + 4(\infty)}}{2} = \frac{1 + \sqrt{\infty}}{2} = \frac{\infty}{2} = \infty$

We know that the calculations are true because  $\sqrt{\infty} = \frac{\infty}{2} = 2\infty = \infty + 1 = \infty - 1 = \infty$ . So this means that

both the value of  $k$  and  $a_{\infty}$  can be infinity. So the following conclusion can be made.  $k = [0, \infty)$ , and

$a_{\infty} = [1, \infty)$

### Conclusion

By first taking a look at two simple infinite surds, we are able to see some similarities in them. Then we are able to apply what we learnt into coming up with the general statements of infinite surds. Graphs and charts were used to aid and made the understanding of infinite surds easier. In conclusion the following general statements about surds can be made:

1. The relation between  $a_n$  and  $a_{n+1}$  for all sequences of infinite surds can be represented by

$$a_{n+1} = \sqrt{k + a_n}.$$

2. The expression  $\frac{1 + \sqrt{1 + 4k}}{2}$  can be used to represent the exact value of all infinite surds. The

restrictions are,  $\sqrt{1 + 4k} \geq 0$ , and that  $\sqrt{1 + 4k}$  can only be a positive odd integer, 0 or  $\infty$ .

3. Infinite surds all have a convergence.
4. The formula  $k = x(x - 1)$  can be used to calculate the values of “k” that make the infinite surd an integer. The product of any two consecutive integers will allow the infinite surd to be an integer.
5. In general, for the infinite surd to have a real solution,  $k \geq 0$ , and the value of the infinite surd when  $k$  is 0,  $\infty$ , or a positive integer is  $\geq 1$ .