The Koch Snowflake

IB Portfolio

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IB HL Math Year 1
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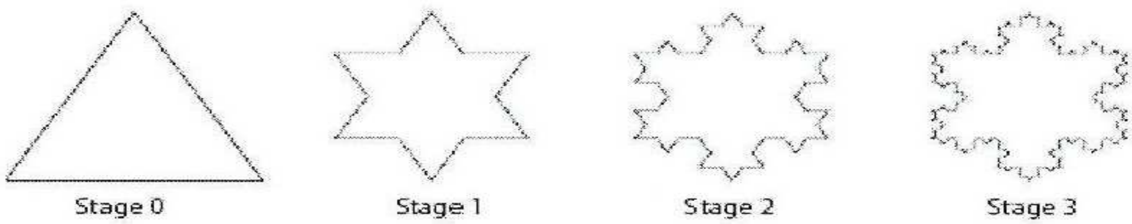
The purpose of doing this investigation is so that we can learn about the different patterns that the Koch Snowflake presents. The simple fact of the Koch Snowflake is a fractal already says a lot about it. A fractal is a figure which repeats itself in smaller scales. For example if stage 12 was looked closer stage two would be found and so



would be stage 3 and so on.

In this investigation we explored the patterns that emerge from one stage to the other. We tried to find any patterns for when n= 0,1,2,3 that could be applied to find any other n term. We specifically looked for patterns in N= the number of sides, L= the length of one side, P= the total perimeter, and A= the total area. And in order to come up with these patterns we used many resources like drawing in order to prove that our results were accurate.

1- Using an initial side length of 1, create a table that shows the value of N_n , l_n ,P_n , and A_n for n= 0, 1, 2, 3. Don't make decimal approximation; use exact values.

n	0	1	2	3
N	3	12	48	192
L	1	,1-3.	,1-9.	,1-27.
P	3	4	,16-3.	,64-9.
A	,,-3.-4.	,,-3.-3.	,10,-3.-27.	,94,-3.-243.

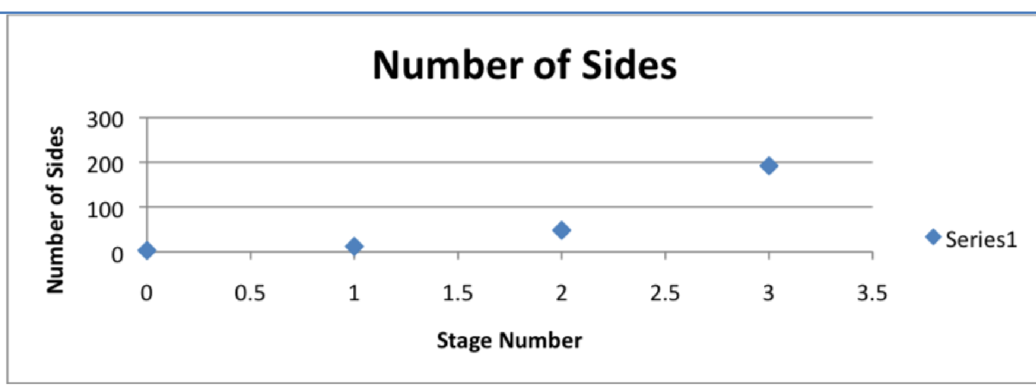
N= I was able to come up with the number of sides (N) by counting the number of sides of the triangles in the drawing of the fractals. When I had to count the number of sides in stage 3 I was able to see that a pattern had already been established. I was able to notice that the answer was increasing constantly by four (4). Using the pattern I was able to find I multiplied 48 which was my answer for n=2 by 4. I got the answer that n=3 is equal to 192. In order to prove that my answer was accurate I had to actually count the number of sides in n=3. In order to do that I got in the internet a high resolution drawing of stage 3 this allowed me to precisely count the number of sides. The fact that I got 192 proved that a pattern had already been established for N.

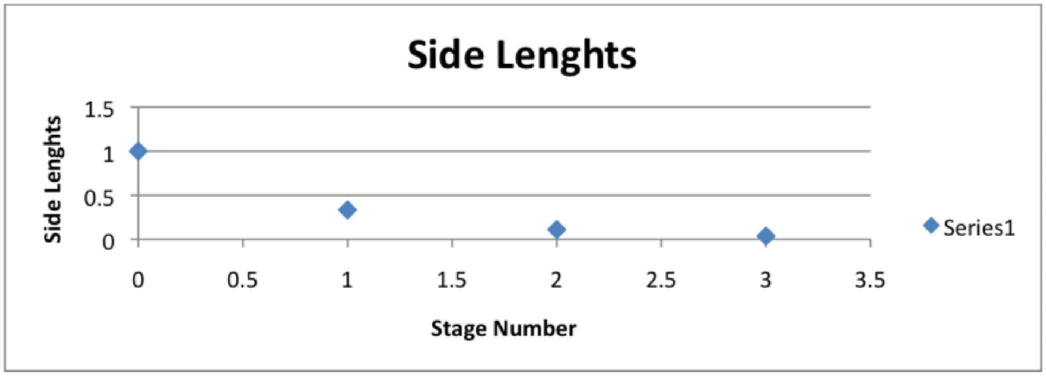
L= I was able to come up with the length of each sides by using the property of fractals. In stage 0 I knew that the side length of each triangle was 1. Since I knew that in order to add a new side I had to make it ,*1-3.*the size of the previous triangle I know the base of the triangle would be ,*1-3.* of the previous triangle and so would be the other sides since it is a equilateral triangle. And so I was able to realize that there was a pattern that each time the side length would get smaller by,*1-3.* . So for the next stages I was able to simply multiply my previous answer by ,*1-3.* . For example the side length in stage 0 is 1. If I multiply that by ,*1-3.* I will get the side length of stage 1 which is,*1-3.*. And if I multiply that by ,*1-3.* once again I will be able to get the side length for stage 2 which is,*1-9.*.

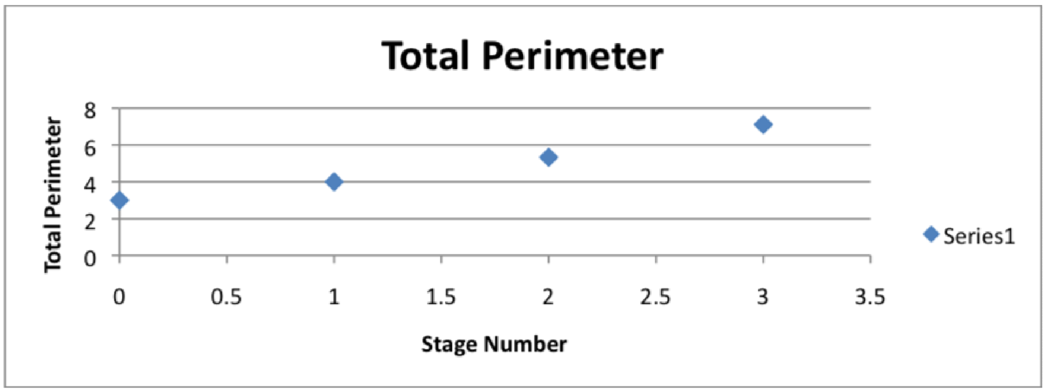
P= In order to come up with the perimeter of each side I simply had to multiply my answers obtained for N and L. Since when finding the perimeter the length of all sides are added together, I was able to realize that if I multiplied the answers I would get my answer for perimeter. In order to prove my idea I estimated my answer for n=2 which was,*16-3.*. I than counted added up the length of every side with my calculator using the drawing of stage 2. And I was able to get the same answers proving my way of quickly getting the perimeter of a fractal.

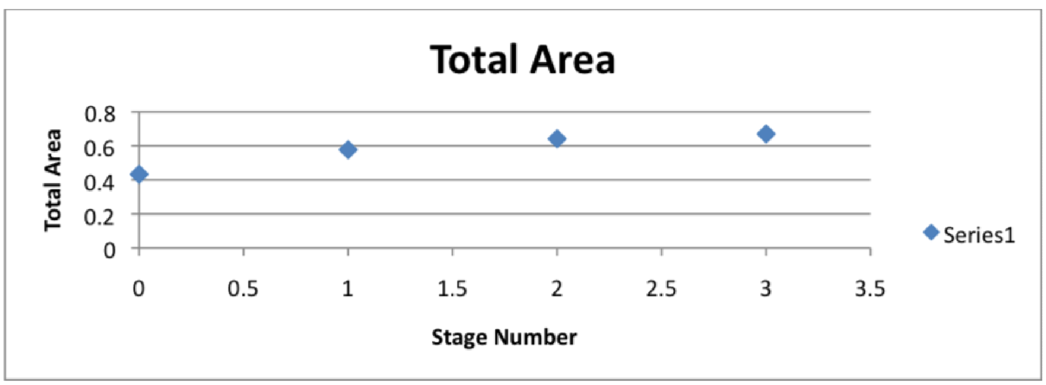
A= In order to calculate the area of each stage I had to use the formula A=,*1-2.*×□×□ which is the formula to find the area of a triangle. In order to find the height of each triangle I had to use the Pythagorean theorem ($a^2+b^2=c^2$) (Note that in the drawings of triangles and when I solved I used b instead of h for the height of the triangle.) For every stage I would find the area of one small triangle than multiply that by the number of new triangles I added in this stage and then I would add the answer to my previous area in order to find my new area.

2- Using Microsoft Excel, create graphs of the four sets of values plotted against the value of n. In your solution, include a printout for each graph.









3- Explain, in words, the relationship between successive terms of the sequence of N_n , l_n , and P_n for n= 0, 1, 2, 3

N= According to the graph the relationship between successive terms of the sequences is exponentially getting bigger every time. As I was able to notice before it is exponentially getting bigger by a factor of 4. This is evident with the graph because we can see that after every stage the difference in the y- axis gets bigger every time. This can probably due explained due to the definition of a fractal. This implies that a new triangle will be drawn for every side. Since, every time it has more sides, consequently the next stage will have even more triangles. And it can clearly be observed by the graph which shows clearly that the number of sides increases more and more every time as we go through stages.

L: According to the graph it is possible to see that the successive terms of the sequence start to converge as we go through the stages. It is possible to see that the numbers converge because the common ration between the terms in the geometric sequence is ,1.<1which a convergent factor. As we move through stages we can see that the difference in the y-axis gets smaller and smaller. Even though the graph does not show this, I would expect the graph to form a line pretty close to straight the higher the stage number is. This probably could be credited to the fact that as we move along through the stages of the sequence the size of each triangle gets smaller, consequently and directly related its side length also gets smaller. Eventually, it will converge because the sides will get so smaller that multiplying it by ,1-3. will give me basically the same result as my previous answer and it is a convergent factor.

P: According to the graph the relationship between successive terms in the sequence for perimeter shows that the difference between one stage to the other slowly increases. IF we look closely to the graph we can see that the line slightly goes up every time. It could be credited to the fact that the ration between each stage is ,4-3 . and that is≈1.3,3.. Since the difference is so close to 1 the difference between the successive terms does not change rapidly, however, a big difference will happen over time, as more triangles are added to the figure.

4- For the sequences of N_n , l_n , and P_n make a conjecture for a statement in terms of n that generalizes the behavior shown in its graph. Explain how you arrived at the statement and verify your conjecture by showing works for n= 0, 1, 2, 3.

$N_n = N_{n-1} \times 4$ (recursive) **N_n= 3 ×,-. (explicit)**

In order to get the conjecture for the number of sides I had to analyze my table. At first I could not realize that if I used t_1=3 times the common ratio which is 4 to the power of n I would get the formula. So I started by realizing that it was possible to get a recursive formula, since I was able to multiply the previous number in the sequence (t_{n-1}) by four and get my answer. But since I wanted to get a explicit formula which would allow me to get any term in the sequence I realized that I would need to keep the four since it was the common ratio. After I also realized that it formed a geometric sequence since it had a common ratio. With that in mind I was able to realize I was able to use the standard geometric sequence formula

(,□-□=,□-1 .. × ,□-□−1.) with just a slight modification that instead of using n-1 for the exponent I would simply use n. I had to make the modification because the formula starts with t_0 instead of t_1 and so I had to adjust to the difference by making everything -1 so my t_0 would be like t_1.

Proof:

,N-0.=3 × ,4-0.→3

,N-1.=3 × ,4-1.→ 12

,N-2.=3 × ,4-2.→ 48

, N-3.=3 ×,4-3.→192

,□-□.=□ × ,□-□.→ □□□

,L-n.=,,1-3..²

In order to come up with this conjecture, I simply tried to find a common ratio or common difference between the terms. Once I realized that it was geometric sequence not a arithmetic sequence I was able to realized that the common ratio was,1-3.. Since the first term of the sequence was one I also realized that I could leave it out. And once again I made the modification to the geometric sequence formula of putting n instead of n-1 for the exponent of the ratio. I had to make the modification because the formula starts with t_0 instead of t_1 and so I had to adjust to the difference by making everything -1 so my t_0 would be like t_1.

Proof:

,L-0.= ,,1-3..⁰→¹

,L-1.= ,,1-3..¹→,1-3.

,L-2.=,,1-3..²→,1-9.

,L-3.=,,1-3..³→,1-27.

,□-□.=,,□-□..⁴→,□-□□.

,□-□.=3 × ,,4-3..ⁿ

I was able to come up with this conjecture because I was able to see that there was a pattern on the way the formulas were created. I was able to notice that I always would use the geometric sequence formula with the modification of using n instead of n-1 for the exponent of the ratio. I was able to prove my conjecture by the fact I had previously noticed there was another pattern which allowed me to get P. The pattern was that **N_n * L_n = P_n** and so that also helped me verify my conjecture.

Proof:

,□-0.=3 × ,,3-4..0→3

,□-1.=3 × ,,3-4..1→4

,□-2.=3 × ,,3-4..2→,16-3.

,□-3.=3 × ,,3-4..3→,64-9.

,□-□.=□ × ,,□-□..4→,□□□-□□.

Note that for all of the of the conjectures n must be a positive integer it also works with 0 though.

5- Now, suign your conjecture for N_n , L_n , and P_n investigate what happens when n=4. Using the triangle graph paper, draw a diagram of one side of the fractal at stage 4, and explain how this verifies your conjectures.

N= With the drawing of stage 4 I'm able to prove that my conjecture that I was able to create in step 4 is correct. I'm able to do so because from my conjecture (**N_n= 3 ×,□-□.)** I got that stage four would have a total number of 768 sides. I'm able to prove that it actually does have a number of 768 sides because I was able to count a total of 64 new triangles. If I multiply that by 3 since every triangle has three sides by definition I'm able to see that there is a grand total of 256 sides in stage 4 of my drawing. Since I only drew ,1-3. of the geometric figure, if I multiply the number of sides I have drawn which is 256 by 3 I will get 768 which is the result that I got from my conjecture making it true.

L= With the drawing for stage 4 I'm able to prove that my conjecture I was able to create for step 4 is correct. I'm able to do so because from my conjecture

(,L-n.=,,1-3..2**)** I got that the side length of each stage would be a total of *,1-81.*I'm able to prove that with my drawing because the length of one side of stage 0 is 27. So I would have to assume that the drawing for stage 4 is 81 times smaller. The smallest triangle in my papers is exactly the outline of the smallest triangle in the paper. If I count the number of outlines of triangles, not actual triangles but little triangles formed by the lines of the paper I will be able to see that there are 81 of them, along the side of stage 0. Since my answer was supposed to be *,1-81.*and I had 81 outlines of triangles I can say that the side length of one triangle of stage 4 is *,1-81.*of stage 0.

P= In order to verify that my value for P_4 is correct I don't really need to use the drawing of my geometric figure. Using the conjecture I created in step 4 I'm to get that the perimeter at stage 4 will be*,256-8.*. I can prove this without using my conjectures because I was already able to prove my values for N_4 and L_4. I was able to realize that there was a pattern when I was doing the chart for step 1 which was that if I multiplied N_n times L_n I was able to get P_n. So in order to prove my conjecture in stage 4 I can simply go through the same process. When I multiply my result for N_4 that is 798 by my result of L_4 that is *,1-81.*I'm able to get the answer that is *,256-81.* .

6- Write down the successive terms of the sequence for A_n in terms of A_0 and the area added in each stage. Explain, in words, the pattern that emerges.

n=0	n=1	n=2	n=3
,,-3.-4.	,,-3.-3.	,10,-3.-27.	,94,-3.-243.
,,-3.-4.	,,-3.-4.+,,-□.-□□.	,,-3.-4.+,,-3.-12.+,,-□.-□□.	,,-3.-4.+,,-3.-12.+,,-3.-27.+,□,-□.-□□□.

After writing down all the successive terms for the sequence of A_n and A_0 I was able to realize that there is a common ratio between the area I'm going to add to my previous area A_{n-1} in order to find my new area A_n. This common ratio is*,4-9.*. Another thing that I was able to realize is that in all of the terms I will always have*,,-3.-4.*. I was also able to notice that there is a common difference between A_0 and A_1 of *,1-3.*.

7- Using Sigma notation make a conjecture for a statement in term of n that generalizes the behavior of A_n for $n \geq 1$. Verify your conjecture by showing it works for n= 1, 2, 3, and then find A_4.

$A_{n-1} + \frac{1}{3}\left(\frac{4}{9}\right)^{n-1} \Delta$ (Recursive)

,A-n.=,,-3.-4.+(,k=1-n-,,-3.-4..×,1-3.×,,,4-9..-n−1.)

↑explicit formula. Note in order for this formula to work n must be a □≥*1* and it must be a integer.

Using the website math world.com (http://mathworld.wolfram.com/KochSnowflake.html) I was able to get a recursive formula for the area of the Koch Snowflake. However, I this was not the explicit formula that we needed for the project. However, using some elements from this recursive formula and some of the patterns I was able to find in step 6 I was able to create my explicit formula. I used the sigma notation sign because in order to get my formula I needed to get the previous area of the triangles at other stages. Than I used the *,1-3.*because that was the ratio between the A_0 and A_1 and I need it because every time I need to move to stage 1 and so I can use my next pattern between the area of the triangles I added which is,*4-9.*. And than I did the n-1 because in a normal geometric sequnce I start with t_1 but since in this equation I start with t_0 I have to make everything -1 so I can get my t_0 to be like t_1. Finally with the area inside my bracket was going to give me just the area of the triangles added but not the area of stage one. So I had to add to my answer the A_0 which was *,,-3.-4.*and all that together was able to give me the answers for the area of the different stages of n.

Proof:

,A-1.=,,-3.-4.+,,k=1-1-,,-3.-4..×,1-3.×,,,4-9..-1−1..=,,-3.-3.≈.5774

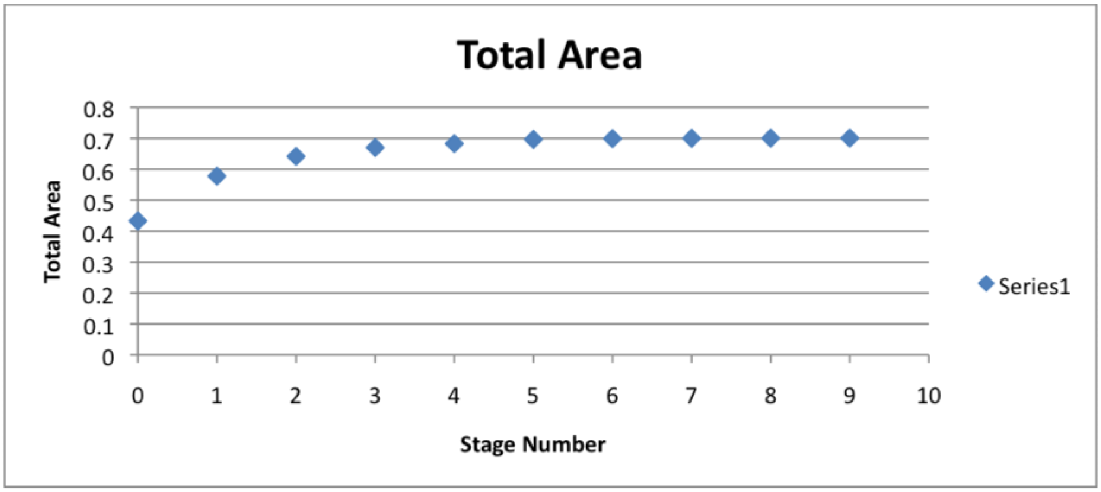
,A-2.=,,-3.-4.+,,k=1-2-,,-3.-4..×,1-3.×,,,4-9..-2−1..=,10,-3.-27.≈.6415

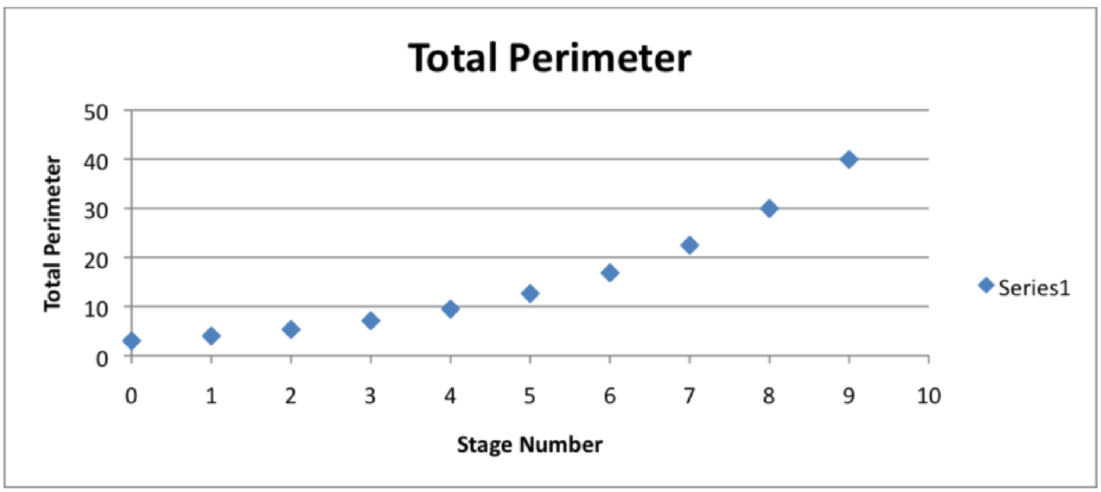
,A-3.=,,-3.-4.+,,k=1-3-,,-3.-4..×,1-3.×,,,4-9..-3−1..=,94,-3.-243.≈.6700

,A-4.=,,-3.-4.+,,k=1-n-,,-3.-4..×,1-3.×,,,4-9..-n−1..≈.6827

8- Using Microsoft Excel, compare graphically what happens to P_n and A_n as n gets

larger. Include a printout for each graph and comment on your results.





P= I can see through my graph that the perimeter increases each time more. It could be credited to the fact that every times the previous perimeter is multiplied by 1.3,3. and it doesn't seem like a lot in the first stages for n, however, as n increases this small difference which it is multiplied gets bigger, and so the difference between the perimeter of each stage increases and by this graph I would expect the perimeter to get a lot larger compared to the previous stage as I increase n.

A= As I had previously predicted the values for A_n will eventually converge because the common ratio is ,1-3.which is a conversion factor. That is the reason why from stage 5 and on the line seems to get straight next to .7.

In conclusion we can finally say that the Koch Snowflake is a geometric figure with many hidden patterns and secrets in it. After extensive analysis, it was possible to create formulas and conjectures which allows for a person to quickly and easily find the Area, Perimeter, Side Length, and Number of sides of each stage of the Koch Snowflake. However, after many patterns were used in order to create the conjectures and the formula, one thing happened in the last step of this investigation. This was the fact that was possible to see that the Area of the fractal converged; however, the perimeter was getting bigger and bigger. So at the end of this extensive investigation one final question can be asked "How can the area of the fractal converge If the perimeter get larger and larger as n increases?"