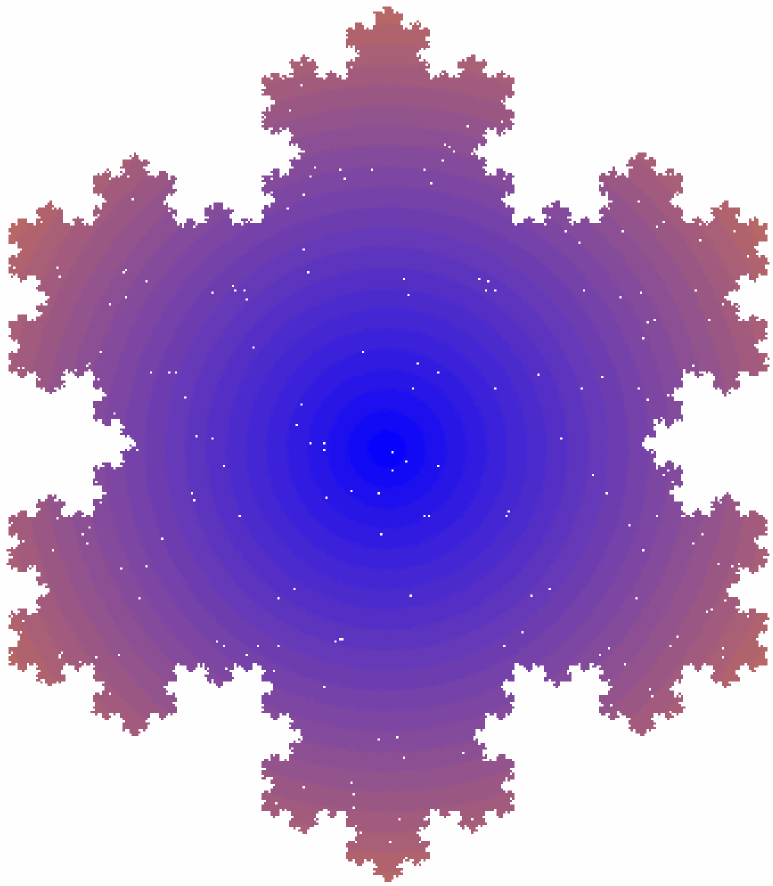
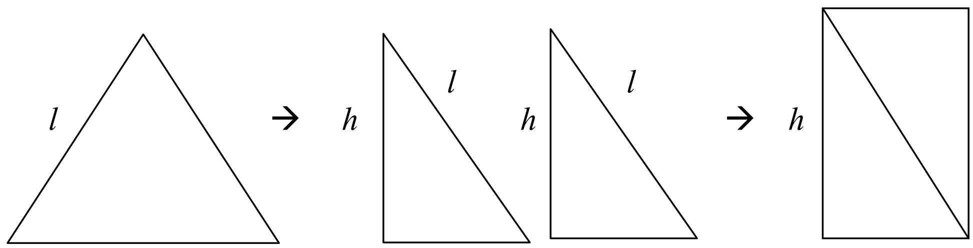
The Koch Snowflake



N_n = the number of sides
l_n = the length of a single side
P_n = the length of the perimeter
A_n = the area of the snowflake

1. Using an initial side length of 1, create a table that shows the values of N_n, l_n, P_n and A_n for n = 0, 1, 2 and 3. Use exact values in

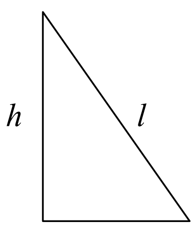
your results. Explain the relationship between successive terms in the table for each quantity N_n , l_n , P_n and A_n.



Equilateral triangle → Two halves → Rectangle

If l is the length of an equilateral triangle and h is the height then the rectangle above has the area of:

$(1 / 2) * l * h$



$l / 2$

Quoting the Pythagorean theorem,

$c^2 = a^2 + b^2$

$l^2 = h^2 + (l / 2)^2$

Thus,
$h = \sqrt{[(3 / 4)l^2]} = \sqrt{(3)} / 2\ l$

Hence the area of the equilateral triangle is

$(1/2) * l * h$

$= \sqrt{(3)} / 4\ l^2$
<u>Stage 0:</u>

<u>Number of sides:</u>
Is determined by counting the sides of the triangle.

$N_0 = 3$

<u>Length of a single side</u>
Is stated through the question.

$L_0 = 1$

<u>Area of the snowflake</u>
Is calculated through the formula established for the equilateral triangle (as stated on page 2).

$A_0 = \sqrt{(3)} / 4\ l^2$

$A_0 = \sqrt{(3)} / 4\ 1^2$

$A_0 = 0.4330127019$

<u>Perimeter of the snowflake</u>
Is calculated through the perimeter of an equilateral triangle formula:

$l_0 \text{ x } N_0 = P_0$

$1 * 3 = 3$

$P_0 = 3$

<u>Stage 1:</u>

Number of sides:
Is determined by multiplying N_0 * 4 *or* proven through counting the sides.

$N_1 = 12$

Length of a single side
Is determined through dividing l_0 by 3.

$l_1 = 1 / 3$

Area of the snowflake
Is calculated through the formula established for the equilateral triangle (as stated on page 2). As well as adding the A_0 onto the answer.

$A_0 = 0.4330127019$

$A_1 = [(3)\ (1 / 3)^2 \sqrt{(3)}] / 4 + A_0$

$A_1 = 0.1443375673$

$A_1 = 0.1443375673 + 0.4330127019$

$A_1 = 0.5773502692$

Perimeter of the snowflake
Is calculated through the perimeter of an equilateral triangle formula:

$l_1 * N_1 = P_1$

$(1 / 3) * 12 = 4$

$P_1 = 4$

Stage 2:

Number of sides:
Is determined by multiplying N_1 * 4 *or* proven through counting the sides.

$N_2 = 48$

Length of a single side
Is determined through dividing l_1 by 3.

$l_2 = 1 / 9$

Area of the snowflake
Is calculated through the formula established for the equilateral tria ngle (as stated on page 2). As well as adding the A_1 onto the answer.

$A_1 = 0.5773502692$

$A_2 = [(3 \text{ x } 4)\ (1 / 9)^2\ \sqrt{(3)} / 4] + A_1$

$A_2 = [(12)\ (1 / 3^2)^2\ \sqrt{(3)} / 4] + A_1$

$A_2 = 0.0641500299 + A_1$

$A_2 = 0.0641500299 + 0.5773502692$

$A_2 = 0.6415002991$

Perimeter of the snowflake
Is calculated through the perimeter of an equilateral triangle formula:

$l_2 * N_2 = P_2$

$(1 / 9) * 48 = 5.333333333$

$P_2 = 5.333333333$

Stage 3:

Number of sides:
Is determined by multiplying N_2 * 4 *or* proven through counting the sides.

$N_3 = 192$

Length of a single side
Is determined through dividing l_2 by 3.

$l_3 = 1 / 27$

Area of the snowflake
Is calculated through the formula established for the equilateral triangle (as stated on page 2). As well as adding the A_2 onto the answer.

$A_2 = 0.6415002991$

$A_3 = [(3 * 4 * 4) (1 / 27)^2 \sqrt{(3)} / 4] + A_2$

$A_3 = [(48) (1 / 3^3)^2 \sqrt{(3)} / 4] + A_2$

$A_3 = 0.0285111244 + A_2$

$A_3 = 0.0285111244 + 0.6415002991$

$A_3 = 0.6700114235$

Perimeter of the snowflake
is calculated through the following formula:

$l_3 * N_3 = P_3$

$(1 / 27) * 192 =$

$P_3 = 7.111111111$

	Stage 0	**Stage 1**	**Stage 2**	**Stage 3**
Number	3	12	48	192

of sides				
Length of a single side	1	1/3 *or* 0.333333333	$1/3^2$ *or* 0.111111111	$1/3^3$ *or* 0.037037037
Length of the perimeter	3	4	5.333333333	7.111111111
Area of the snowflake	0.4330127019	0.5773502692	0.6415002991	0.6700114235

3. For each of the three graphs above develop a statement in terms if the four sets of values shown in this graph. Explain how you arrived

at your generalizations. Verify that your generalizations apply consistently to the sets of values produced above.

The number of sides:

192 / 48 = 4

The number of sides is determined by multiplying the N_{n-1} (the stage before) by 4.

$N_{n-1} * 4 = N_n$

The length of a single side:

(1 / 9) / (1 / 3) = 0.333333333

The length of a single side is determined by dividing the l_{-1} (the stage before) by 3.

$l_{n-1} / 3 = l_n^2 \sqrt{(3)} / 4$

The length of the perimeter:

7.1111111111 / 5.333333333 = 1.333333333

Suggests the perimeter increases by (4/3) by every iteration.

$P_n = N_n * l_n * (4/3)$

$P_n = 12 * (1/3)$
$P_n = 4$ (the length of a single side)

$P = 4 * (4/3)$
$P = 5.333333333$ (the next stage)

Thus we can determine that through using $(4/3)^n$ the perimeter of any internal stage can be found.

$P_n = 3 * (4/3)^n$
The area of the snowflake:

$A_3 = [(3 * 4 * 4) (1 / 27)^2 \sqrt{(3)} / 4] + A_2$

$A_3 = [(48) (1 / 3^3)^2 \sqrt{(3)} / 4] + A_2$

$A_3 = 0.0285111244 + A_2$

$A_3 = 0.0285111244 + 0.6415002991$

Through this equation it can be reduced to the following formula.

$A_n = N_{n-1} * l_n^2 * \sqrt{(3)} / 4 + A_{n-1}$

4. Investigate what happens at n = 4. Use your conjectures from part 3 to obtain values for N_4 , l_4 , P_4 , A_4 . Now draw a large diagram of

one side (i.e. one side of the original triangle that has been transformed) of the fractal at Stage 4 and clearly verify your predictions.

Stage 4:

The number of sides:

$N_{n-1} * 4 = N_n$
$N_{4-1} * 4 = N_4$
$N_3 * 4 = 768$

$N_4 = 768$

The length of a single side:

$l_{n-1} / 3 = l_n$
$l_{4-1} / 3 = l_4$
$l_3 / 3 = l_4$
(1 / 27) / 3 = 0.012345679 *or* 1 / 81

$l_4 = 0.012345679$ *or* 1 / 81

The length of the perimeter:

$P_n = 3 * (4/3)^n$
$P_4 = 3 * (4/3)^4$
$P_4 = 9.481481481$

Proven by:
$N_4 * l_4 = P_4$
768 * (1 / 81) = 9.481481481
$P_4 = 9.481481481$

The area of the snowflake:

$A_n = N_{n-1} * l_n^2 * \sqrt{(3)} / 4 + A_{n-1}$
$A_4 = 192 * (1/81)^2 * \sqrt{(3)} / 4 + A_{n-1}$
$A_4 = 0.0126716108 + 0.6700114235$
$A_4 = 0.6826830343$

5. Calculate values for N_6 , l_6 , P_6 and A_6 . You need not verify these answers.

<u>The number of sides:</u>
$N_{n-1} * 4 = N_n$
$N_5 * 4 = N_6$
$3072 * 4 = N_6$

$N_6 = 12288$

<u>The length of a single side:</u>

$l_{n-1} / 3 = l_n$
$l_5 / 3 = l_6$
(1 / 243) / 3 = 0.0013717421 or 1 / 729

$l_6 = 0.0013717421$ or 1 / 729

<u>The length of the perimeter:</u>

$P_n = 3 * (4/3)^n$
$P_6 = 3 * (4/3)^6$

$P_6 = 16.85596708$

<u>The area of the snowflake:</u>

$A_n = N_{n-1} * l_n^2 * \sqrt{(3)} / 4 + A_{n-1}$
$A_6 = 3072 * (1 / 729)^2 * \sqrt{(3)} / 4 + A_5$
$A_6 = 0.0025030342 + 0.6883148613$

$A_6 = 0.6908178955$

6. Write down successive values of A_n , in terms of A_{0}- .What pattern occurs?

Total Area:
A_0 = 0.4330127019
A_1 = 0.5773502692
A_2 = 0.6415002991
A_3 = 0.6700114235
A_4 = 0.6826830343
A_5 = 0.6883148613
A_6 = 0.6908178955

Triangle areas added at each internal:

A_1 = 0.1443375673
A_2 = 0.0641500299
A_3 = 0.0285111244
A_4 = 0.0126716108
A_5 = 0.005631827
A_6 = 0.0025030342

The pattern that emerges is that through each stage the area increases in size, however to a lesser extent. This is because during the first stages the addition of triangles would result in a large amount of area being added to the initial triangle. However as the stages continued, at around the second stage the triangle began to gain less and less area with each iteration. This statement is supported by the triangle areas, where they would decrease throughout each iteration.

7. Explain what happens to the perimeter and area, as n gets very large. What conclusions can you make about the area as n → infinity? Comment on your results.

As *n* gets very large the perimeter continues growing exponentially. There is not limit to the number that *n* can reach due to the calculation formula having no boundary. Eventually the snowflake would be made up of only sharp corners with no smooth lines connecting them, however this would not limit the infinite series because as long as the perimeter of the snowflake is connected the fractal will maintain its shape.

The area of the snowflake would also follow this trend. Although as n reached infinity the increase in area would become microscopic and possibly unmeasurable. However theoretically speaking the area would still increase.