INTERNATIONAL BACCALAUREATE
II. GIMNAZIJA MARIBOR

Portfolio mathematics HL
Assignment 1

# The segments of a polygon

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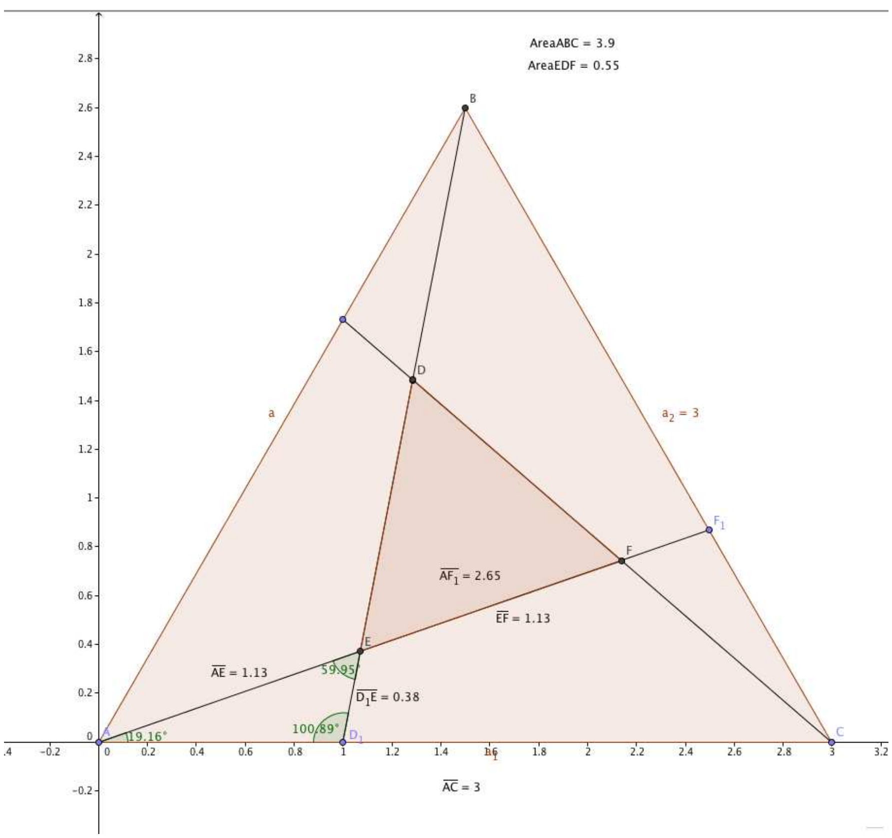
***Introduction***

First Mathematics HL Portfolio is about investigating the segments of a polygon, using graphical methods and analytical proofs. The task is to find conjecture between ratio of the sides and the ratio of the areas, first of triangles developing it to general conjecture of polygons.

1. In an equilateral triangle ABC, a line segment is drawn from each vertex to a point on the opposite side so that the segment divides the side in the ratio 1:2, crating another equilateral triangle DEF.

(a) What is the ratio of the areas of the two equilateral triangles? To answer this question,

(i) create the above diagram with your geometry package.



(ii) measure the lengths of the sides of the two equilateral triangles

$\overline{AB} = \overline{BC} = \overline{CA} = 3$ units

$\overline{ED} = \overline{DF} = \overline{FE} \doteq 1.13$ units

(iii) find the areas of the two equilateral triangles and the ratio between them.

Formula for area of triangle: $A = \frac{a^2\sqrt{3}}{4}$

$A_1 = A_{ABC} = \frac{3^2\sqrt{3}}{4} = 3.897u^2 \doteq 3.9u^2$

(u=unit)

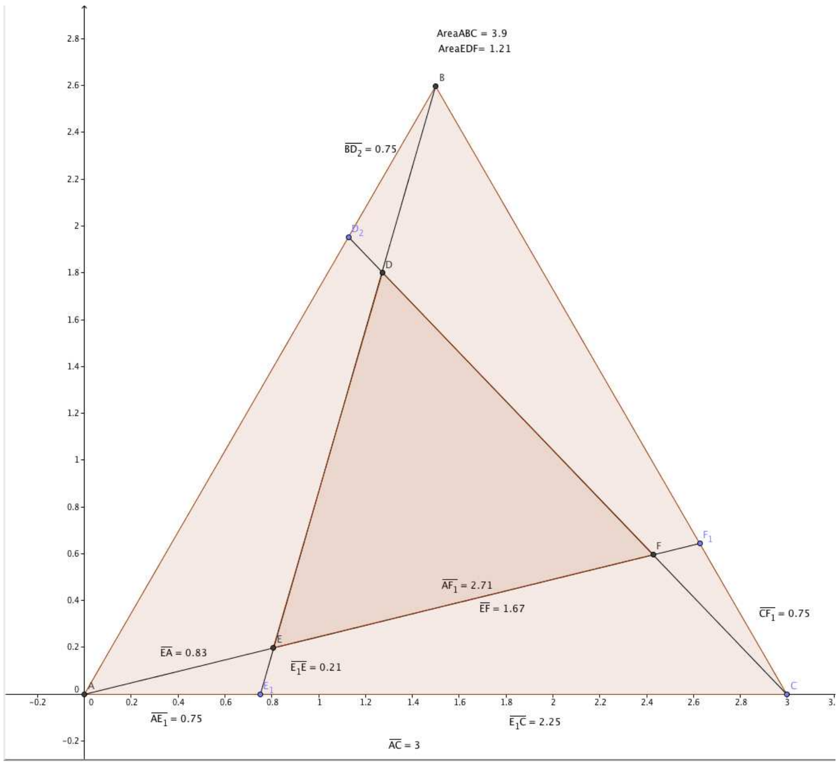
$A_2 = A_{EDF} = \frac{1.13^2\sqrt{3}}{4} = 0.5529u^2 \doteq 0.55u^2$

I got the same results for the areas, as before calculated with program.
(see the picture above)

$\frac{A_1}{A_2} = \frac{3.9\cancel{u^2}}{0.55\cancel{u^2}} = \frac{7.09}{1} \doteq \frac{7.1}{1}$

(b) Repeat the procedure above for at least two other side ratios, 1:n.

I decided to draw triangle with ratio: 1:3




$\overline{AB} = \overline{BC} = \overline{CA} = 3$ units

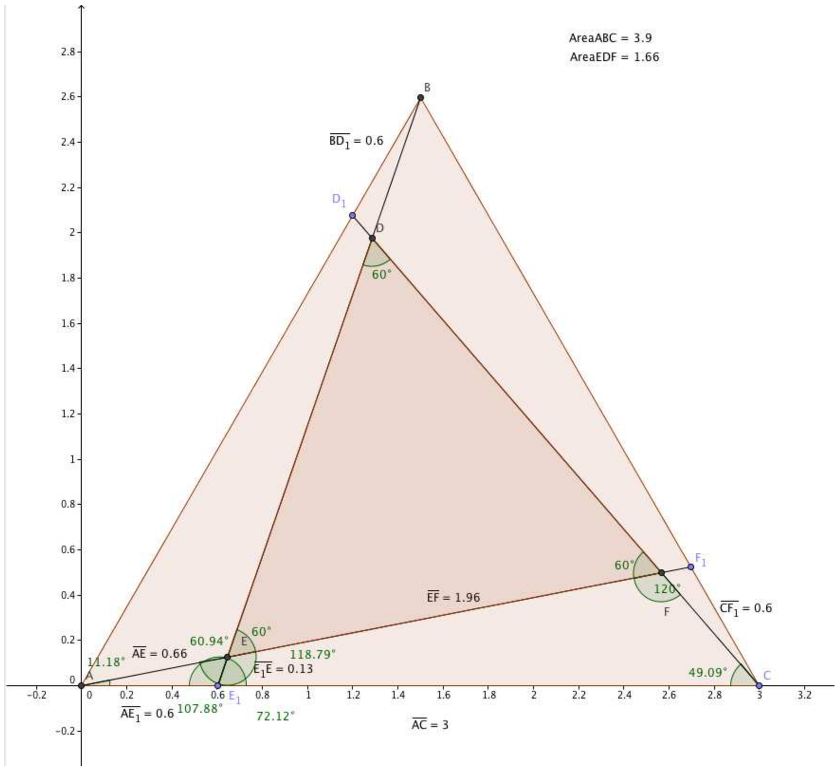
$\overline{ED} = \overline{DF} = \overline{FE} \doteq 1.67$ units

$$A_1 = A_{ABC} = \frac{3^2\sqrt{3}}{4} = 3.897u^2 \doteq 3.9u^2$$

$$A_2 = A_{EDF} = \frac{1.67^2\sqrt{3}}{4} = 1.2076u^2 \doteq 1.21u^2$$

$$\frac{A_1}{A_2} = \frac{3.9u^2}{1.21u^2} = \frac{3.9}{1.21} \doteq \frac{3.2}{1}$$

Third ratio: 1:4




$\overline{AB} = \overline{BC} = \overline{CA} = 3$ units

$\overline{ED} = \overline{DF} = \overline{FE} \doteq 1.96$ units

$$A_1 = A_{ABC} = \frac{3^2\sqrt{3}}{4} = 3.897u^2 \doteq 3.9u^2$$

$$A_2 = A_{EDF} = \frac{1.96^2\sqrt{3}}{4} = 1.6634u^2 \doteq 1.66u^2$$

$$\frac{A_1}{A_2} = \frac{3.9u^2}{1.66u^2} = \frac{3.9}{1.66} \doteq \frac{2.34}{1} \doteq \frac{2.3}{1}$$

(c) For:

n=2 we get ratio of areas 7

n=3 we get ratio of areas 3.2

n=4 we get ratio of areas 2.3

To see connection between this results the best way is to draw a graph on which n will be represented on x-axis and ratio of areas on y-axis. I use program to get the function of a curve, which goes through this points. And the program gives me this function:

$$f(x)=\frac{x^2+x+1}{x^2-2x+1}$$

Now I have to replace x with n and f(x) with ratio of areas $\frac{A_1}{A_2}$ and I get:

$$\frac{A_1}{A_2}=\frac{n^2+n+1}{n^2-2n+1}$$

(d) Prove this conjecture analytically.

$$\frac{A_1}{A_2}=\frac{a_1^2}{a_2^2}$$

For simplification in calculating I will use :

$A_1=A_{ABC}$ , $A_2=A_{EDF}$ , $a_1=\overline{AB}$ , $a_2=\overline{ED}$

y= $\overline{AE}=\overline{CF}=\overline{BD}$

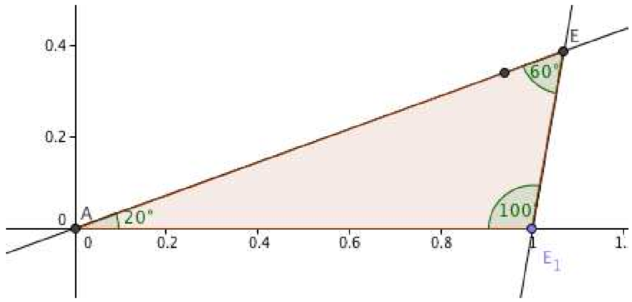
x= $\overline{FF_1}=\overline{EE_1}=\overline{DD_2}$

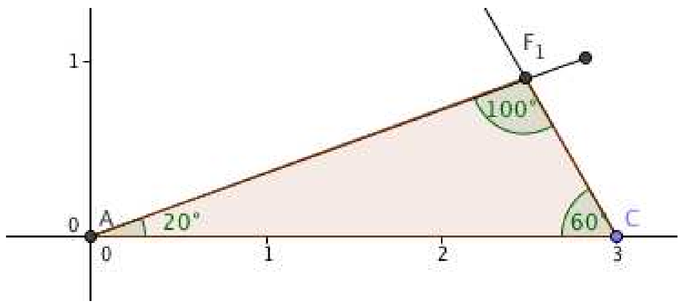
p= $\overline{AF_1}=\overline{BE_1}=\overline{CD_2}$

$$\overline{AE_1}=\overline{CF_1}=\overline{BD_2}=\frac{a_1}{n+1}$$

To get the relationship between the ratios of the sides and the ratio of the areas I have to first get relationship between $a_1$ and $a_2$ . If I look on the first triangle above, I see that $a_2=p-y-x$ , so I have to calculate p, y and x to get $a_2$ .

Triangles $\triangle AEE_1$ and $\triangle AF_1C$ are similar (if two angles of one triangle are same to two angel in second triangle, they are similar). So I can get the relationship of sides of triangles.




$$\frac{X}{y}=\frac{\frac{\not{a}_1}{n+1}}{\not{a}_1}\rightarrow x=\frac{y}{n+1}$$

To calculate p I have to use cosine's rule:

$$p^2=a_1^2+\left(\frac{a_1}{n+1}\right)^2-2a_1\left(\frac{a_1}{n+1}\right)\cos\ 60^\circ=$$

$$a_1^2+\frac{a_1^2}{(n+1)^2}-\frac{\not{2}a_1^2}{\not{2}(n+1)}=a_1^2\left(1+\frac{1}{(n+1)^2}-\frac{1}{(n+1)}\right)=$$

$$a_1^2\left(\frac{(n+1)^2+1-(n+1)}{(n+1)^2}\right)=a_1^2\left(\frac{n^2+2n+1+\not{1}-n-\not{1}}{(n+1)^2}\right)\rightarrow$$

$$p=\frac{a_1}{n+1}\sqrt{n^2+n+1}$$

Now I go again back to relations of sides in similar triangles to calculate x.

$$\frac{x}{\frac{a_1}{n+1}} = \frac{\frac{a_1}{n+1}}{p} \rightarrow$$

$$x = \frac{(\frac{a_1}{n+1})^2}{p} = \frac{a_1 / (n+1)}{a_1 (n+1) / \sqrt{n^2+n+1}} = \frac{a_1}{(n+1)\sqrt{n^2+n+1}}$$

$x = \frac{y}{n+1} \rightarrow y = x(n+1)$ now I enter x that I get above into the equation for y and get:

$$y = \frac{a_1 (n+1)}{(n+1)\sqrt{n^2+n+1}} = \frac{a_1}{\sqrt{n^2+n+1}}$$

And now we have all components to calculate $a_2$. We enter them in the equation(I have already wrote it down above) : $a_2 = p - y - x$

$$a_2 = \frac{a_1\sqrt{n^2+n+1}}{n+1} - \frac{a_1}{(n+1)\sqrt{n^2+n+1}} - \frac{a_1}{\sqrt{n^2+n+1}} =$$

$$a_1 \frac{n^2+n+1-1-n-1}{(n+1)\sqrt{n^2+n+1}} = a_1 \frac{n^2-1\ (n+1)(n-1)}{(n+1)\sqrt{n^2+n+1}} \rightarrow$$

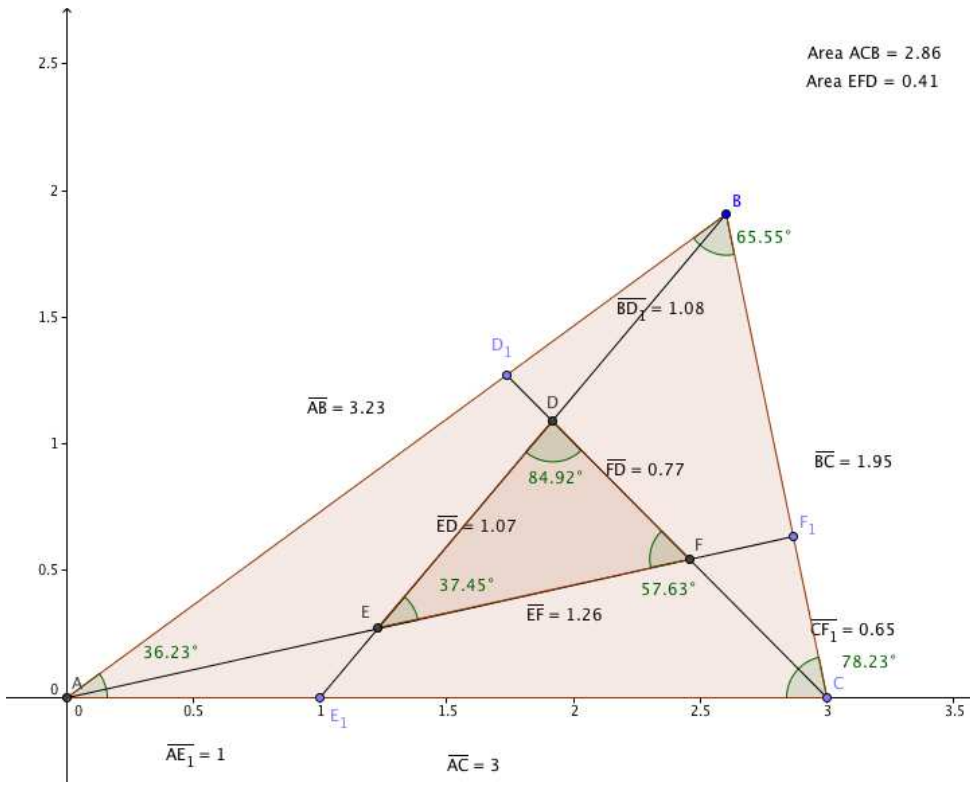
$$a_2 = a_1 \frac{(n-1)}{\sqrt{n^2+n+1}}$$

$\rightarrow \frac{a_1}{a_2} = \frac{\sqrt{n^2+n+1}}{(n-1)}$ since $\frac{A_1}{A_2} = \frac{a_1^2}{a_2^2}$ I get:

$$(\frac{a_1}{a_2})^2 = (\frac{\sqrt{n^2+n+1}}{(n-1)})^2 = \frac{n^2+n+1}{(n-1)^2} = \frac{n^2+n+1}{n^2-2n+1} = \frac{A_1}{A_2}$$

So I prove the conjecture that I get from program in task c.

2. Does this conjecture hold for non-equilateral triangles? Explain.




To easily understand this, I draw a non-equilateral triangle which sides are split in ratio 1:2. I measure the areas of ABC and EDF and get those results:

$A_1$ =Area ABC= $2.86u^2$

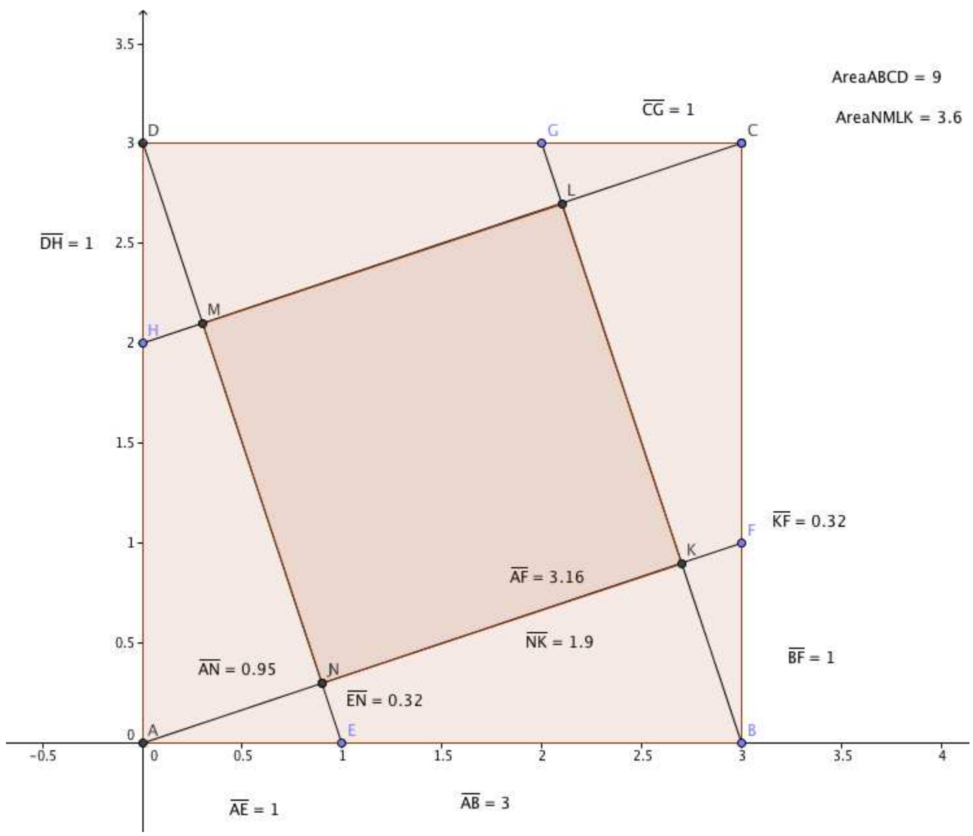
$A_2$ =Area EDF $0.41u^2$

I calculated the ratio of areas between this two triangles

($\frac{A_1}{A_2} = \frac{2.86u^2}{0.41u^2} \doteq \frac{7}{1} = \frac{n^2+n+1}{n^2-2n+1}$) and I get the same ratio as for equilateral triangle which sides are also split in ratio 1:2. I have done this comparison for at least 20 other non-equilateral triangles, which sides were split in different ratios (but all three sides in same) and my conjecture holds for those. So for those it was only important fact that all three sides where split in same ratio (no matter the side length) in order to get the same area ratio as it can be calculated from my conjecture above. From this I can conclude that conjecture holds for the majority of non-equilateral triangles, but as I didn't find out analytical proof, as there are no geometrical identities (different inner angles of triangle ABC and EDF), there may be some exceptions.

3.

(a) Do a similar construction in a square where each side is divided into the ratio of 1:2. Compare the area of the inner square to the area of the original square.

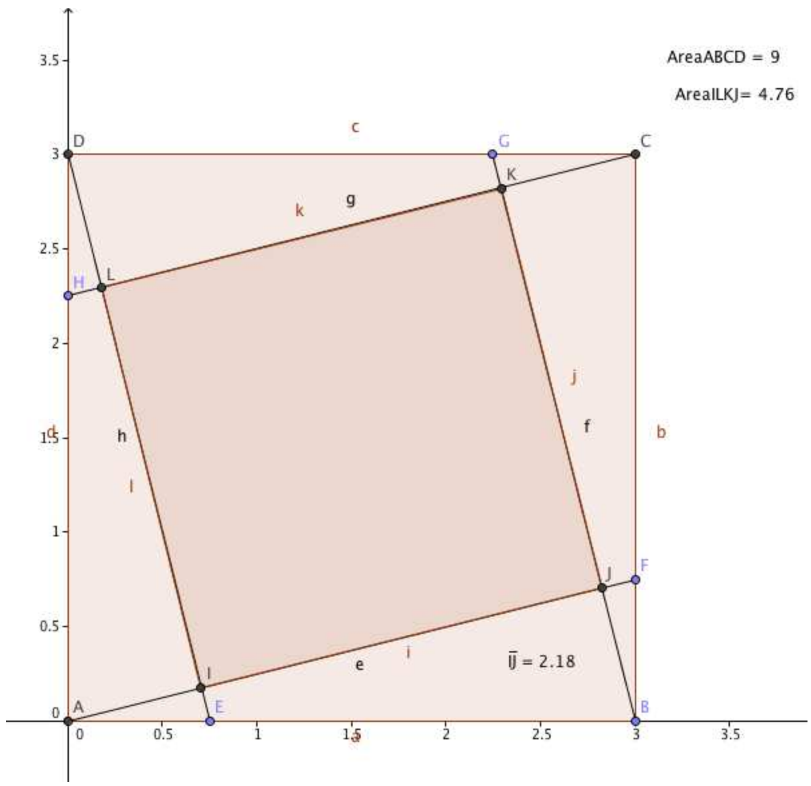



I calculate the areas with program:

$$A_1 = A_{ABCD} = 9u^2$$
$$A_2 = A_{NMLK} = 3.6u^2 \rightarrow \frac{A_1}{A_2} = \frac{9u^2}{3.6u^2} = \frac{2.5}{1}$$

(b) How do the areas compare if each side is divided in the ratio 1:n? Record your observations, describe any patterns noted, and formulate a conjecture.

I have again constructed a square in which each side is divided in the ratio 1:3.




$A_1 = A_{ABCD} = 9u^2$

$A_2 = A_{ILKJ} = 4.76u^2$

$\rightarrow \frac{A_1}{A_2} = \frac{9u^2}{4.76u^2} \doteq \frac{1.9}{1}$

$\frac{A_1}{A_2} = \frac{a_1^2}{a_2^2}$

I do again same procedure as in first task and program gives me relation:

$\frac{A_1}{A_2} = (\frac{a}{a_2})^2 = \frac{(n+1)^2+1}{n^2}$

(c) Prove the conjecture.

To prove the conjecture we have to calculate the side of the inner square $a_2$ . For calculations I will use the first square (1:2).

Again this can be done like this: $a_2 = \overline{AF} - \overline{AN} - \overline{EN}$

So I use Pythagoras theory to calculate $\overline{AF}, \overline{AN}, \overline{EN}$ .

$$\overline{AB} = a$$

$$\overline{AE} = \overline{BF} = \overline{CG} = \overline{DH} = \frac{a}{n+1}$$

$$\overline{AF}^2 = a^2 + (\frac{a}{n+1})^2 \rightarrow \overline{AF} = \sqrt{a^2 + (\frac{a}{n+1})^2}$$

Then I use again the theory for similar triangles (like in 1 task) and calculate $\overline{AN}$

$$\frac{\overline{AN}}{\overline{AE}} = \frac{\overline{AB}}{\overline{AF}} \rightarrow \overline{AN} = \frac{\overline{AB}\,\overline{AE}}{\overline{AF}}$$

$$\overline{AN} = \frac{a(\frac{a}{n+1})}{\sqrt{a^2 + (\frac{a}{n+1})^2}} = \frac{\frac{a^2}{n+1}}{\sqrt{a^2 + \frac{a^2}{(n+1)^2}}} =$$

$$\frac{\frac{a^{\not 2}}{n+1}}{\not a\sqrt{1 + \frac{1}{(n+1)^2}}} = \frac{\frac{a}{n+1}}{\sqrt{1 + \frac{1}{(n+1)^2}}} = \frac{a}{(n+1)\sqrt{\frac{(n+1)^2+1}{(n+1)^2}}} =$$

$$\frac{a}{\not(n+1)\frac{\sqrt{(n+1)^2+1}}{\not(n+1)}} \rightarrow \overline{AN} = \frac{a}{\sqrt{n^2+2n+2}}$$

Now we have to get $\overline{EN}$ , again with Pythagoras theory.

$$\overline{EN} = \sqrt{\overline{AE}^2 - \overline{AN}^2} = \sqrt{(\frac{a}{n+1})^2 - (\frac{a}{\sqrt{n^2+2n+2}})^2} =$$

$$= \sqrt{\frac{a^2}{(n+1)^2} - \frac{a^2}{(n+1)^2+1}} = \sqrt{a^2(\frac{1}{(n+1)^2} - \frac{1}{(n+1)^2+1}} = a\sqrt{(\frac{((n+1)^2+1)-(n+1)^2}{(n+1)^2((n+1)^2+1)}} =$$

$$= \frac{a}{n+1}\sqrt{\frac{\not n^2 + \not 2n + 1 + \not 1 - \not n^2 - \not 2n - \not 1}{(n+1)^2+1}} = \frac{a}{n+1}\sqrt{\frac{1}{(n+1)^2+1}} \rightarrow$$

$$\overline{EN} = \frac{a}{(n+1)\sqrt{(n+1)^2+1}}$$

Now we enter all components that we get to $a_2 = \overline{AF} - \overline{AN} - \overline{EN}$ .

$$a_2 = \sqrt{a^2 + (\frac{a}{n+1})^2} - \frac{a}{\sqrt{(n+1)^2+1}} - \frac{a}{(n+1)\sqrt{(n+1)^2+1}}$$

$$= a(\sqrt{1 + \frac{1}{(n+1)^2}} - \frac{1}{\sqrt{(n+1)^2+1}} - \frac{1}{(n+1)\sqrt{(n+1)^2+1}}) \rightarrow$$

$$\frac{a_2}{a} = \sqrt{\frac{(n+1)^2+1}{(n+1)^2}} - \frac{1}{\sqrt{(n+1)^2+1}} - \frac{1}{(n+1)\sqrt{(n+1)^2+1}} =$$

$$= \frac{\sqrt{(n+1)^2+1}\sqrt{(n+1)^2+1}(n+1) - \sqrt{(n+1)^2}(n+1) - \sqrt{(n+1)^2}}{(n+1)^2\sqrt{(n+1)^2+1}} =$$

$$= \frac{((n+1)^2+1)(n+1) - (n+1)(n+1) - (n+1)}{(n+1)^2\sqrt{(n+1)^2+1}} =$$

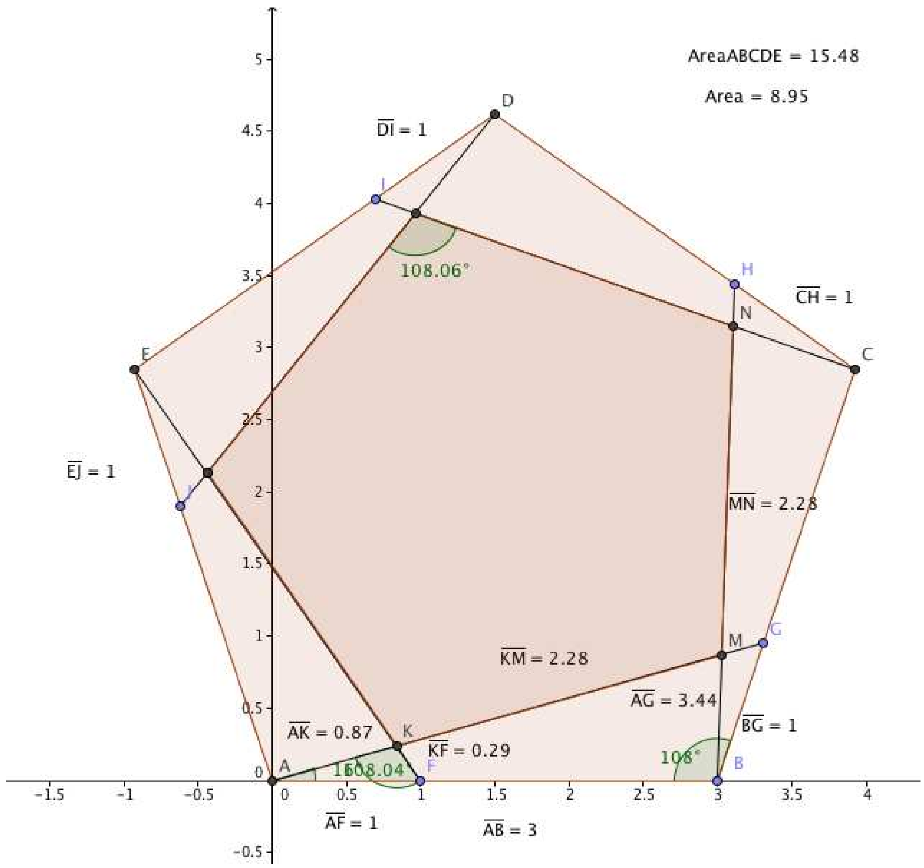
$$\frac{(n+1)^2 + 1 - (n+1) - 1}{(n+1)\sqrt{(n+1)^2+1}} = \frac{n+1-1}{\sqrt{(n+1)^2+1}} \rightarrow \frac{a_2}{a} = \frac{n}{\sqrt{(n+1)^2+1}}$$

$$\frac{A_1}{A_2} = (\frac{a}{a_2})^2 = \frac{(n+1)^2+1}{n^2}$$

I prove conjecture.

4. If segments were constructed in a similar manner in other polygons (e.g. pentagons, hexagons, etc.) would a similar relationship exist? Investigate the relationship in another regular polygon.

I draw a pentagon which sides are divided in ration 1:2




Firs I have to determine  the sides:

$\overline{AB} = \overline{BC} = \overline{CD} = \overline{DE} = \overline{EA} = a$

$\overline{KM} = \overline{AG} - \overline{AK} - \overline{MG} = a_2$

$\overline{AK} = \overline{BM} = \overline{CN}...$

$\overline{KF} = \overline{MG} = \overline{HN}...$

$\overline{GB} = \frac{a}{n+1}$

So as in previous examples I have to get AG, AK and MG to calculate a

I will start with calculating $\overline{AG}$ with using the cosine's rule:

$$\overline{AG}^2 = a^2 + \frac{a^2}{(n+1)^2} - 2a\frac{a}{n+1}\cos\theta$$

In my sketch $\cos\theta$ would be 108 but as I want to get relationship for all regular polygons I will use the formula for sum of all inside angles:

$\theta_{all} = (s-2)180^{\circ}$

s represents the number of sides of regular polygon.

To get one inside angle I divide $\theta_{all}$ with number of sides and I get:

$$\theta = \frac{(s-2)180^{\circ}}{s}$$

$$\overline{AG}^2 = a^2\left(1 + \frac{1}{(n+1)^2} - \frac{2\cos\theta}{n+1}\right) = a^2\left(\frac{(n+1)^2 + 1 - 2(n+1)\cos\theta}{(n+1)^2}\right) \rightarrow$$

$$\overline{AG} = \frac{a}{n+1}\sqrt{n^2 + 2n + 2 - (2n\cos\theta + 2\cos\theta)} = \frac{a}{n+1}\sqrt{n^2 + 2n(1-\cos\theta) + 2(1-\cos\theta)} \rightarrow$$

$$\overline{AG} = \frac{a}{n+1}\sqrt{n^2 + (1-\cos\theta)(2n+2)}$$

Now I will again use the rule for similar triangles to calculate all components needed for $a_2$ :

$$\frac{\overline{AG}}{a} = \frac{\frac{a}{n+1}}{\overline{AK}} \rightarrow \overline{AK} = \frac{a^2}{(n+1)\overline{AG}}$$

$$\overline{AK} = \frac{a^{2}(n+1)}{a(n+1)\sqrt{n^2 + (1-\cos\theta)(2n+2)}} = \frac{a}{\sqrt{n^2 + (1-\cos\theta)(2n+2)}}$$

$$\frac{a}{\frac{a}{n+1}} = \frac{\overline{AK}}{\overline{KF}} \rightarrow \overline{KF} = \frac{\overline{AK}}{n+1} \rightarrow \overline{KF} = \frac{a}{(n+1)\sqrt{n^2 + (1-\cos\theta)(2n+2)}} = \overline{MG}$$

I have all components of $a_2$ so I can calculate it by formula
$\overline{KM} = \overline{AG} - \overline{AK} - \overline{MG} = a_2$

$$a_2 = \frac{a\sqrt{n^2 + (1-\cos\theta)(2n+2)}}{n+1} - \frac{a}{\sqrt{n^2 + (1-\cos\theta)(2n+2)}} - \frac{a}{(n+1)\sqrt{n^2 + (1-\cos\theta)(2n+2)}} =$$

$$= a\left(\frac{n^2 + (1-\cos\theta)(2n+2) - (n+1) - 1}{(n+1)\sqrt{n^2 + (1-\cos\theta)(2n+2)}}\right) = a\left(\frac{n^2 + 2n - 2n\cos\theta + 2 - 2\cos\theta - n - 2}{(n+1)\sqrt{n^2 + (1-\cos\theta)(2n+2)}}\right) =$$

$$a\frac{n^2 + n - 2\cos\theta(n+1)}{(n+1)\sqrt{n^2 + (1-\cos\theta)(2n+2)}} = a\frac{n(n+1) - 2\cos\theta(n+1)}{(n+1)\sqrt{n^2 + (1-\cos\theta)(2n+2)}} \rightarrow$$

$$a_2 = a\frac{n - 2\cos\theta}{\sqrt{n^2 + (1-\cos\theta)(2n+2)}}$$

$$\frac{A_1}{A_2} = \left(\frac{a}{a_2}\right)^2 = \frac{n^2 + (1-\cos\theta)(2n+2)}{(n-2\cos\theta)^2}$$

So I get the general formula for all regular polygons:

$\frac{A_1}{A_2}=(\frac{a}{a_2})^2=\frac{n^2+(1-\cos\theta)(2n+2)}{(n-2\cos\theta)^2}$ $\theta=\frac{(s-2)180^\circ}{s}$ (s is number of sides)

To prove this formula I can put in data for my pentagon, which I get with program and compare ratio that I get with my formula and ratio tha t I get from results of program.

$A_{ABCDE}=15.48$
$A_{KMDH_1J_1}=8.95$
$\frac{A_1}{A_2}\doteq 1.7$

and results from my formula

$\frac{A_1}{A_2}=\frac{11.85}{6.85}\doteq 1.7$

**Conclusion**

Through four tasks I found the conjecture between the ratios of the sides and the ratio of the areas for all regular polygo ns. I found also some similarities in relationship between areas and sides in non-equilateral triangles, but as I didn't find the analytical proof, I can't claim that it holds for all of them.

I find this task as very interesting and instructively becaus e I learned a lot of new methods (graphical) how to solve problems using advanced computer programs (I also learned how to use these programs as well). Furthermore I also refreshed my knowledge about geometry and came to some useful findings.

**Software used:**

**GeoGebra(http://www.geogebra.org/cms/)**

**Graph 4.3 (http://www.padowan.dk/graph/)**

**MATLAB (http://www.mathworks.com/)**