The Fibonacci Numbers and the Golden Ratio

The Fibonacci Numbers

The Fibonacci numbers are sequence of numbers. They are named after the Italian mathematician Leonardo of Pisa, known as Fibonacci. He published a book called "Liber Abaci", and he was the first person to publish a book in Western Europe that used the Indian numerals 9, 8, 7, 6, 5, 4, 3, 2, 1. Fibonacci was perhaps the greatest mathematician of his time. But he is most famous for the numbers which has his name.

These are the first ten terms in the Fibonacci sequence.

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}
1	1	2	3	5	8	13	21	34	55

To get the next term in the sequence the two previous terms must be added. This can be written as:

$$\mathbf{U_n = U_{n-1} + U_{n-2}}$$

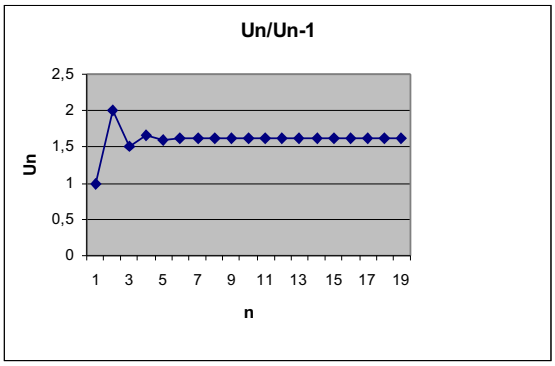
A number in the sequence is the sum of its two predecessors.

The Golden Ratio

To find the Golden Ratio, a term has to be divided with the previous term (i.e. . $\frac{u_2}{u_1}, \frac{u_3}{u_2}$, .. etc). By doing this a certain amount of times you'll reach the Golden Ratio. It can be done, no matter what the starting numbers are. To prove my assumption I made several different tables and graphs that shows the Golden Ratio. I've used different numbers; negative and positive (look in appendix 1).

These are the "original" start numbers 1 and 1.

n	U_n	U_{n+1}/U_n
1	1	1
2	1	2
3	2	1,5
4	3	1,666667
5	5	1,6
6	8	1,625
7	13	1,615385
8	21	1,619048
9	34	1,617647
10	55	1,618182
11	89	1,617978
12	144	1,618056
13	233	1,618026
14	377	1,618037
15	610	1,618033
16	987	1,618034
17	1597	1,618034
18	2584	1,618034
19	4181	1,618034
20	6765	0



By investigating the Fibonacci numbers, I can make a conjecture, that the ratio of two consecutive terms gets closer, as they increase, to the Golden Ratio. By changing the start numbers I proved that it doesn't make a difference which numbers we use.

To prove my conjecture I've changed the equation and solved it. Afterwards I found the discriminant (5), which leads me to find the roots. I plugged in the discriminant to the quadratic equation for finding the roots. The results for the positive root is, 1.618

and the negative root is, -0.618. This proves my conjecture. Look below to see my calculations and explanations.

I divided both sides of this equation $\mathbf{U_n = U_{n-2} + U_{n-1}}$, by $\mathbf{U_{n-1}}$.

$\frac{u_{n-1}}{u_{n-2}} = 1 + \frac{u_{n-2}}{u_{n-1}}$ = Why we do it?

$$\frac{u_n}{u_{n-1}} = 1 + \frac{1}{u_{n-1}/u_{n-2}}$$

Here, $\frac{u_n}{u_{n-1}}$ is one of the numbers in the column above, it gets closer and closer to 1.618, which I will call x. On the other side of the equation, $\frac{1}{u_{n-1}/u_{n-2}}$ is the reciprocal of one of the numbers in the same column, so it gets closer to $\frac{1}{x}$.

Now the equation looks like this;

$$x = 1 + \frac{1}{x}$$

By rearranging the equation into a quadratic one, I can find the discriminant, which leads to the finding of the roots.

$x^2 - x - 1 = 0$

Calculate the discriminant:

$d = b^2 - 4 \cdot a \cdot c$

$d = (-1)^2 - 4 \cdot 1 \cdot -1$

$d = 5$

Calculate the roots:

$x = -b +- \sqrt{\frac{d}{a \cdot c}}$

$x = -1 +- \sqrt{\frac{5}{2 \cdot 1}} =$

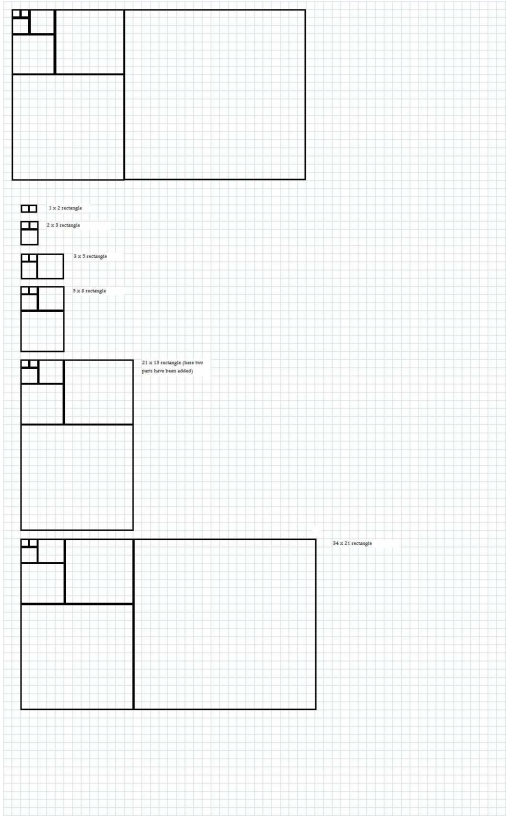
Roots are **1.618** and **-0.618**.

After calculating the roots, I observed that the negative reciprocal of the Golden Ratio was found. Actually, - 0,618 is the limit of consecutive ratios. This is done by going backwards in the table i.e. U_3, U_2, U_1, U_0, U_{-1}, U_{-2}, U_{-3} etc. I noticed that you´ll reach the inverted Golden ratio -0.618 faster than its reciprocal. I also made table that proves my statement above.

n	U_n	U_n/U_{n-1}
1	1	1
0	-1	-0,5
-1	2	-0,66667
-2	-3	-0,6
-3	5	-0,625
-4	-8	-0,61538
-5	13	-0,61905
-6	-21	-0,61765
-7	34	-0,61818
-8	-55	-0,61798
-9	89	-0,61806
-10	-144	-0,61803
-11	233	-0,61804

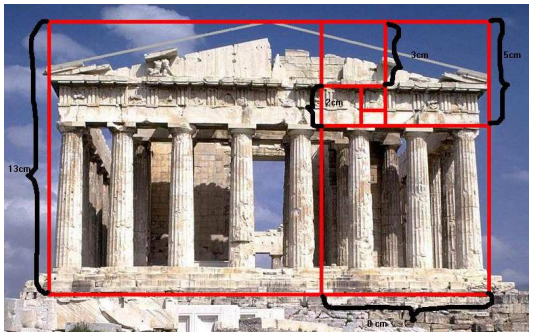
-12	-377	-0,61803
-13	610	-0,61803
-14	-987	-0,61803
-15	1597	-0,61803
-16	-2584	-0,61803
-17	4181	-0,61803
-18	-6765	-0,61803
-19	10946	-0,61803
-20	-17711	-0,61803
-21	28657	-0,61803
-22	-46368	-0,61803
-23	75025	-0,61803
-24	-121393	-0,61803
27	196418	########

The Fibonacci numbers and the Golden Ratio can also be found in nature. They appear in the leaf arrangement in plants, in the pattern of a flower, the bracts of a pinecone, or the scales of a pineapple. The Fibonacci numbers are called the Natures Numbering system. It´s still a mystery why they appear in nature. The ratio between squares is also connected to the Golden ratio. I made some squares that show that the ratio between them is the Golden Ratio.



These are the Fibonacci squares. The ratio between the sides of the squares is 1.618 – the golden ratio. It starts by two small 1 by 1cm. The next one is 1 by 2, 2 by 3 and so on.

The Greeks used the golden ratio in the design and construction of the Greek temple (Parthenon).



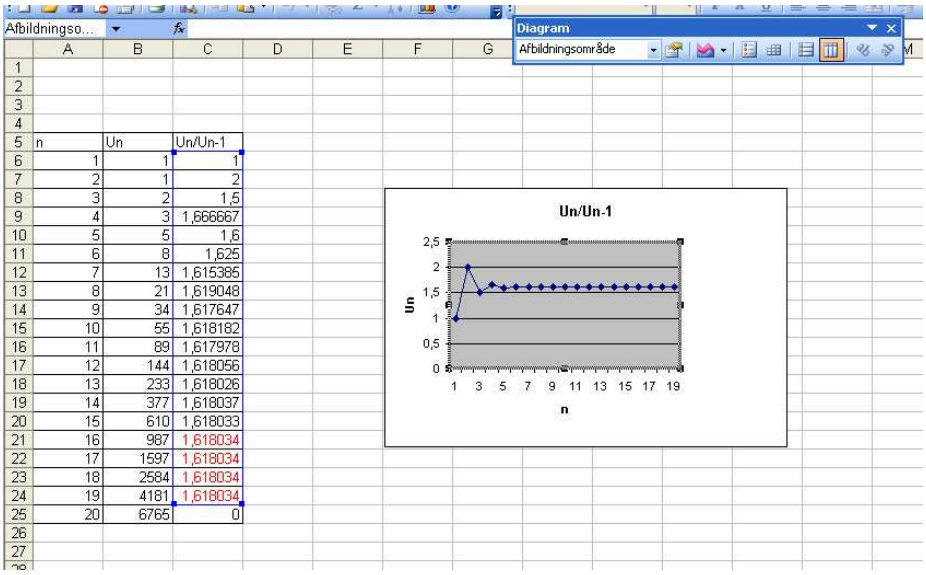
To prove that the Greeks really used the golden ratio in the design, I've measured the sides. The first square is 1cm by 1cm; the next is 1cm by 2 cm and so on. Look in the table below to see the ratio between each side.

n	U_n	U_{n+1}/U_n
1	1	1
2	1	2
3	2	1,5
4	3	1,666667
5	5	1,6
6	8	1,625
7	13	1,615385

Appendix

1)

I changed to two "start numbers" to 1 and 7, and by making a table and a graph I proved that it doesn't matter which numbers we use to get the Golden Ratio.



n	Un	Un/Un-1
1	1	1
2	1	2
3	2	1,5
4	3	1,666667
5	5	1,6
6	8	1,625
7	13	1,615385
8	21	1,619048
9	34	1,617647
10	55	1,618182
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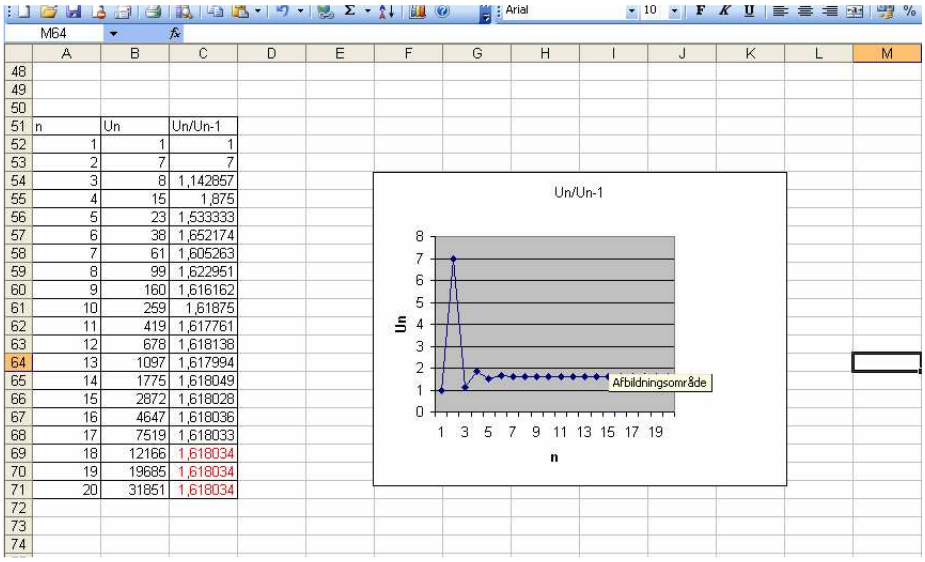
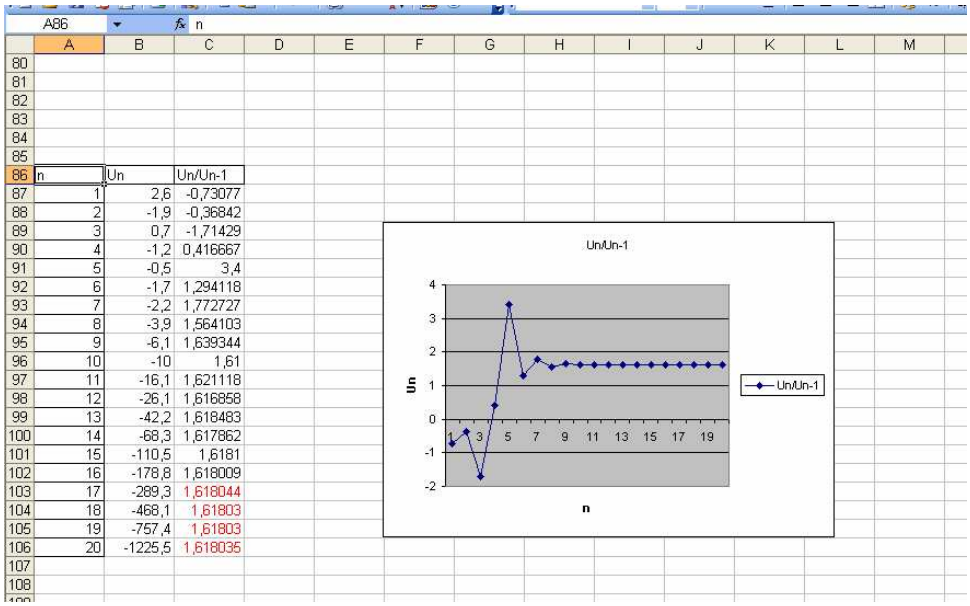
15	610	1,618033
16	987	1,618034
17	1597	1,618034
18	2584	1,618034
19	4181	1,618034
20	6765	0

To show that we would eventually get the same result, no matter what "start numbers" were, I plugged in -9 and 5. To prove that with even a negative number, we would reach the Golden Ratio.



n	Un	Un/Un-1
1	1	1
2	7	7
3	8	1,142857
4	15	1,875
5	23	1,533333
6	38	1,652174
7	61	1,605263
8	99	1,622951
9	160	1,616162
10	259	1,61875
11	419	1,617761
12	678	1,618138
13	1097	1,617994
14	1775	1,618049
15	2872	1,618028
16	4647	1,618036
17	7519	1,618033
18	12166	1,618034
19	19685	1,618034
20	31851	1,618034

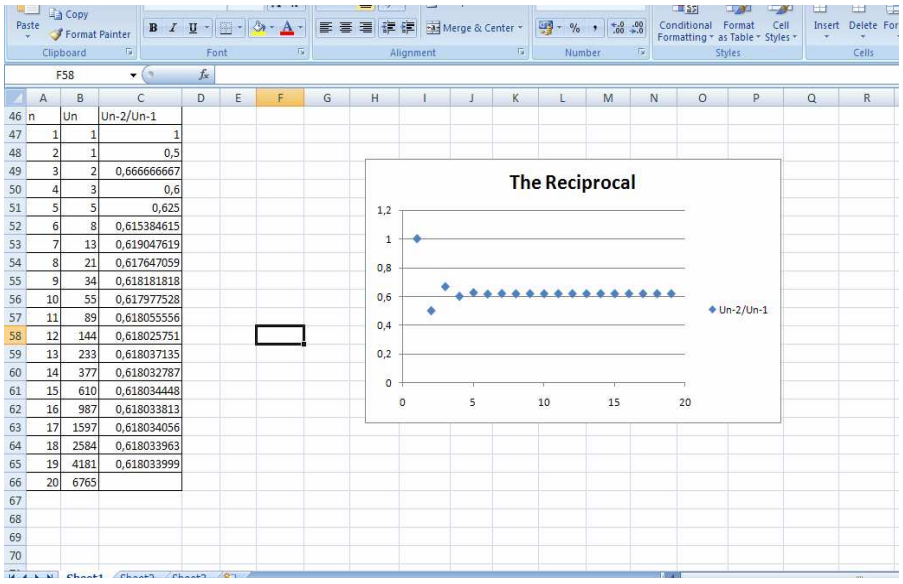
I also tried to plug in decimal numbers to see what happened. I plugged in 2,6 and -1,9.



n	Un	Un/Un-1
1	2,6	-0,73077
2	-1,9	-0,36842
3	0,7	-1,71429
4	-1,2	0,416667
5	-0,5	3,4
6	-1,7	1,294118
7	-2,2	1,772727
8	-3,9	1,564103
9	-6,1	1,639344
10	-10	1,61
11	-16,1	1,621118
12	-26,1	1,616858
13	-42,2	1,618483
14	-68,3	1,617862
15	-110,5	1,6181
16	-178,8	1,618009
17	-289,3	1,618044
18	-468,1	1,61803
19	-757,4	1,61803
20	-1225,5	1,618035

2)

The inverted Golden Ratio can be found by dividing U_{n-1} by U_{n-2}. As you can see below, the inverted ratio is reached much faster, than the opposite.



n	Un	Un-2/Un-1
1	1	1
2	1	0,5
3	2	0,666666667
4	3	0,6
5	5	0,625
6	8	0,615384615
7	13	0,619047619
8	21	0,617647059
9	34	0,618181818
10	55	0,617977528
11	89	0,618055556
12	144	0,618025751
13	233	0,618037135
14	377	0,618032787
15	610	0,618034448
16	987	0,618033813
17	1597	0,618034056
18	2584	0,618033963
19	4181	0,618033999
20	6765	