The Koch snowflake

Introduction: The Koch snowflake fractal is built by starting with an equilateral triangle, and removing the inner third of each side, building another equilateral triangle at the location where the side was removed. When n = 0, 1, 2and 3 each value of N_n, l_n, P_n and A_n can be shown as the following table:

n	0	1	2	3
N_n	3	12	48	192
l_n	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$
P_n	3	4	$\frac{16}{3}$	$\frac{64}{9}$
A_n	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{3}$	$\frac{10\sqrt{3}}{27}$	$\frac{94\sqrt{3}}{243}$

And the process I use to get the value are shown below:

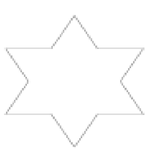
The number of the side:

To get the next snowflake, we can found that, each side of the triangle will break into 4 new sides, hence the number of each side will always 4 times than previous one. Thus we get

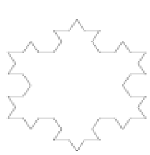
$$N_n=4N_{n-1}$$

Thus we assume that Nn is the single of the number of sides and get the process below:

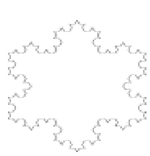
Shape1: we get a triangle. N1=3



Shape2: We break into 4 new sides. So the N2=3×4



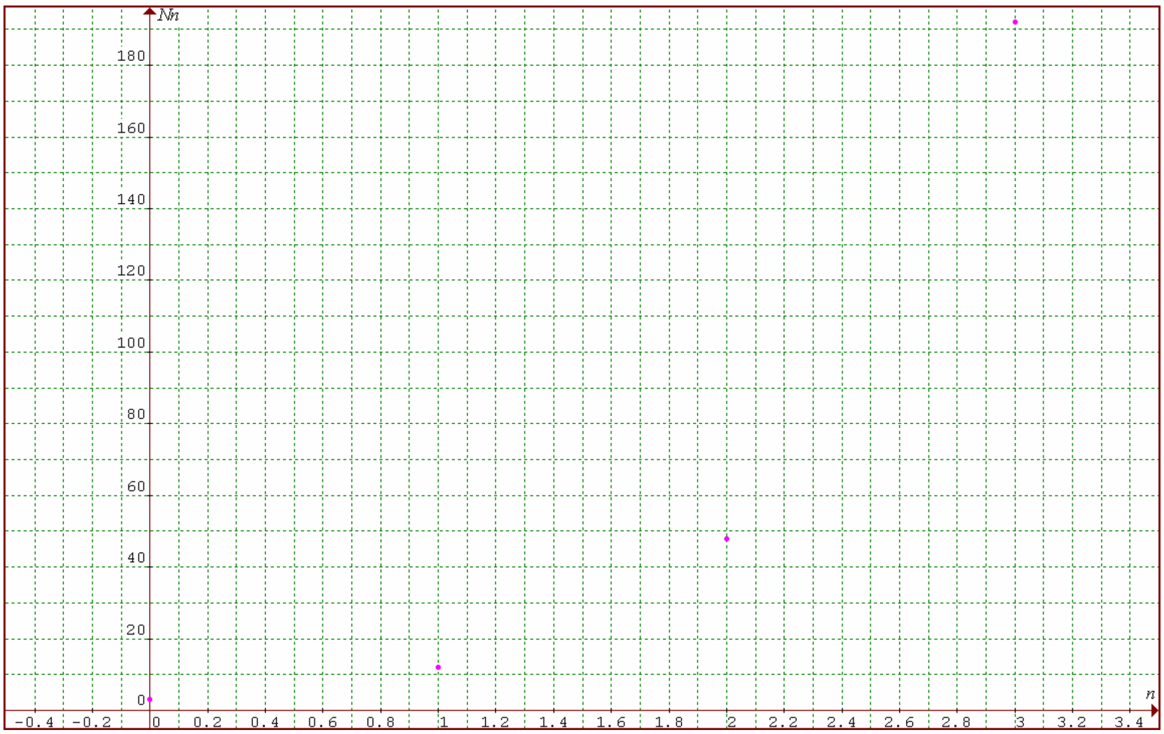
Shape 3: Each side then brake into 4 new sides, So N3=3×4 ×4



Shape 4:We repeat the same step, thus we get N4=3×4×4×4

And then we graph our data:

The number of the side



We can found that the shape of the graph is a geometric series. Thus, if we want to get the next shape, we should time the same number. Thus it's a geometric series which has a first term of 3 and a common ratio of 4, obviously. So we can get the geometric formula:

$$Nn = 3 \times 4^n$$

Then, to verify the formula, we place

n=0 n=1 n=2 n=3

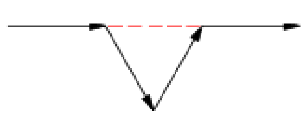
And then we get:

$N0 = 3\times 4^0 = 3$ $N1 = 3\times 4^1 = 12$ $N2 = 3\times 4^2 = 48$ $N3 = 3\times 4^3 = 192$

The results are already proved above.

The length of each side:

Because each side was break into 3 equal part, and the inner one is moved (As shown in the picture below).



Therefore the length of each side is decrease and become one-third of its previous length in each stage of generation. So ln can be presented in terms of ln-1:

$$ln=\frac{1}{3}ln-1$$

To get next generation, each length should times one-third of the previous one. Thus we calculate length of each shape:

$$l_0=(\frac{1}{3})^0=1$$

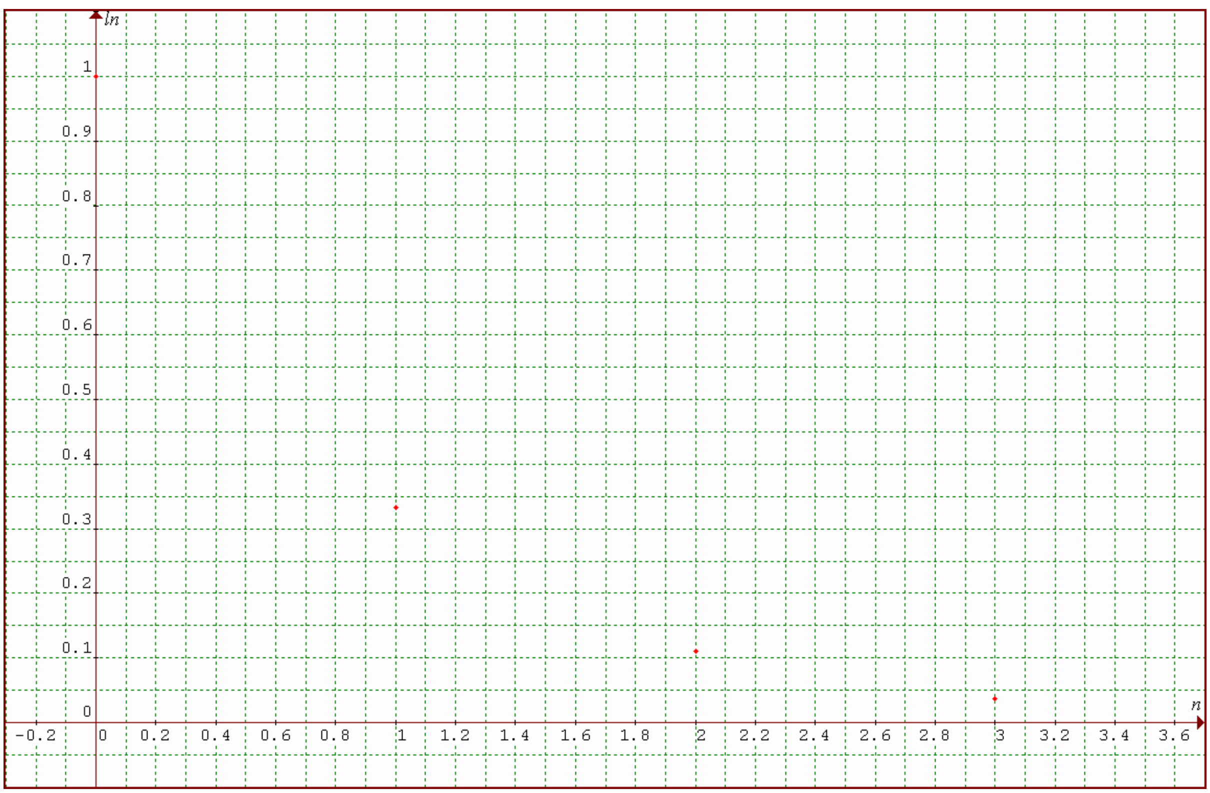
$$l_1=(\frac{1}{3})^1=1\times\frac{1}{3}$$

$$l_2=(\frac{1}{3})^2=1\times\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$$

$$l_3=(\frac{1}{3})^3=1\times\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}=\frac{1}{27}$$

The length of each side

Then we can put our data into the diagram below:



We can found that that ln is reducing in geometric, and then we can get the first term and common ratio, Hence it can written in geometric series:

$$l_n = (\frac{1}{3})^n$$

To verify the formula, we put n=0, 1,2and 3 into the formula. So we get:

$$l_0 = (\frac{1}{3})^0 = 1 \quad l_1 = (\frac{1}{3})^1 = \frac{1}{3} \quad l_2 = (\frac{1}{3})^2 = \frac{1}{9} \quad l_3 = (\frac{1}{3})^3 = \frac{1}{27}$$

The results are already proved above.

The length of perimeter:

According to the statement above, we can know that to get next generation, Each side was break into 4 part and the length of each part is equal to each other, and the length of graph is decrease at one over four times of the previous one. Thus, the perimeter of the graph is increasing and become four-third of its length in next generation. So in terms of Pn-1, we can present Pn as:

$$P_n = \frac{4}{3} P_{n-1}$$

To get next generation, each length should times one-third of the previous one. Thus we calculate perimeter of each shape:

$$P_0 = 3 \times \frac{4}{3} = 3 \qquad P_1 = 3 \times \frac{4}{3} \times \frac{4}{3} = 4$$

$$P_2 = 3 \times \frac{4}{3} \times \frac{4}{3} = \frac{16}{3} \qquad P_3 = 3 \times \frac{4}{3} \times \frac{4}{3} \times \frac{4}{3} = \frac{64}{9}$$

Then we can put our data into the diagram below:

The length of perimeter

We found that if we link all point together, we may get an up-sloping curve which is a geometric series. Thus, if we want to get the next shape's perimeter, we should time the same number. Thus it's a geometric series which has a first term of 3 and a common ratio of $\frac{4}{3}$, obviously. So we can get the geometric formula:

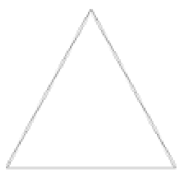
$$P_n=3\times\left(\frac{4}{3}\right)^n$$

To verify the formula, we put n=0, 1,2and 3 into the formula. So we get:

$$P_0=3\times\left(\frac{4}{3}\right)^0=3 \qquad P_1=3\times\left(\frac{4}{3}\right)^1=4 \qquad P_2=3\times\left(\frac{4}{3}\right)^2=\frac{16}{3} \qquad P_3=3\times\left(\frac{4}{3}\right)^3=\frac{64}{9}$$

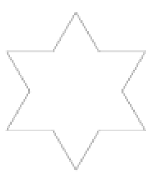
The results are already proved above.

The area of the snowflake:



From the observation, we can found a rule between the number of the side and the number of new triangle appeared.

Firstly, N_n=3

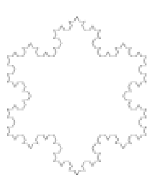


Secondly, N_n=12

The number of new triangle appeared=3

Thirdly, N_n=48

The number of new triangle appeared=12



Fourthly, N_n=192

The number of new triangle appeared=12

We can found that, the number of new equilateral triangle appeared in the shape is equal to the number of the side in the previous triangle (N_{n-1}).

And then we show the area of the each addition one.

We can found that, the new triangles which add to the original triangle are equal.

And we know that each length of the new triangle is equal to the $\frac{1}{3}$ times of the previous

one. Thus the length of each side can be write as

$$l_{new} = \frac{1}{3} l_{n-1}$$

According to the sin formula

$$A_{new} = \frac{1}{2} AB\sin C$$

Hence the area of each additional triangle is equal to

$$A_{new} = \frac{1}{2} (\frac{1}{3} l_{n-1}) (\frac{1}{3} l_{n-1}) \sin C$$

Because each side of the triangle is equal to each other, every angle is equal to each other which is 60°. Hence $\sin C = \frac{\sqrt{3}}{2}$.

$$A_{new} = = \frac{1}{2} (\frac{1}{3} \ln - 1)^{2} \times \frac{\sqrt{3}}{2}$$

In terms of Nn-1, ln-1 and An, we can determine the area of Nn

$$An = An\text{-}1 + Nn\text{-}1 \times A_{new}$$

And in terms of N_{n-1}, l_{n-1} and A_n, we can determine the area of N_n

$$A_n = A_{n-1} + N_{n-1} \times A_{new}$$

$$= A_{n-1} + N_{n-1} \times \frac{\sqrt{3}}{4} \times (\frac{1}{3} l_{n-1})^2$$

Since we have find out the formula for N_{n-1} and l_{n-1} in terms of n, we can substitute these functions into the expression of An:

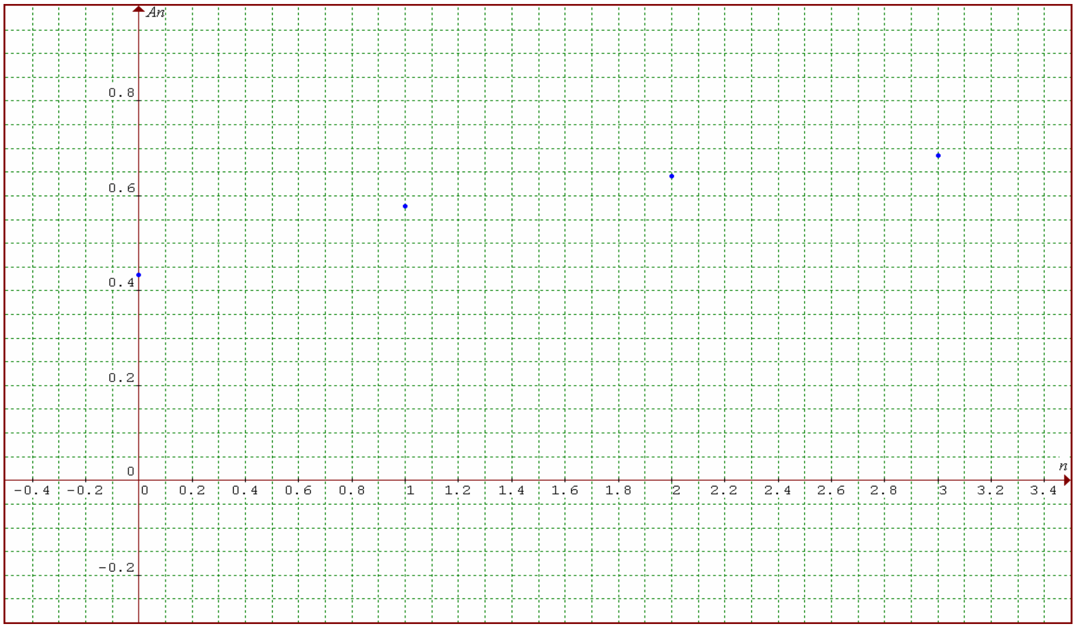
$$A_n = A_{n-1} + 3 \times 4^{n-1} \times \frac{\sqrt{3}}{4} \times (\frac{1}{3} (\frac{1}{3})^{n-1})^2$$

$$= A_{n-1} + \frac{\sqrt{3}}{12} \times 4^{n-1} \times (\frac{1}{9})^{n-1}$$

$$= A_{n-1} + \frac{\sqrt{3}}{12} \times (\frac{4}{9})^{n-1}$$

From the graph above we can see that the area is increasing, but the ratio of new area added in becomes smaller as the value of n getting bigger. The following diagram shows the data:

The area of the snowflake



The new area added in first shape

$A_1=\frac{\sqrt{3}}{12}\times(\frac{4}{9})^{1-1}=\frac{\sqrt{3}}{12}\times(\frac{4}{9})^{0}$

The new area added in second shape

$$A_2=\frac{\sqrt{3}}{12}\times(\frac{4}{9})^{2-1}=\frac{\sqrt{3}}{12}\times(\frac{4}{9})^{1}$$

The new area added in n^{th} shape

$$A_n=\frac{\sqrt{3}}{12}\times(\frac{4}{9})^{n-1}$$

Hence the new area added to the origin is:

$$A^1+A^2+\cdots\cdots+A^n=\frac{\sqrt{3}}{12}\times(\frac{4}{9})^0+\frac{\sqrt{3}}{12}\times(\frac{4}{9})^1+\cdots\cdots+\frac{\sqrt{3}}{12}\times(\frac{4}{9})^{n-1}$$

$$=\frac{\sqrt{3}}{12}\times((\frac{4}{9})^1+(\frac{4}{9})^2+\cdots\cdots(\frac{4}{9})^n)$$

$$=\frac{\sqrt{3}}{12}\times\sum_{k=0}^{n-1}\left(\frac{4}{9}\right)^k$$

So we add the total area:

$A^n=A^0+(A^1+A^2+\cdots\cdots+A^n)$

$$\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{12}\times\sum_{k=0}^{n-1}\left(\frac{4}{9}\right)^{k} \quad (n=1, 2, 3\ldots\ldots)$$

To verifying the above formula, <u>for n=0, it's not make any sense. That is because it does have a previous term which is n-1, so in this case we shouldn't place n=0 into the formula.</u> However when we put n=1, 2, 3 we can see that the results are all the same as the table said.

$$A1=\frac{\sqrt{3}}{3} \quad A2=\frac{10\sqrt{3}}{27} \quad A3=\frac{94\sqrt{3}}{243}$$

The results are already proved above.

Because we have got the formula above , we are able get what happen when n=4.

$$N_4 = 4N_3 = 4\times 192 = 768$$

$$l_4 = \frac{1}{3}\times l_3 = \frac{1}{3}\times \frac{1}{27} = \frac{1}{81}$$

$$P_4 = \frac{4}{3}\times P_3 = \frac{4}{3}\times \frac{64}{9} = \frac{256}{27}$$

$$A_4 = A_3 + \frac{\sqrt{3}}{12}\times (\frac{4}{9})^{4-1} = \frac{94\sqrt{3}}{243} + \frac{\sqrt{3}}{12}\times (\frac{4}{9})^{4-1} = \frac{862\sqrt{3}}{2187}$$

The following figure shows one side of the triangle when N=4

n	An	Nn	In	Pn
0	0.433013	3	1	3
1	0.57735	12	1	4
2	0.6415	48	0.333333	5.333333
3	0.670011	192	0.111111	7.111111
4	0.682683	768	0.037037	9.481481
5	0.688315	3072	0.012346	12.64198
6	0.690818	12288	0.004115	16.85597
7	0.69193	49152	0.001372	22.47462
8	0.692425	196608	0.000457	29.96616
9	0.692645	786432	0.000152	39.95488
10	0.692742	3145728	5.08E-05	53.27318
11	0.692786	12582912	1.69E-05	71.03091
12	0.692805	50331648	5.65E-06	94.70788
13	0.692813	201326592	1.88E-06	126.2772
14	0.692817	805306368	6.27E-07	168.3696
15	0.692819	3221225472	2.09E-07	224.4927
16	0.69282	12884901888	6.97E-08	299.3237
17	0.69282	51539607552	2.32E-08	399.0982
18	0.69282	2.06158E+11	7.74E-09	532.1309

To find out the value for n where An+1 equals An to six places of decimals. We get the help from the Excel software to work out the first 18 term to check if we get the number we need.

From the result we can see that the A_n for n=16 is equal to for n=17 to six decimal places. So the value of n is 16 where A_n is equal to An+1 to six decimal places.

So when n = 16:
$N_{16} = 12884901888$ $\quad l_{16} = 2.323057\times10^{-8}$
$P_{16} = 299.323656$ $\quad A_{16} = 0.692820$

Then we consider the expression for Pn which:

$$Pn=\frac{4}{3}Pn\text{-}1$$

We can see that Pn is always increasing in the same ratio of $\frac{4}{3}$. So the value of Pn increase to positive infinity as n→∞.

For the expression of An which is:

$$An= An\text{-}1+\frac{\sqrt{3}}{12}\times\ (\frac{4}{9})^{n-1}$$

According to the formula of sum of geometric series to infinity:

$$Sn=\frac{U1(1-r^n)}{1-r}\ (r\neq 1)$$

$$\text{If} -1 < r < 1, r^{n} \to 0 \text{ as } n \to \infty$$

Thus, $S_n = \lim_{n\to\infty} \frac{u_1}{1-r}$

$$A_\infty = \lim_{n\to\infty} \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \times \frac{1}{1-\frac{4}{9}} = \frac{2}{5}\sqrt{3}$$

Thus we know: in this case as n→∞ we can have an infinity length of perimeter in a limited area.

we use the math induction to prove this conclusion:

As we know $A_n = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \times \sum_{k=0}^{n-1} \left(\frac{4}{9}\right)^k$

P(1) is true since $A_1 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \times (\frac{4}{9})^0 = \frac{\sqrt{3}}{3}$

Assume that n=m is true for integer t≥1.

$$A_m = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \times \sum_{k=0}^{m-1} \left(\frac{4}{9}\right)^k$$

Consider n=m+1

$$A_{m+1} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \times \sum_{K=0}^{m} \left(\frac{4}{9}\right)^k$$

Prove $A_{m+1} = A_m + \frac{\sqrt{3}}{12} \times (\frac{4}{9})^m$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \times \sum_{k=0}^{m-1} \left(\frac{4}{9}\right)^k + \frac{\sqrt{3}}{12} \times (\frac{4}{9})^m$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \times \left(\sum_{k=0}^{m-1} \left(\frac{4}{9}\right)^k + (\frac{4}{9})^m\right)$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \times \sum_{k=0}^{m} \left(\frac{4}{9}\right)^k$$

Thus P(m+1) is true when P(m) is true. Therefore P(n) is true for all integers m≥1.