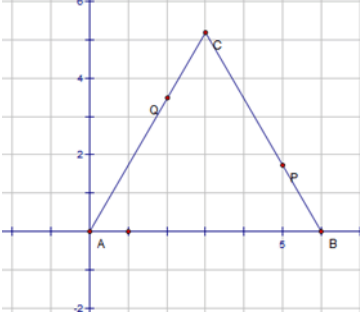
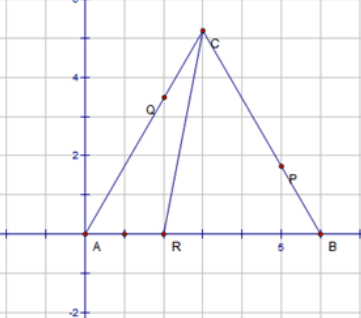
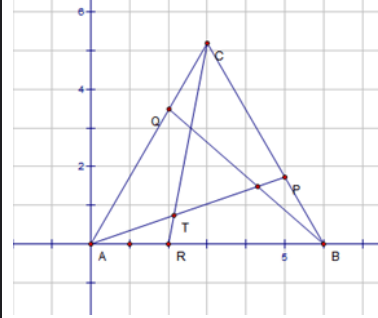


THE SEGMENT OF A POLYGON

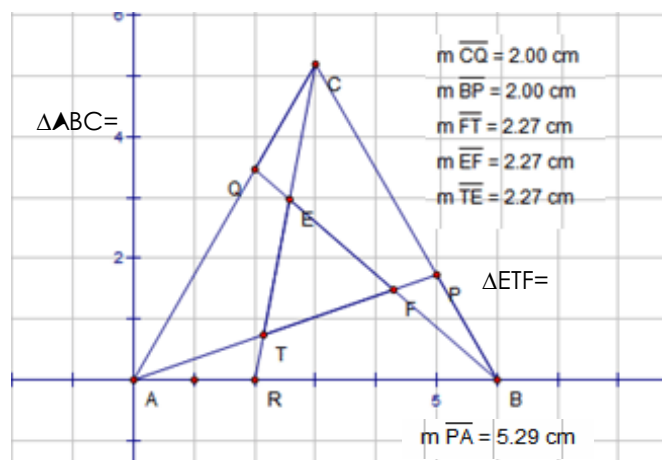
BY: RISA SANTOSO 11DP-2

The purpose of this investigation is to find the relationship of the ratios of sides and the ratios of area in a polygon. By doing this investigation, we will be able to acquire more skills and knowledge about shapes and the relationship between the lines of sides and area of a shape with the help of trigonometry.

To find the relationship of the ratio of sides and area in a polygon, we use the "Geometer's Sketchpad" program.

		
<p>Draw an equilateral triangle using the straightedge tool in the "Geometer's Sketchpad".</p>	<p>Draw a line from the vertex to the opposite side with the ratio of 1:2</p>	<p>Do the same with all of the other vertex.</p>

SIDES RATIO 1 : 2



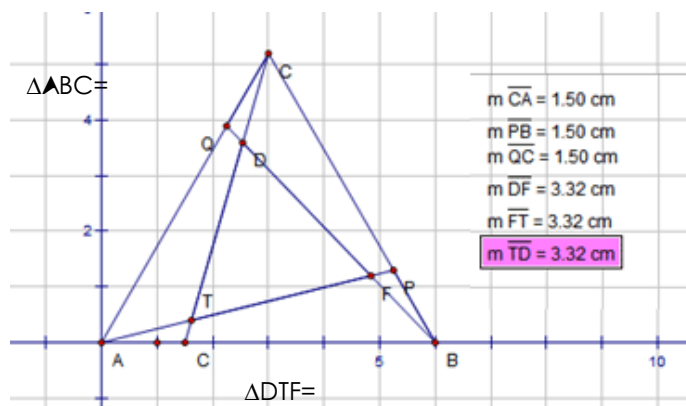
The results of the drawings are shown bellow. By letting the sides be 6 cm, the length of the AR (that made the ratio of 1:2 to RB) and the other two sides of PB and QC is 2 cm.

The side of the smaller (inner) triangle is 2.27. Which have a ratio of 2.64:1 or 1:0.378 to the side length of the outer triangle (outer:inner) .

The outer equilateral triangle's area obtained from the "Geometer's Sketchpad" is 15.59 cm² while the

inner equilateral triangle's area is 2.227cm^2 . The sketchpad ratio of the two area are 1:0.143 or 7:1 (outer : inner).

SIDES RATIO 1 : 3

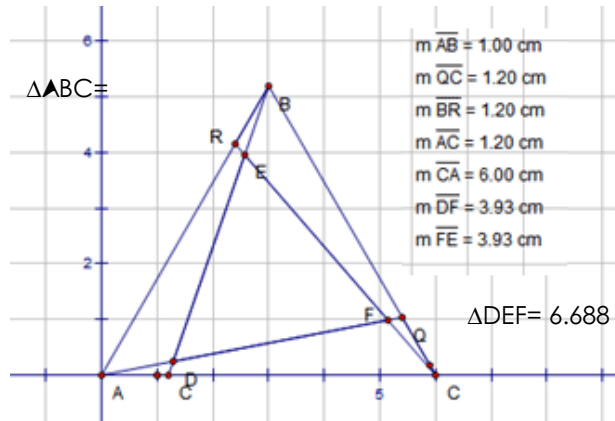


This is the results of the drawings of the ratio 1:3 (length of sides). Like the graph on 1:2, let the sides be 6 cm, the length of the ΔR (that made the ratio of 1:3 to RB) and the other two sides of PB and QC is 1.5 cm (obtained from $\frac{6}{4}$).

The side of the smaller (inner) triangle is 3.32 cm. It has the ratio of 1.81:1 or 1:0.55 to the side length of the outer triangle (outer:inner) .

The outer equilateral triangle's area obtained from the "Geometer's Sketchpad" is 15.59cm^2 while the inner equilateral triangle's area is 4.77cm^2 . The sketchpad ratio of the two area is 3.268:1 or 13:4 (outer : inner).

SIDES RATIO 1 : 4



This is the results of the drawings of the ratio 1:4 (length of sides). Like the graph on 1:2 and 1:3, let the sides be 6 cm, the length of the \triangle (that made the ratio of 1:4 to RB) and the other two sides of PB and QC is 1.2 cm (counted from $\frac{6}{5}$).

The side of the smaller (inner) triangle is 3.93cm. It has the ratio of 1.527:1 or 1:0.655 to the side length of the outer triangle (outer:inner) .

The outer equilateral triangle's area obtained from the "Geometer's Sketchpad" is 15.59 cm^2 while the inner equilateral triangle's area is 6.688 cm^2 . The sketchpad ratio of the two area is 1.53:1 or 7:3 (outer : inner).

Table of result

Ratio of sides	Length of the outer (big) sides of triangle	Length of inner (smaller) sides of triangle	Ratio between the length of sides of the outer and the inner triangle	Area of outer (big) triangle	Area of the inner (small) triangle	Ratio between the area of the outer and inner triangle.
1:2	6 cm	2.27cm	1:0.378	15.59 cm^2	2.227 cm^2	7:1
1:3	6 cm	3.32 cm	1:0.55	15.59 cm^2	4.77 cm^2	13:4
1:4	6 cm	3.93 cm	1:0.655	15.59 cm^2	6.688 cm^2	21:9 or 7:3

From the table above, we can spot clearly that the more 'n' (from 1:n) in the ratio, the more length increases in the sides and the area of the inner (the small) triangle. However, the ratio between the area of the outer and the inner triangle's gap became smaller (1:7 (inner: outer) can be written as 1: 0.143, 4: 13; 1: 0.308 and 3: 7; 0.429). This shows that the area of the inner triangle increases when the ratio sides increases while the outer triangle stays the same. There is also a pattern shown in the ratio of the area

of the outer and inner triangle. The difference of the bigger and ratio is increased by 2 (6 and 8) while the pattern of the inner triangle is the square of the number before 'n' $((2-1)^2=1, (3-1)^2=4, (4-1)^2=9)$

Moreover, it should be noted that the ratios of the sides are the square root of the ratio between the areas. The square of 0.378 for example is 0.1429, while 1 divide by 7 is also 0.1429. This also happen in the 1:3. 0.55^2 is 0.30, same with 4 divide by 7. For 1:4, 0.655^2 is 0.429, the same with 3 divide by 7. So, basically, like other shapes, if the two shapes are similar, the ratio of the area is the square of the ratio between the sides.

Using the data above, we can obtained a conjecture of;

$$\text{Outer triangle: inner triangle} = (n+1)^2 - n : (n-1)^2$$

This conjecture can be used to obtain the ratio result between the area of the outer and the inner triangle. 'n' represent the number of ratio of sides. We can prove the conjecture by trialing the ratio of sides.

Ratio of 1:2

$$(n+1)^2 - n : (n-1)^2$$

$$(2+1)^2 - 2 : (2-1)^2$$

$$(3)^2 - 2 : (1)^2$$

$$9 - 2 : 1$$

$$7 : 1$$

Ratio of 1:3

$$(n+1)^2 - n : (n-1)^2$$

$$(3+1)^2 - 3 : (3-1)^2$$

$$(4)^2 - 3 : (2)^2$$

$$16 - 3 : 4$$

$$13 : 4$$

Ratio of 1:4

$$(n+1)^2 - n : (n-1)^2$$

$$(4 + 1)^2 - 4 : (4 - 1)^2$$

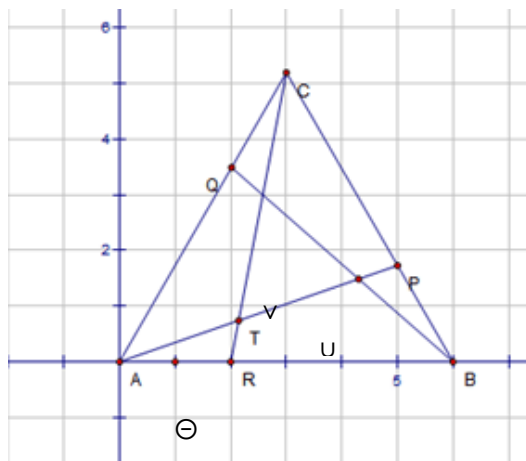
$$(5)^2 - 4 : (3)^2$$

$$25 - 4 : 9$$

$$21 : 9$$

$$7 : 3$$

From the proofs and trials above, the conjecture is supposedly true. We can also prove that the answer is true by using manual calculation. To prove that the conjecture is true, the answer of the manual calculation should match the geometry tool. This is to prevent the conjecture from the possible error on the geometry tool.



In this drawing of triangle 1:2, let the sides be 'S'.

The area of $\triangle CAB$ can be determined as below

$$\triangle CAB = \frac{1}{2} \times S \times S \times \sin 60^\circ$$

$$\triangle CAB = \frac{1}{2} \times S^2 \times \frac{1}{2} \sqrt{3}$$

$$\triangle CAB = \frac{1}{4} S^2 \sqrt{3}$$

$$AR = PB = QC$$

$$\triangle PBU = \triangle ATR = \triangle QCV \text{ let the smallest triangular be 't'}$$

$$\triangle TUV = \triangle CAB - \triangle ABP - (\triangle ARC - t) - (\triangle BCQ - 2t)$$

$$\triangle TUV = \triangle CAB - \frac{1}{3} \triangle CAB - \frac{1}{3} \triangle CAB + t - \frac{1}{3} \triangle CAB + 2t$$

$$\triangle TUV = 3t$$

Using the sine rule we continue and get

$$\frac{BP}{\sin \theta} = \frac{AB}{\sin(120^\circ - \theta)} = \frac{AP}{\sin 60^\circ}$$

$$\frac{\frac{1}{3}S}{\sin\theta} = \frac{S}{\sin 120^\circ \cos\theta + \cos 120^\circ \sin\theta}$$

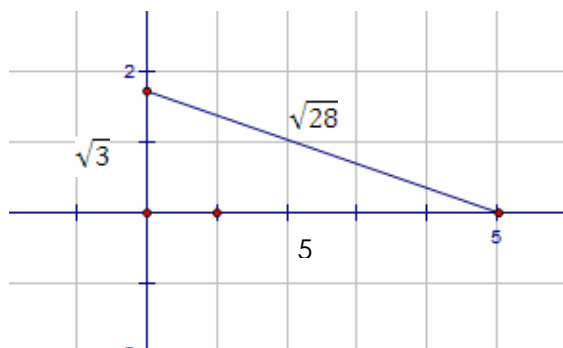
$$\sin\theta = \frac{1}{3} \sin 120^\circ \cos\theta - \cos 120^\circ \sin\theta$$

$$3\sin\theta = \frac{1}{2}\sqrt{3}\cos\theta + \frac{1}{2}\sin\theta$$

$$2\frac{1}{2}\sin\theta = \frac{1}{2}\sqrt{3}\cos\theta$$

$$\tan\theta = \frac{\sqrt{3}}{5}$$

From the Pythagoras theory ($\tan = \frac{\text{opposite}}{\text{adjacent}}$) I got



$$\text{hypotenuse} = \sqrt{(\sqrt{3})^2 + 5^2}$$

$$\text{hypotenuse} = \sqrt{3 + 25}$$

$$\text{hypotenuse} = \sqrt{28} = 2\sqrt{7}$$

from this point, we know that $\sin\theta = \frac{\sqrt{3}}{2\sqrt{7}}$ ($\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$)

while $\cos\theta = \frac{5}{2\sqrt{7}}$ ($\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$)

To find $\angle T$ we use the sine rule.

$$\frac{AR}{\sin 60^\circ} = \frac{AT}{\sin (120^\circ - \theta)}$$

$$\frac{AR}{\sin 60^\circ} = \frac{AT}{\sin 120^\circ \cos \theta - \cos 120^\circ \sin \theta}$$

$$\frac{\frac{1}{3}S}{\frac{1}{2}\sqrt{3}} = \frac{AT}{\frac{1}{2}\sqrt{3} \frac{5}{2\sqrt{7}} + \frac{1}{2} \frac{\sqrt{3}}{2\sqrt{7}}}$$

$$AT = \frac{\frac{1}{3}S \frac{3\sqrt{3}}{2\sqrt{7}}}{\frac{1}{2}\sqrt{3}} = \frac{S}{2\sqrt{7}} \cdot 2 = \frac{S}{\sqrt{7}}$$

$$t = \frac{1}{2} \times \frac{S}{\sqrt{7}} \times \frac{1}{3} \times S \times \frac{\sqrt{3}}{2\sqrt{7}}$$

$$t = \frac{\frac{1}{6}S^2\sqrt{3}}{14}$$

$$3t = \frac{1}{28}S^2\sqrt{3}$$

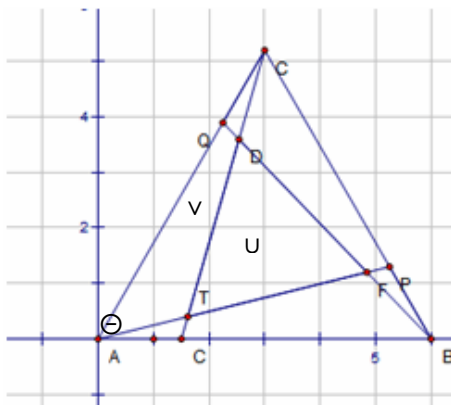
The ratio of the bigger equilateral triangle to the smaller equilateral triangle is acquired from

$$\frac{1}{4}S^2\sqrt{3} : \frac{1}{28}S^2\sqrt{3}$$

So, the ratio (after crossing the $S^2, \sqrt{3}, etc$) is 4:28 or simplified to be 1:7. The calculation of the "Geometric's Revolution" and the formula is true.

Of course, checking the other ratios (of the triangle shaped polygons) are a bit different from the one calculated above. Every one of the triangle have only 3 parts overlaying another part. When it is 1:2, the portion of the section is $\frac{1}{3}$, which makes the whole part of the triangle covered if not minus the overlaying parts. In other ratios though, an additional calculation is needed.

▲An example is for the 1:3



In this drawing of triangle 1:3, let the sides be 'S'.

Exactly the same as the 1:2, the area of $\triangle CAB$ can be determined as below

$$\triangle CAB = \frac{1}{2} \times S \times S \times \sin 60^\circ$$

$$\Delta CAB = \frac{1}{2} \times S^2 \times \frac{1}{2} \sqrt{3}$$

$$\Delta CAB = \frac{1}{4} S^2 \sqrt{3}$$

$$\Delta R = \Delta P = \Delta Q = C$$

$\Delta PBU = \Delta ATR = \Delta QCV$ let the smallest triangular be 't'

$$\Delta TUV = \Delta CAB - \Delta ABP - (\Delta ARC - t) - (\Delta BCQ - 2t)$$

$$\Delta TUV = \Delta CAB - \frac{1}{4} \Delta CAB - \frac{1}{4} \Delta CAB + t - \frac{1}{4} \Delta CAB + 2t$$

$$\Delta TUV = \frac{1}{4} \Delta CAB + 3t$$

There will be an additional $\frac{1}{4} \Delta CAB$ to get the area of the smaller triangle.

For 1:4 it is basically

$$\Delta TUV = \frac{2}{5} \Delta CAB + 3t$$

Since

$$\Delta TUV = \Delta CAB - \Delta ABP - (\Delta ARC - t) - (\Delta BCQ - 2t)$$

$$\Delta TUV = \Delta CAB - \frac{1}{5} \Delta CAB - \frac{1}{5} \Delta CAB + t - \frac{1}{5} \Delta CAB + 2t$$

$$\Delta TUV = \frac{2}{5} \Delta CAB + 3t$$

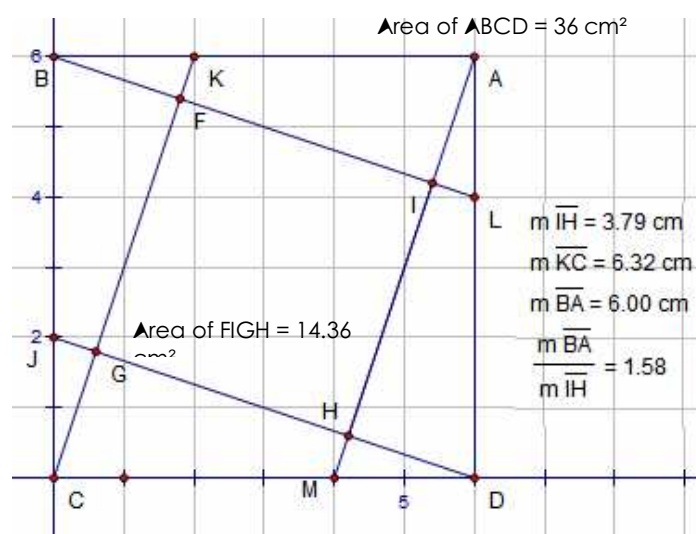
Therefore, to find the area of the smaller triangle, we are able to construct a formula of $\frac{n-2}{n+1} \times \frac{1}{4} S^2 \sqrt{3} + 3t$. $\frac{1}{4} S^2 \sqrt{3}$ is the formula of the smaller triangle (S for the length of the sides) and t is for the area of the smallest triangle which usually are on the one being overlaid in the edge of the object. This conjecture is to find the area of the inner triangle.

Of course the area of t is not always the same. Because of that, we should follow the procedure mentioned above for the case of 1:2 (by obtaining the triangle over layering areas and adding it with the percentage mentioned from the formula above.

The conjecture of $(n+1)^2 - n : (n-1)^2$ though, will only work for an equilateral triangle. It will not work for non equilateral triangle because the sides and the angle are not the same. We therefore cannot calculate the degree and the sides properly. Even though the sine rule can still be use, the equation would not have the same conjecture like the one that I have made. The formula of the smaller triangle in the middle ($\frac{n-2}{n+1} \times \frac{1}{4} S^2 \sqrt{3} + 3t. \frac{1}{4} S^2 \sqrt{3}$) would not work for a non equilateral triangle because of two reasons. One is because the area of the triangle will not be the same (since the degrees are different) and two is because the area ratio of the smaller triangle and the bigger triangle wouldn't be the same for every side.

For a similar construction of inner segments in a square which is divided in the ratio of 1:2, the result will be as show in the diagram below.

RATIO OF 1:2



The results of the drawings are shown below. By letting the sides be 6 cm, the length of the BK (that made the ratio of 1:2 to BA) and the other three sides of portion AL, MD and JC are 2 cm, the lines created another inner square of FGHI.

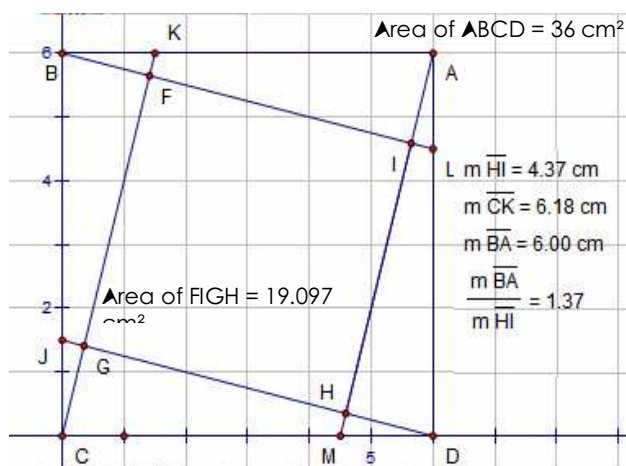
The sides of the smaller (inner) square are 3.79 cm. Which have a ratio of 1.58:1 (inner: outer) or 1:0.632 to the side length of the outer square (outer: inner).

The outer square's area obtained from the "Geometers Sketchpad" is 36 cm² while the inner square's area is 14.36 cm². The sketchpad ratio of the two area are 1:0.399 or 5:2 (outer : inner).

Like the triangle shape, the ratios of the sides are the square root of the ratio of the area. 0.632² is

0.399 while 1.58^2 is also 5:2

RATIO OF 1:3



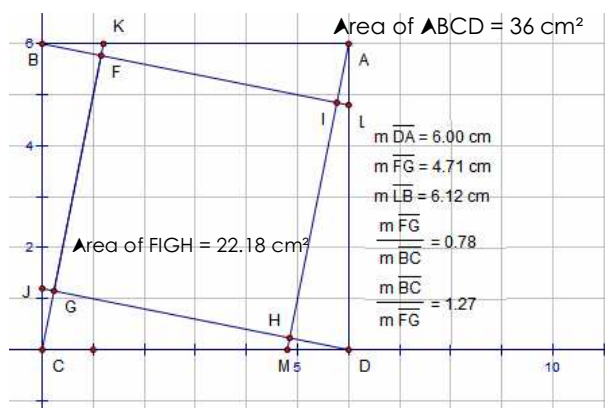
By letting the sides be 6 cm, the length of the BK (that made the ratio of 1:3 to BA) and the other three sides of portion AL, MD and JC are 1.5 cm, the lines created another inner square of FGHI.

The sides of the smaller (inner) square are 4.37 cm. Which have a ratio of 1.37:1 (inner: outer) or 1:0.738 to the side length of the outer square (outer: inner).

The outer square's area obtained from the "Geometer's Sketchpad" is 36 cm^2 while the inner square's area is 19.097 cm^2 . The sketchpad ratio of the two area are 1:0.535 (inner: outer) or 17:9 (outer : inner).

Like the triangle shape and the square above, the ratios of the sides are the square root of the ratio of the area. 0.738^2 is 0.535 while 1.37^2 is also 17:9

RATIO OF 1:4



By letting the sides be 6 cm, the length of the BK (that made the ratio of 1:4 to BA) and the other three sides of portion AL, MD and JC are 1.2 cm. the lines created another inner square of FGHI.

The sides of the smaller (inner) square are 4.31 cm. Which have a ratio of 1.274:1 (inner: outer) or 1:0.78 to the side length of the outer square (outer: inner).

The outer square's area obtained from the "Geometer's Sketchpad" is 36 cm^2 while the inner square's area is 22.18 cm^2 . The sketchpad ratio of the two area are 1:0.616 (inner: outer) or 13:8 (outer : inner).

Like the triangle shape and the squares above, the ratios of the sides are the square root of the ratio of the area.
 0.78^2 is 0.61 while 1.374^2 is also 13:8 (or 1.62)

From the result above, I was able to make a table to sum it all up.

Ratio of sides	Length of the outer (big) square's sides	Length of inner (small) square's sides	Ratio between the length of sides of the outer and the inner square	Area of outer (big) square	Area of the inner (small) square	Ratio between the area of the outer and inner square
1:2	6 cm	3.79 cm	1:0.632	36 cm ²	13.36 cm ²	5:2
1:3	6 cm	4.37 cm	1:0.738	36 cm ²	19.097 cm ²	17:9
1:4	6 cm	4.71 cm	1:0.78	36 cm ²	22.18 cm ²	26:16 or 13:8

Similar to the triangle shaped, the length of the inner square's sides increases as the 'n' of 1: n increases. The area of the inner square also increases, however, the gaps between the ratios of the area become smaller. 5:2 can be written as 1: 2.5 (inner: outer), 17:9; 1: 1.88 and 13.8; 1: 525. From the ratio of 1: 2.5, to 1: 1.88 and to 1: 525, we are able to know that the areas of the inner square indeed become bigger (since the area of the bigger square stays the same, the ratio decreases). Like what I have mentioned above, the ratios of the length of sides are the square root of the ratio between the area. There is also a pattern shown in the ratio of the area of the outer and inner square. Similar to the triangle pattern, the difference of the bigger and ratio is increased by 2 but starts with an odd number (7 and 9) while the pattern of the inner square is the square of the number 'n' ($2^2=4$, $3^2=9$, $4^2=16$)

The conjecture that I formulated from seeing the results above from the relationship of 1: n and the ratio between the area of the outer square and inner square is;

$$\text{Outer square: inner square} = (n+1)^2 + 1 : n^2$$

Proving the conjecture;

The ratio of 1:2

$$(n+1)^2 + 1 : n^2$$

$$(2 + 1)^2 + 1 : 2^2$$

$$3^2 + 1 : 2^2$$

$$10 : 4$$

$$5 : 2$$

The ratio of 1:3

$$(n + 1)^2 + 1 : n^2$$

$$(3 + 1)^2 + 1 : 3^2$$

$$4^2 + 1 : 3^2$$

$$17 : 9$$

The ratio of 1:4

$$(n + 1)^2 + 1 : n^2$$

$$(4 + 1)^2 + 1 : 4^2$$

$$5^2 + 1 : 4^2$$

$$26 : 16$$

$$13 : 8$$

This proves that my conjecture is correct and that it works for squares in the same state as these ones.

I believe that if segments were constructed in a similar manner in other regular (same length of sides and same angles) polygons such as pentagons, hexagon, octagon, etc, a similar relationship will exist. Even though the conjecture will be different, since the calculation of the triangles and the manner of the overlaid triangles are the same, a similar relationship will exist. Even with a different length of sides, the ratio will stay the same that's why it doesn't matter how long the examples of the sides are for the formula to work out.

To investigate the relationship of the polygons, I have made a table such as below to clearly see the patterns of the area ratio.

Ratio of 1:n for the sides segment	Ratio of the areas (outer: inner)		Ratio of the outer object		Ratio of the inner object	
	Triangle	Square	Triangle	Square	Triangle	Square
1:2	7:1	10:4	7	10	1	4
1:3	13:4	17:9	13	17	4	9
1:4	21:9	26:16	21	26	9	16

From the table above, I have found out a pattern that states that the smallest segments possible to be constructed in this manner is triangle and the starting point of the ratio on the smaller ratio is the square of 1 for triangle, the square for 2 for square, and I believe that the one with 5 sides will be the square of 3, and so on. While for the starting point of the bigger ratio starts with 7 for triangle, 13 for square and I believe it will keep increasing like the manner of the smaller ratio. So if polygon were constructed the ratio bigger ratio will be 14 and the smaller ratio will be 9 (14:9) and if a hexagon were constructed the ratio will be 19:16 (outer: inner).

For the ratio of 1: n, we can know that the difference of bigger triangle is 6 (13-7) and 8 (21-13) while for the smaller ones are the square of 1 (1), 2 (4), and 3 (9). While for the ratio of 1: n of square, the difference of the bigger square and the smaller square is 7 (17-10) and 9 (26-17), whereas the difference of triangle and square was 3 (for 1:2 segment side ratio), 4 for 1:3 and 5 for 1:4.

I believe that the pattern of 1: n for other polygons will be similar. So if the sides are 5 (pentagon), the difference should be 8 and 10 so after the first side segment ratio of 1:2 is 14:9, the next ratio would be 22:16 and for the side segment ratio of 1:3 is 32:25. While for hexagon, the difference will be 9 and 11. So for the segment ratio of 1:2 is 19:16, 1:3 is 28:25 and for 1:4 side segment ratios is 39:36.

So from the predictions of relationship above, I made this table below to summarize my findings and my predictions.

Side segment ratio	Triangle area ratio (outer : inner)	Square area ratio (outer : inner)	Pentagon area ratio (outer : inner)	Hexagon area ratio (outer : inner)
1:2	7:1	10:4	14:9	19:16
1:3	13:4	17:9	22:16	28:25

1:4	21:9	26:16	32:25	39:36
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The ratio of the outer shape addition should increase since the inner shape also increases (the square of the next number). If we look at the outer ratio of the triangle and square, the pattern value increases every time, from 4 (17-13) for the 1:2, 5 (22-17) for 1:3. That is why I predict the pattern of the other polygons would be similar to that. Even though I cannot verify whether the patterns of the other kinds of polygon are exactly like my prediction, I know that the segments were constructed in a similar manner for other regular polygons since to find the triangles inside the shapes; we basically use the same formula of trigonometry (sine rule, etc).