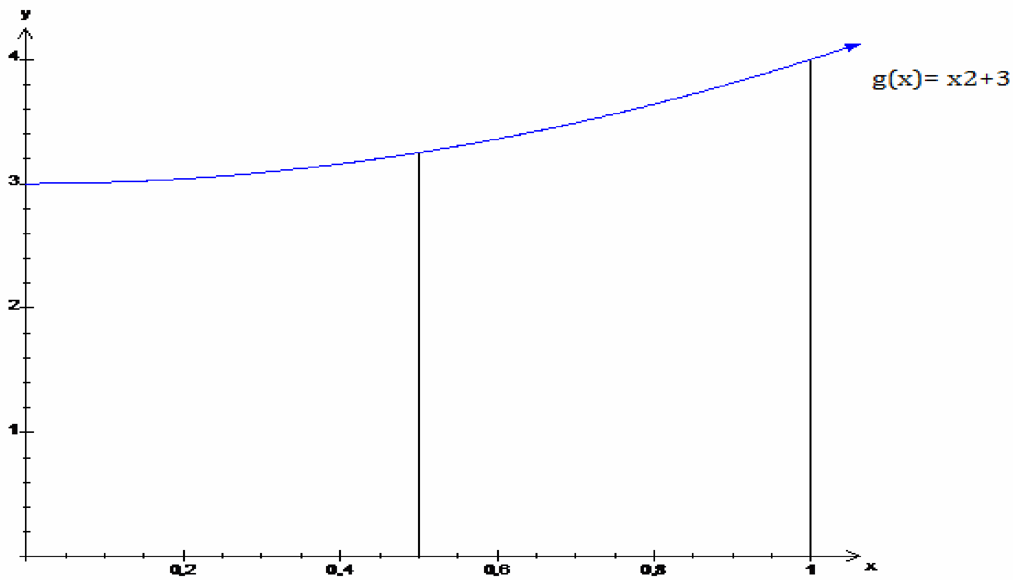
Shady areas

The aim of this particular portfolio investigation is to find the area under a certain curve by dividing it into trapeziums and adding the sum of the areas of these trapeziums to approximate the actual area. This is done instead of using the usual method of integration. Let us consider the following function to begin with:-



The above graph shows the function $g(x) = x^2 + 3$ from interval of x=0 and x=1. The area under this curve, as seen above, is divided into 2 trapeziums. If the area under the curve were to be found using the usual method of integration it would be as follows

$$\int_0^1 g(x) = x^2 + 3\ dx$$
$$= \left[\frac{x^3}{3} + 3x\right]_0^1$$
$$= \frac{1^3}{3} + 3(1) - 0$$
$$= \frac{10}{3}\ \boldsymbol{or\ 3.33\ldots}$$

Another way in which the above value can be approximated is by adding the sum of the area of the two trapeziums. One of the aims of this investigation is to see if there is some kind of relation between the accuracy of the approximation and the number of trapeziums involved to find it.

Area of trapezium $=(a + b)\frac{h}{2}$ ………………….where a and b = parallel sides of trapezium
h = height of trapezium

Using the above formula the area of each trapezium can be found and then added together to find the approximation of the actual area. The height of each trapezium is found by subtracting the lower limit, which is zero, from the upper limit, which is one, and dividing it by the number of trapeziums that is two in this case. Hence, h will be 0.5.To find the parallel sides of the trapezium the x values of each trapezium are inserted into the function of $g(x) = x^2 + 3$. Using the trapeziums the area under the curve could be calculated as follows:-

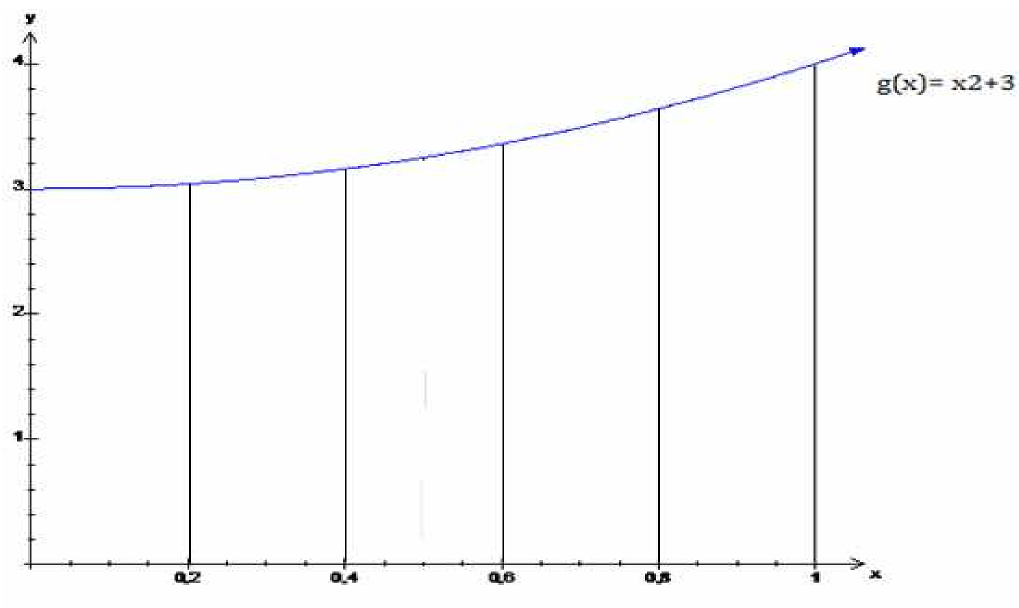
Area = area of first trapezium + area of second trapezium

$$= \frac{(g(0) + g(0.5))\,0.5}{2} + \frac{(g(0.5) + g(1))\,0.5}{2}$$

$$= \frac{(3+ 3.25)0.5}{2} + \frac{(3.25 + 4)0.5}{2}$$

$$= \frac{27}{8} \quad \textit{or } 3.375$$

It can be seen from the above calculation that a very close value was found to that of the actual value found using integration. The graph below shows the same curve with interval of x=0 and x=1 divided into 5 trapeziums.



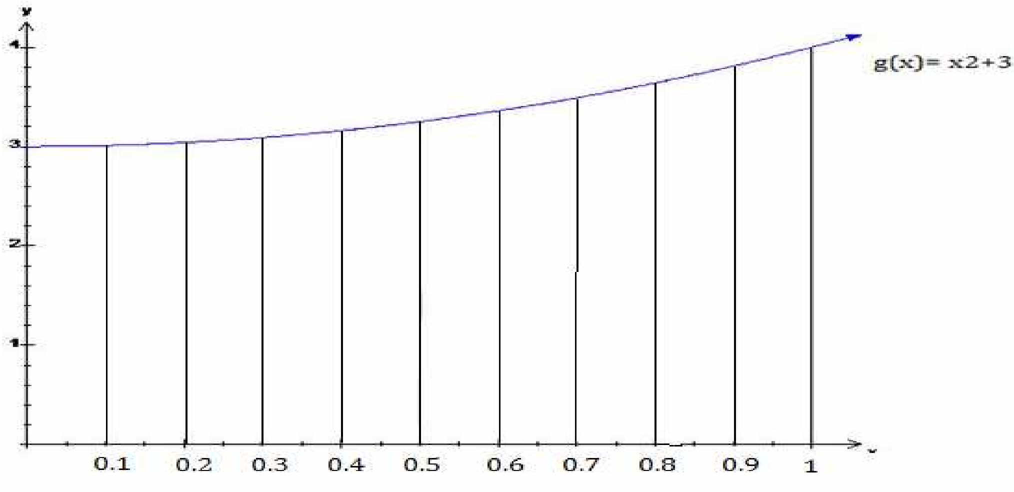
Height of each trapezium will be, h= $\frac{1-0}{5} = 0.2$. the area under the curve can be approximated as follws with the use of 5 trapeziums.

Area $= \frac{(g(0) + g(0.2)\ 0.2)}{2} + \frac{(g(0.2) + g(0.4)\ 0.2)}{2} + \frac{(g(0.4) + g(0.6)\ 0.2)}{2} + \frac{(g(0.6) + g(0.8)\ 0.2)}{2} + \frac{(g(0.8) + g(1)\ 0.2)}{2}$

$= \left((3+ 3.04)+ (3.04+ 3.16)+(3.16+ 3.36) +(3.36+ 3.64)+ (3.64+ 4)\right)\frac{0.2}{2}$ ……….factorizing

$= \frac{167}{50}$ ***or 3.34***

The area under the curve approximated using five trapeziums was closer to the actual value than when using only two trapeziums. In the following graph, the number of trapeziums used to divide the area under the curve was increased to 10 trapeziums.



Height of each trapezium will be, h= $\frac{1-0}{10} = 0.2$. Approximating the area under the graph using 10 trapeziums can be calculated as follows:-

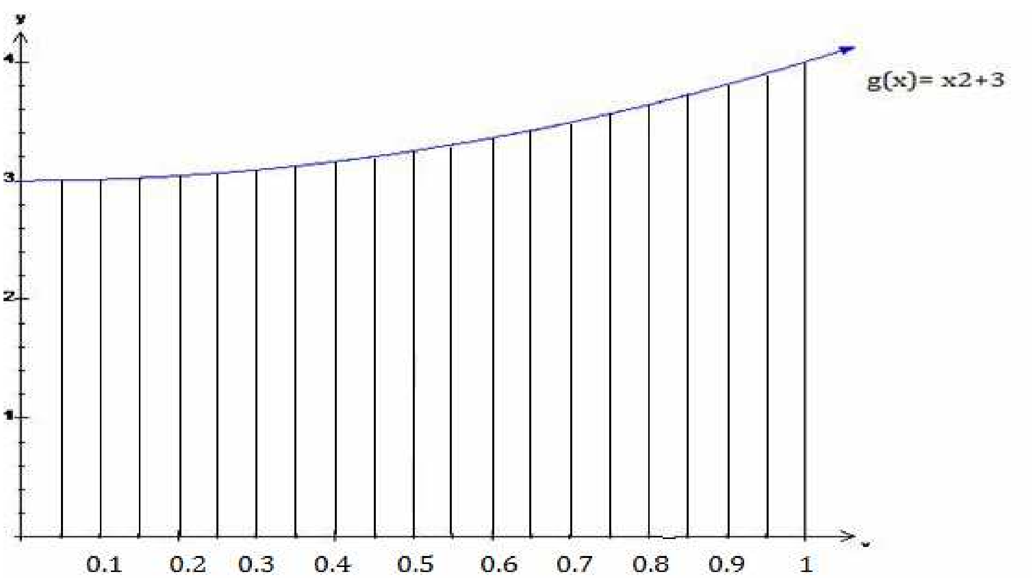
Area = area of first trapezium + area of second trapezium+area of third trapezium+……………+ area of tenth trapezium.

= $\frac{h}{2}$(g(x) values of1st trap. + g(x) values of 2nd trape.+g(x) values of 3rd trape…………+ g(x)values of 10th trapezium) ……………………………………………………………***factorizied***

= $\frac{0.1}{2}$((g(0)+g(0.1))+ (g(0.1)+g(0.2))+ (g(0.2)+g(0.3))+…………+(g(0.9)+g(1)))

= $\frac{0.1}{2}$((3+3.01)+(3.01+3.04)+(3.04+3.09)+……………..+(3.81+4))

= $\frac{667}{200}$ ***or 3.335***



Even further to see the effect that the number of trapezium has on the approximation of the area, the above graph was divided into 20 trapeziums. Height of each trapezium will be, h= $\frac{1-0}{20} = 0.05$ and the area was calculated as follows:-

Area = area of 1st trapezium + area of 2nd trapezium+area of 3rd trapezium+……………+ area of 20th trapezium.

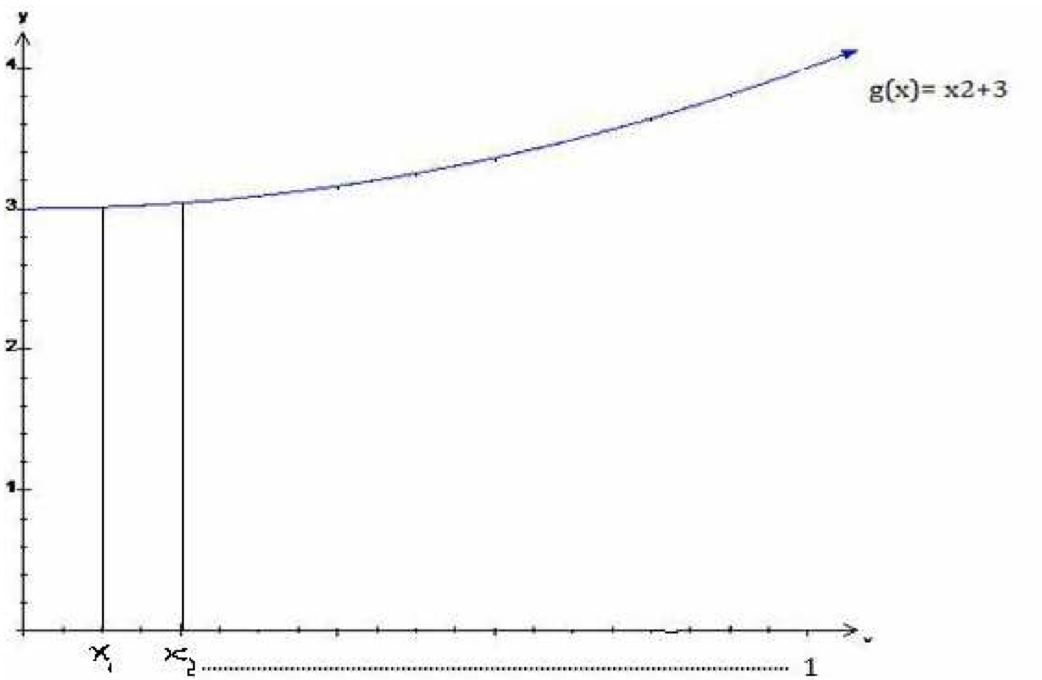
= $\frac{h}{2}$ (g(x) values of first trap. + g(x) values of second trape.+g(x) values of third trape…………+ g(x)values of twenthyth trapezium) ……………………………………***factorizied***

= $\frac{0.05}{2}$((g(0)+g(0.05)) + (g(0.05)+g(0.1))+ (g(0.1)+g(0.15))+…………+(g(0.95)+g(1)))

= $\frac{0.05}{2}$((3+3.0025)+(3.0025+3.01)+(3.01+3.0225)+……………..+(3.9025+4))

= $\frac{2667}{800}$ ***or*** **3.33375**

It can be concluded from the above calculations that by increasing the number of trapeziums under the area of the curve the approximation will get closer to the actual value. This is because increasing number of trapeziums will lead to smaller areas to be calculated by each trapezium thus minimizing the error formed. A general expression for the area under the curve of $g(x) = x^2 + 3$, from x=0 to x= 1, using n trapeziums can be deducted as follows:



Given n as the number of trapeziums, the interval (0, 1) is divided into equally sized n intervals. The height, h, will be h= (1-0)/n. also the x values of the trapeziums could be expressed as below

$0 = X_0, X_1, X_2, X_3 \ldots\ldots\ldots X_{n-1}, X_n = 1$

And from the above it can be derived that each x value is the sum of the first interval which is zero in this case and the value of height times r where r=0,1,2,3……..n.

$$X_r = a + rh$$

Then the area of each trapezium will be-

$\frac{1}{2}(g(x_0) + g(x_1)) + \frac{1}{2}(g(x_1) + g(x_2)) + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2}(g(x_{n-2}) + g(x_{n-1})) + \frac{1}{2}(g(x_{n-1}) + g(x_n))$

Adding all of the above together, we get the expression:

Area = $\boldsymbol{\frac{h}{2}}$ $\boldsymbol{\mathit{(g(x_0) + 2g(x_1) + 2g(x_2) + 2g(x_3) + \ldots\ldots\ldots\ldots\ldots + 2g(x_{n-1}) + g(x_n))}}$

The expression above can be developed further into a general statement where the area under any curve $y = f(X)$, from x=a to x=b can be calculated using n trapeziums. The height of will be h= (a-b)/n. the area under the curve will be divided into n interval where each interval expresses as:

$$X_r = a + rh \ldots\ldots \text{where } r= 0, 1, 2, 3 \ldots\ldots\ldots n.$$

Then the approximation area of each trapezium will be

$$h/2(f(xr - 1) + f(xr)$$

Summing the above expression for n number of trapeziums will give

$\int_b^a f(x)\,dx \cong {}^h/_2\ (f(x_0) + 2f(x_1) + 2f(x_2) + \ldots\ldots\ldots\ldots + 2f(x_{n-1}) + f(x_n))$

One might wonder where the number two appeared during the summation for all f(x) values except in those of $f(x_0)$ and $f(x_n)$. This is because every side, except $f(x_0)$ and $f(x_n)$, repeats itself because each side it is used as a parallel side for two consecutive trapeziums. The general statement can generally be written as:-

$$\int_b^a f(x)\,dx \cong \sum_{r=1}^{n} \frac{h}{2}\ (f(x_{r-1}) + 2f(x_r))$$

To test the above general statement the following three function with integral between ***a=1*** and ***b= 3***, where n= 8, are used. These functions are first integrated to find the area under the intervals so that it is easier to compare these results with those found using the general statement.

- $y = (\frac{x}{2})^{\frac{2}{3}}$ when integrated as follows gives the area under the curve from ***a=1*** and ***b= 3***

$$\int_1^3 y = (\frac{x}{2})^{\frac{2}{3}}\,dx$$
$$= \left[(\frac{x}{2})^{\frac{5}{3}}\,x\,\frac{3}{5}\right]_1^3$$
$$= (\frac{3}{2})^{\frac{5}{3}}\,x\,\frac{3}{5} - (\frac{1}{2})^{\frac{5}{3}}\,x\,\frac{3}{5}$$
$$= \boldsymbol{1.980}$$

Using eight trapeziums, the same area can be approximated using the general statement derived above:-

$$\int_b^a y = \left(\frac{x}{2}\right)^{\frac{2}{3}} dx \cong \sum_{r=1}^{8} \frac{h}{2}\ (f(x_{r-1}) + f(x_r))$$

$\boldsymbol{\mathit{from\ previous}}$ deductions h $= \frac{3-1}{8}$ which is $\frac{1}{4}$. it then becomes $\frac{1}{8}$ when divided by 2

In addition, the value of each y(x) can be found from

$$\mathbf{y(x_r) = a + rh \ldots. where\ r = 0, 1, 2 \ldots. n.}$$

$$= \frac{1}{8}(y(x_0) + 2y(x_1) + 2y(x_2) + \ldots\ldots\ldots\ldots + 2y(x_{n-1}) + y(x_n))$$

$$= \frac{1}{8}\left(y(1) + 2y\left(\frac{9}{8}\right) + 2y\left(\frac{10}{8}\right) + \ldots\ldots\ldots\ldots + 2y\left(\frac{15}{8}\right) + y(2)\right)$$

= ***1.835***

The approximated value here is less than that of the actual value. This particular function is raised to a certain power and by increasing the number of trapezium; a closer value can be obtained.

- $y = \frac{9x}{\sqrt{x^3+9}}$ when integrated as follows gives the area under the curve from ***a=1*** and ***b= 3***

$$\int_1^3 y = \frac{9x}{\sqrt{x^3+9}}\ dx$$

$$= \left[\frac{9x^2}{2} x2(x^3+9)^{\frac{1}{2}}\right]_1^3$$

$$= \left[9x^2 x(x^3+9)^{\frac{1}{2}}\right]_1^3$$

= **8.25**

Using eight trapeziums, the same area can be approximated using the general statement derived above:-

$$\int_b^a y = \frac{9x}{\sqrt{x^3+9}}\ dx \cong \sum_{r=1}^{8} \frac{h}{2}\ (f(x_{r-1}) + f(x_r))$$

$$\mathrm{h} = \frac{3-1}{8} \text{which is } \frac{1}{4}\text{. it then becomes } \frac{1}{8} \text{ when divided by 2}$$

$$= \frac{1}{8}(y(x_0) + 2y(x_1) + 2y(x_2) + \ldots\ldots\ldots\ldots + 2y(x_{n-1}) + y(x_n))$$

$$= \frac{1}{8}\left(y(1) + 2y\left(\frac{9}{8}\right) + 2y\left(\frac{10}{8}\right) + \ldots\ldots\ldots\ldots + 2y\left(\frac{15}{8}\right) + y(2)\right)$$

= ***8.162***

- $y = 4x^3 - 23x^2 + 40x - 18$ when integrated as follows gives the area under the curve from ***a=1*** and ***b= 3***

$$\int_1^3 y = 4x^3 - 23x^2 + 40x - 18\ dx$$

$$= \left[x^4 - \frac{23}{3}x^3 + 20x^2 - 18x\right]_1^3$$

$= \mathbf{4.666}$

Using eight trapeziums, the same area can be approximated using the general statement derived above:-

$$\int_b^a y = 4x^3 - 23x^2 + 40x - 18\ dx \cong \sum_{r=1}^{8} \frac{h}{2}\ (f(x_{r-1}) + f(x_r))$$

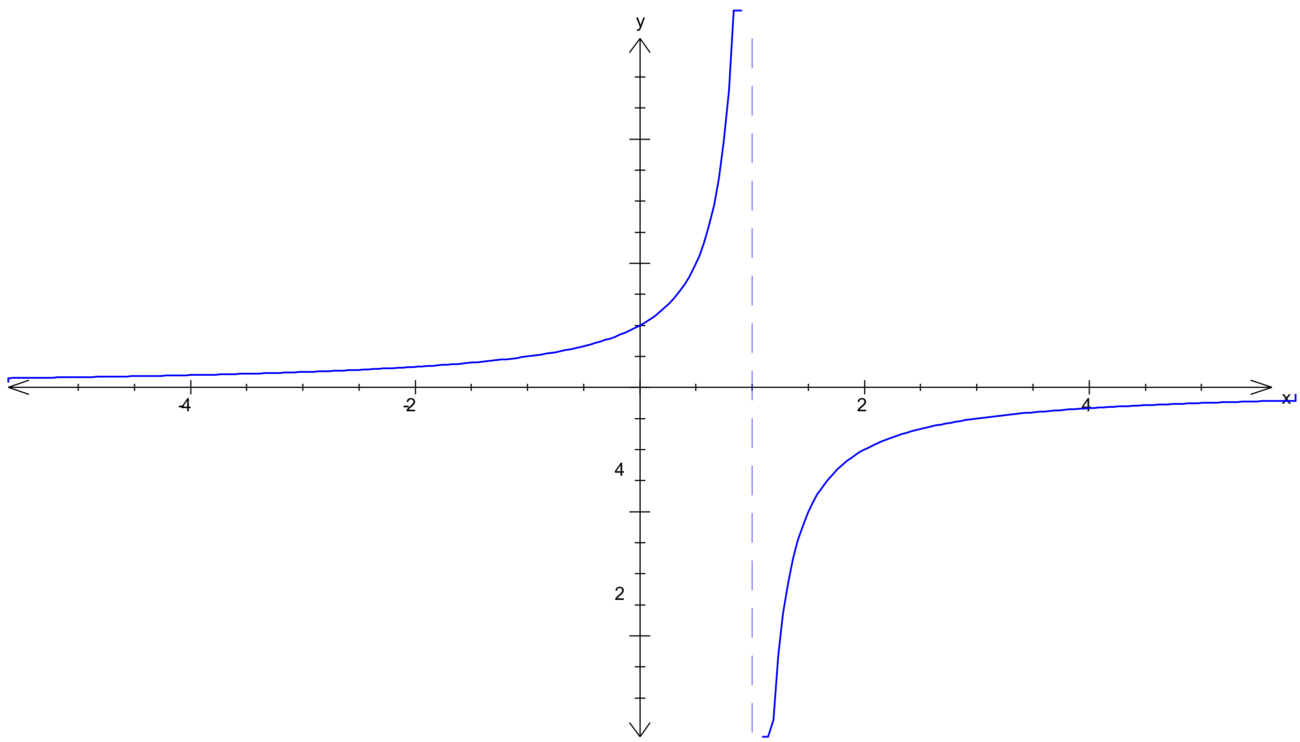
$$\mathrm{h} = \frac{3-1}{8} \text{which is } \frac{1}{4}\text{. it then becomes } \frac{1}{8} \text{ when divided by 2}$$

$$= \frac{1}{8}(y(x_0) + 2y(x_1) + 2y(x_2) + \ldots\ldots\ldots\ldots + 2y(x_{n-1}) + y(x_n))$$

$$= \frac{1}{8}\left(y(1) + 2y\left(\frac{9}{8}\right) + 2y\left(\frac{10}{8}\right) + \ldots\ldots\ldots\ldots + 2y\left(\frac{15}{8}\right) + y(2)\right)$$

=***4.6681***

One of the uses of this general statement in other types of functions is it enables to find the area under a graph where integration is not the ideal formula to use. Asymptote graphs are good example. The graph below the function of $f(X) = \frac{1}{1-x}$ where x can never be 1.



If the area under the curve of x=1 and x=2 is asked then instead of integration the trapezium rule can be used since x can be approximated to close values such as x= 0.0000001 and so on. This is also true for functions such as f(x) = tan x. The approximation can be as accurate as desired by just increasing the number of trapeziums under a certain curve. One of the limitations of this rule is it can only be applied to continuous functions of interval (a, b).