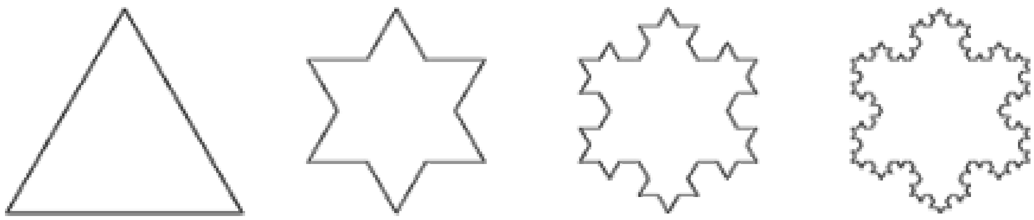
The Koch Snowflake:

Description:

In 1904 the swedish mathematician Helge Von Koch created the Koch snowflake. The Koch snowflake is a fractal curve, a geometric shape which can be subdivided into various parts. This fractal curve is built starting with a normal equilateral triangle (stage 0) and then to get to the following stage the inner third of each side is removed and another equilateral triangle is built where the side was removed. This process can be repeated various times. The first 4 stages are shown below:



Stage 0 Stage 1 Stage 2 Stage 3

The aim of this investigation is to find an expression linking the area of all stages.

Method:

Part 1:

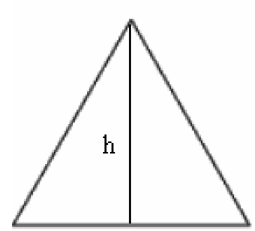
We'll start by creating a table that shows various values at each different stage
N_{n} = the number of sides of each stage
L_{n} = the length of a single side of each stage
P_{n} = The length of the perimeter of each stage
A_{n} = The area of the snowflake of each stage

	Stage 0	Stage 1	Stage 2	Stage 3
N_n	3	12	48	192
L_n	1	1/3	1/9	1/27
P_n	3	4	5 1/3	7 1/9
A_n	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{3}$	$\frac{10\sqrt{3}}{27}$	$\frac{94\sqrt{3}}{243}$

(Assuming an initial side length of 1)

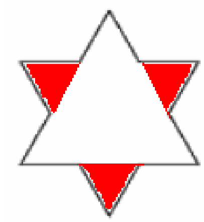
Area of Stage 0 = $\frac{\text{base x height}}{2}$

Height (h) = $\cos^{-1}$ (½ / 1) = 60º = ½
= tan 60 ($\sqrt{3}$) x ½ = $\frac{\sqrt{3}}{2}$



Stage 0

Area = $\dfrac{\frac{\sqrt{3}}{2}}{2}$ = $\dfrac{\sqrt{3}}{4}$



Stage 1

Area of Stage 1 = Area of Stage 0 + Area of 3 shaded triangles

Area of shaded small triangle = $\dfrac{\text{base x height}}{2}$

Height = $\cos^{-1}((\frac{1}{6}) / (\frac{1}{3})) = 60^{\circ} = ½$

$= \tan 60\ (\sqrt{3}) \text{ x } \frac{1}{6} = \frac{\sqrt{3}}{6}$

Area of shaded small triangle

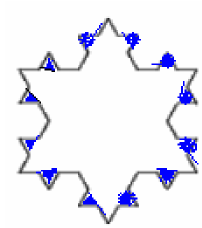
$= \dfrac{\frac{1}{3} \text{ x } \frac{\sqrt{3}}{6}}{2}$

$= \dfrac{\sqrt{3}}{36}$

Total area of Stage 1 = $\dfrac{\sqrt{3}}{4} + \dfrac{3\sqrt{3}}{36}$

$= \dfrac{12\sqrt{3}}{36}$

$= \dfrac{\sqrt{3}}{3}$



Stage 2

Area of Stage 2 = Area of Stage 1 + Area of 12 small shaded triangles

Area of shaded small triangle = $\dfrac{\text{base x height}}{2}$

Height= $\cos^{-1}((\frac{1}{18}) / (\frac{1}{9})) = 60^{\circ} = ½$

$= \tan 60\ (\sqrt{3}) \text{ x } \frac{1}{18} = \frac{\sqrt{3}}{18}$

Area of shaded small triangle

$= \dfrac{\frac{1}{9} \text{ x } \frac{\sqrt{3}}{18}}{2}$

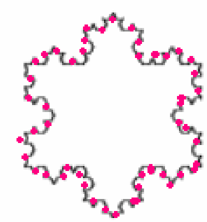
$= \frac{\sqrt{3}}{324}$

Total area of Stage 2 = $\frac{\sqrt{3}}{3} + \frac{12\sqrt{3}}{324}$

$= \frac{120\sqrt{3}}{324}$

$= \frac{10\sqrt{3}}{27}$

Area of Stage 3 = Area of Stage 2 + Area of 48 small shaded triangles


Stage 3

Area of shaded small triangle = $\frac{\text{base x height}}{2}$

Height= $\cos^{-1}\left(\left(\frac{1}{54}\right) / \left(\frac{1}{27}\right)\right) = 60^{o} = ½$

$= \tan 60\ (\sqrt{3})\ x\ \frac{1}{54} = \frac{\sqrt{3}}{54}$

Area of shaded small triangle

$= \frac{\frac{1}{27}\ x\ \frac{\sqrt{3}}{54}}{2}$

$= \frac{\sqrt{3}}{2916}$

Total area of Stage 3 = $\frac{10\sqrt{3}}{27} + \frac{48\sqrt{3}}{2916}$

$= \frac{1128\sqrt{3}}{2916}$

$= \frac{94\sqrt{3}}{243}$

N_n) The number of sides increases from stage to stage by a factor of 4. This is because we remove the inner third of each stage and we then build an equilateral triangle where it was removed, therefore the number of sides of each stage is equal to the number of sides of the previous stage, plus 3 (number of sides of the equilateral triangle) times the number of sides of the previous stage. This is equal to 4 times the number of sides of the previous stage.

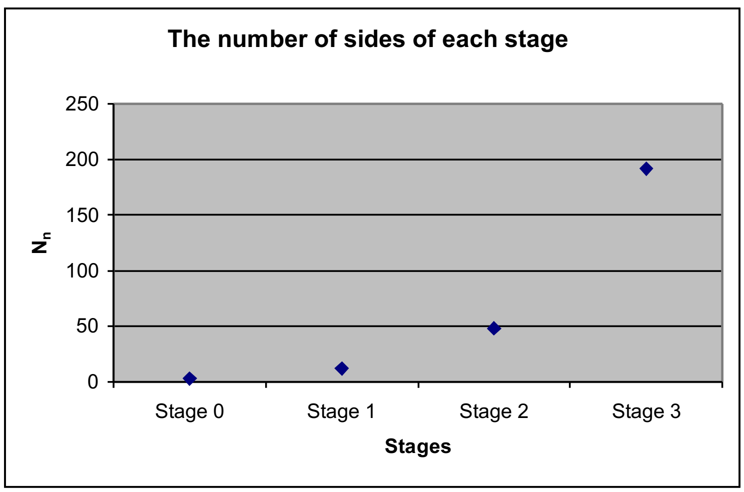
L_n) The length of a single side decreases by a factor of 3 because the inner third of each side is removed therefore if the original length is 1 and you decrease it by a factor of 3 (multiply by 1/3) you'll get the following length to be 1/3

P_n) The length of the perimeter is simply N_n times L_n and since N_n is increasing by a factor of 4 and L_n is decreasing by a factor of 3, then P_n is increasing by a factor of 4/3

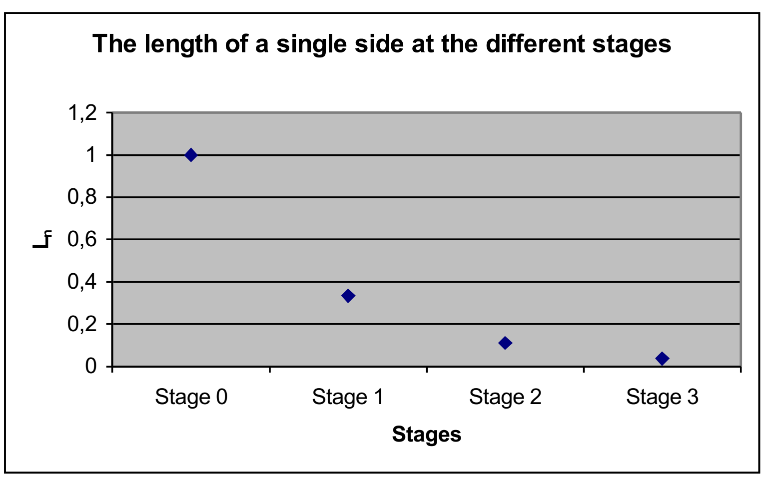
A_n) The area of each stage was found by adding the area of the small triangles to the area of the shape at the previous stage.

Part 2 :

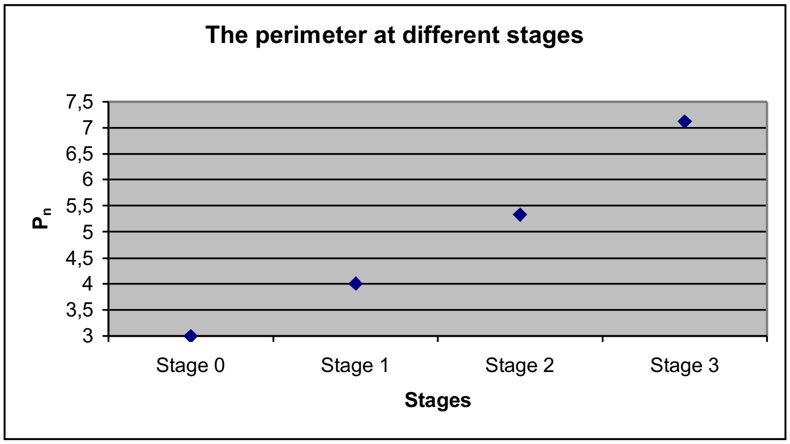
Graph 1



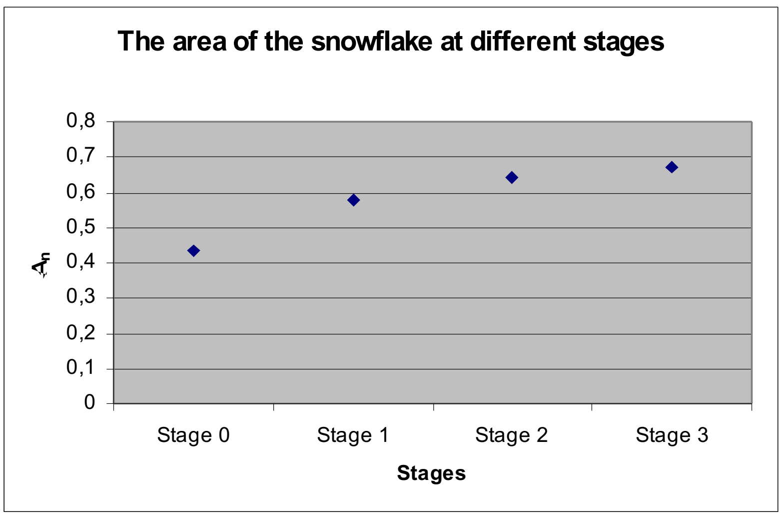
Graph 2



Graph 3



Graph 4



Part 3:

In graph 1, which shows the number of sides of each stage, we can conclude that the formula for calculating the number of sides at the n^{th} stage is 3×4^n (where n is the number of the stage) since the gradient of the graph is 4 and the first term is 3.

In graph 2, which shows the length of a single side of each stage, we can conclude that the formula for calculating the number of sides at the n^{th} stage is $1 \times (1/3)^n$ (where n is the number of the stage) since the gradient of the graph is -3 (1/3) and the first term is 1

In graph 3, which shows the perimeter of the different stages, the formula for calculating the perimeter at the n^{th} stage is N_n x L_n . $N_n = 3 \times 4^n$ and $L_n = 1 \times (1/3)^n$ for the reasons previously explained, therefore the formula for the perimeter is $3 \times (4/3)^n$.

In graph 4, which shows the area of each shape, we can see that as we progress in stage the area increases. To find the formula in this case we'll need to look at the similarities between each stage and perform some calculations as shown below:

$$\text{Area Stage } 0 = \frac{\sqrt{3}}{4}.$$

$$\text{Area Stage } 1 = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{36}$$

$$\text{Area Stage } 2 = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{36} + \frac{12\sqrt{3}}{324}$$

$$\text{Area Stage } 3 = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{36} + \frac{12\sqrt{3}}{324} + \frac{48\sqrt{3}}{2916}$$

$$A_n = A_{n-1} + \frac{L_n \times \frac{N_n}{4} \times \sqrt{3}}{\frac{\frac{2}{L_n}}{2}}$$

$$A_n = A_{n-1} + \frac{L_n^2 \times \frac{N_n}{4} \times \sqrt{3}}{4}$$

$$A_n = A_{n-1} + \frac{(L_n^2 N_n \sqrt{3})}{16}$$

Part 4:

	Stage 4
N_n	768
L_n	$\frac{1}{81}$
P_n	$9\frac{13}{27}$
A_n	$\frac{862\sqrt{3}}{2187}$

$\mathbf{N_n} = 3 \times 4^n$
$= 3 \times 4^4$
$= 768$

$\mathbf{L_n} = 1 \times (1/3)^n$
$= 1 \times (1/3)^4$
$= \frac{1}{81}$

$\mathbf{P_n} = 3 \times (4/3)^n$
$= 3 \times (4/3)^4$
$= 3 \times \frac{256}{81}$
$= \frac{768}{81}$
$= 9\frac{13}{27}$

$$\mathbf{A_n} = A_{n-1} + \frac{(L_n^2\, N_n \sqrt{3}\,)}{16}$$

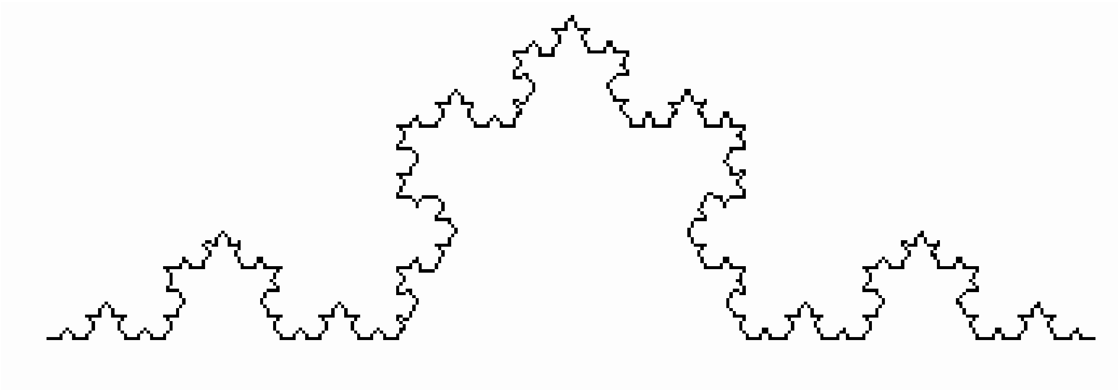
$$= \frac{94\sqrt{3}}{243} + \frac{\left(\frac{1}{6561} \times 768 \times \sqrt{3}\,\right)}{16}$$

$$= \frac{94\sqrt{3}}{243} + \frac{\frac{768\sqrt{3}}{6561}}{16}$$

$$= \frac{94\sqrt{3}}{243} + \frac{768\sqrt{3}}{104976}$$

$$= \frac{94\sqrt{3}}{243} + \frac{16\sqrt{3}}{2187}$$

$$= \frac{862\sqrt{3}}{2187}$$



Drawing of the Koch snowflake at the 4th stage:

Part 5:

The value of n where A_{n+1} is equal to A_n to six places of decimals is 17. This was found using a spreadsheet on Microsoft Excel.

n	15	16	17	18	19
N_n	3221225472	12884901888	51539607552	2,06158E+11	8,24634E+11
L_n	6,96917E-08	2,32306E-08	7,74352E-09	2,58117E-09	8,60392E-10
P_n	224,4927416	299,3236555	399,0982074	532,1309432	709,5079242
A_n	0,692818968	0,692819721	0,692820055	0,692820204	0,69282027

Part 6:

The formula for calculating the perimeter is $3 \times (4/3)^n$. As n gets very large or in other words as $n \to \infty$, the perimeter becomes larger and larger (we can prove that using the spreadsheet above) therefore we can conclude that the perimeter is infinite:

$$\lim_{n \to \infty} 3 \times \left(\frac{4}{3}\right)^n = \infty$$

The formula for calculating the area is $A_{n-1} + \dfrac{(L_n^2 N_n \sqrt{3})}{16}$ if we look back to the end of part 3 we can see that the Area of the nth stage is equal to the area of the previous stage plus the area of the smaller triangles added at each stage. This new area added, as $n \to \infty$, becomes smaller and smaller and will be converging to zero therefore the Area for the nth stage must be converging to a certain number so it must be finite. In order to find the number to which it converges we can simply use the spreadsheet and from looking at a part of the spreadsheet below we can see that as n is getting larger the value for A_n is remaining the same therefore it must be converging to that number.

n	24	25	26	27	28	29	30
N_n	8,44425E+14	3,3777E+15	1,35108E+16	5,40432E+16	2,16173E+17	8,64691E+17	3,45876E+18
L_n	3,54071E-12	1,18024E-12	3,93412E-13	1,31137E-13	4,37124E-14	1,45708E-14	4,85694E-15
P_n	2989,860553	3986,480737	5315,30765	7087,076866	9449,435822	12599,24776	16798,99702
A_n	0,692820322	0,692820323	0,692820323	0,692820323	0,692820323	0,692820323	0,692820323

We can see that as n becomes larger than 25 (as $n \to \infty$) the area remains to be 0.692820323 therefore it's converging to that value. To find an exact value for this I can simply use my trigonometry knowledge… $\sqrt{\frac{1}{2}} = 0.7071067812$, therefore by trial and improvement I discovered that 0.692820323 = $\sqrt{\frac{12}{25}}$ which in turn is equal to $\frac{2\sqrt{3}}{5}$.

In conclusion we can state that the Koch Snowflake has an infinite perimeter and a finite area as $n \to \infty$. We can also see that as $n \to \infty$, the length of each side (L_n) decreases and the number of sides N_n increases.