Mathematical Investigation

Investigate Systems of linear equations where the system constants have well known mathematical patterns.

Here is a 2x2 system of linear equations $\left.\begin{array}{l} x + 2y = 3 \\ 2x - y = -4 \end{array}\right\}$

It is easy to notice that if one starts from the coefficient of x, they can generate the coefficient of y by adding 1 to it and the constant by adding 1 to the coefficient of y. Another way to look at this is to start from the constant and add -1 to it you can generate the y coefficient and add -1 to this to get the x coefficient. The same is true for the second equation except that instead of adding 1 to the coefficient of x, we add -3. Similarly, we can start from the constant and add 3 each time to get the y and x coefficients.

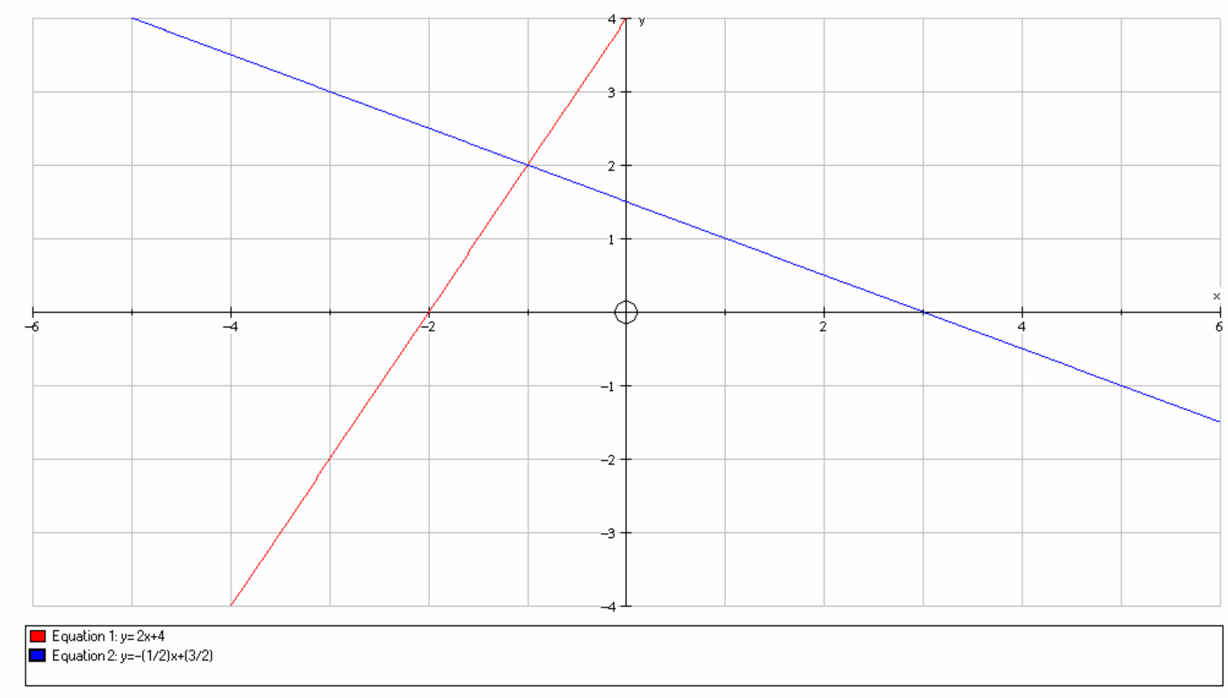
$$\left.\begin{array}{l} x + 2y = 3 \\ 2x - y = -4 \end{array}\right\} \quad \left.\begin{array}{l} x = -2y + 3 \\ 2(-2y + 3) - y = -4 \end{array}\right\} \quad \left.\begin{array}{l} \\ -4y + 6 - y = -4 \end{array}\right\} \quad \left.\begin{array}{l} \\ -5y = -10 \end{array}\right\} \quad \left.\begin{array}{l} x = -2(2) + 3 \\ y = 2 \end{array}\right\}$$

x= -1

y= 2

The solution x equals -1 and y equals 2 means that that point satisfies both equations. As we can see in the graph the two equations intersect at a point and that point is the same as the solution we found when we solved the problem algebraically. So the point that the two equations intersect is (-1, 2).

Graph



Here are some examples using the same pattern:

Example 1:

$x + 3y = 5$ $\quad$ $x = 5 - 3y$ $\quad$ $x = 5 - 3(2)$ $\quad$ **x= -1**

$5x - y = -7$ $\quad$ $5(5 - 3y) - y = -7$ $\quad$ $25 - 16y = -7$ $\quad$ $-16y = -32$ $\quad$ $y=2$ $\quad$ **y= 2**

Example 2:

$2x + 6y = 10$ $\quad$ $x = 5 - 3y$ $\quad$ $x = 5 - 3(2)$ $\quad$ **x= -1**

$7x + 3y = -1$ $\quad$ $7(5 - 3y) + 3y = -1$ $\quad$ $35 - 18y = -1$ $\quad$ $-18y = -36$ $\quad$ $y = 2$ $\quad$ **y= 2**

In the first example, the pattern in the coefficients for the first equation was adding two. In the second equation the pattern was subtracting six or adding negative six. The solution to the first example was the same as the solution in the original system of equations. In my second example, the pattern in the coefficients for the first equation was adding four. In the second equation the pattern was subtracting 4 or adding negative four. The solution to the second example was the same as the first example and the original system of equations. This seems to indicate that all 2x2 systems that follow this pattern will have the same solutions, but in order to prove that we need to solve a 2 by 2 system of equations in its most general form; which I will do next.

My conjecture for all equations that have similar patterns is that they all have exactly the same solutions: $x = -1$ and $y = 2$.

To prove or disprove this conjecture, I will consider the most general form. The most general form of the equation is:

$ax + (a + b)y = (a + 2b)$ $\quad \dfrac{a + 2b - (a + b)y}{} = x$

$cx + (c + d)\,y = (c + 2d)$ $\quad \dfrac{c(a + 2b - (a + b)y}{a} + (c + d)\,y = c + 2d$

$c(a + 2b - (a + b)y + ay(c + d) = a\,(c + 2d)$ $\quad ca + 2cb - (a + b)y + ay(c + d) = a(c + 2d)$

c(a+ b) – cay – cby + cay + day= a(c + 2d)   ca + 2cb – cby + day= ca + 2ad
2cb – cby + day= 2ad

2a – 2b

day – cby= 2ad – 2cb   y(ad – bc)= 2(ad – bc)   y= 2

ax= a + 2b – (a + b)(2)   ax= a + 2b –

y=2

X= a(1 – 2) / a   **x= -1**

y=2   **y= 2**

We can see that even in the most general form the answers are the same.

Originally I guessed that the 3x3 system of equations would work the same way as the 2x2 system. I would have thought that all the 3x3 systems would have a common solution. Unlike the 2x2 system of equations for which I used the method of substitution to solve, I used matrix algebra to solve the 3x3 system.

Let's consider a 3x3 system of linear equations:

x + 2y + 3z= 4

2x – y – 4z= -7

3x + 5y + 7z= 10

Also can be written as

When using algebra. → **AX**=**B**

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & -4 \\ 3 & 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ 10 \end{pmatrix}$$

Or

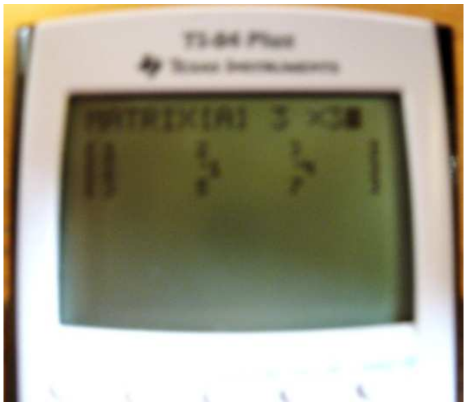
Where **A**, X and B are the corresponding matrices

By multiplying both sides by A^{-1} , the inverse matrix of **A**, we get the solution. In other words,

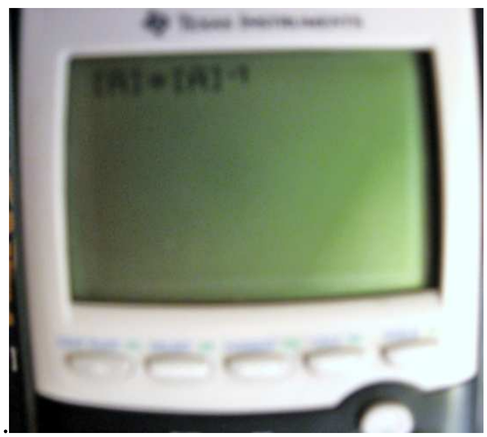
$A^{-1}AX = A^{-1}B \Leftrightarrow \quad IX = A^{-1}B \quad\quad X = A^{-1}B$ (where I is the 3x3 identity matrix)

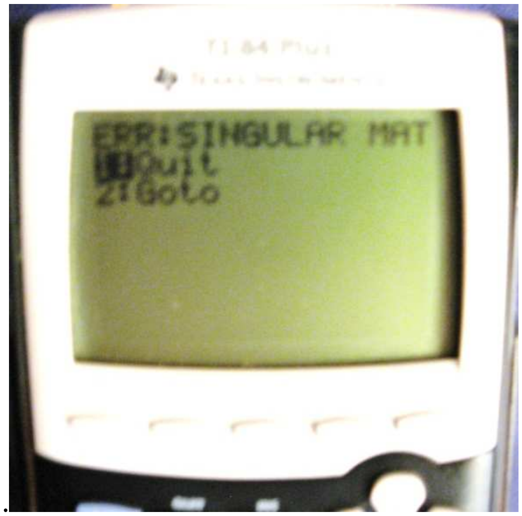
All we have to do is find the inverse matrix of A, which I did using a graphing calculator, as shown in picture 1. To my surprise I found that the inverse did not exist. I then tried another system of equations that followed the same pattern. Again the inverse did not exist.

Picture 1:

A. 

This image shows the 3x3 matrix with the coefficients of the variables that I inputted into my calculator to find its inverse.

B. 

C. 

This image shows that it doesn't exist.

Since the inverse of A did not exist in both of the above cases, I decided to find the determinant of matrix A and in both cases I got zero indicating that both 3x3 system of equations had no solutions.(As shown in picture 2).

At that time, I decided to look at the most general form of the 3x3 system of equations.

The most general form of the 3x3 system that follow the same pattern is of the following form:

$$
\left.\begin{aligned}
ax + (a + b)y + (a + 2b)z &= a + 3b \\
cx + (c + d)y + (c + 2d)z &= c + 3d \\
ex + (e + k)y + (e + 2h)z &= e + 3k
\end{aligned}\right\}
$$

Finding the determinant:

$$\begin{vmatrix} a & (a+b) & (a+2b) \\ c & (c+d) & (c+2d) \\ e & (e+k) & (e+2h) \end{vmatrix} = a\begin{vmatrix} (c+d) & (c+2d) \\ (e+k) & (e+2h) \end{vmatrix} - c\begin{vmatrix} (a+b) & (a+2b) \\ (e+k) & (e+2h) \end{vmatrix} + e\begin{vmatrix} (a+b) & (a+2b) \\ (c+d) & (c+2d) \end{vmatrix} =$$

$$= a((c+d)(e+2h) - (c+2d)(e+k)) - c((a+b)(e+2k) - (a+2b)(e+k)) + e((a+b)(c+2d) - (a+2b)(c+d)) =$$

$$= ack \cancel{- ade} + bce + \cancel{ack} + 2bck \cancel{- 2ack} \cancel{- 2bck} + ade - bce = 2ack - 2ack = \mathbf{0}$$

Even the determinant of the coefficients of the most general form turns out to be zero indicating that the system has no solution.

Here is a 2x2 system of linear equations: $\begin{cases} x + 2y = 4 \\ 5x - y = □5 \end{cases}$

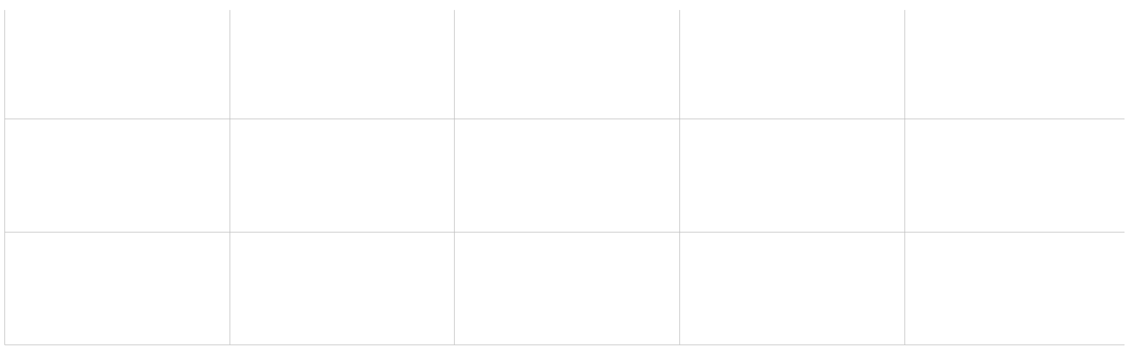
If you start from the coefficients of x one can see that we can generate the coefficient of y by multiplying by 2 and then we can find the constant by multiplying by 2 again. It is a similar pattern for the second equation, to get the coefficient of y you need to divide the x coefficient by 5 or you could multiple the x coefficient by □5 because it is the same thing. To get the constant you would divide the y coefficient by 5 or multiple the y coefficient by □5.

If you re-write the equations in the form of $y = ax + b$ you get:

$$\left.\begin{array}{l} x + 2y = 4 \\ 5x - y = □5 \end{array}\right\} \quad \begin{array}{l} y = 2 - .5x \\ y = 5x - □5 \end{array}$$

We can see that the product of a and b is negative one. In the first equation the product of 2 and a negative half is negative one. In the second equation the product of 5 and negative □5 is negative one.

This graph below represents a family of linear equations similar to the example shown above.



The generalized form of the two equations is:

$$\left.\begin{array}{l} x + ay = a^2 \\ bx - y = \frac{1}{b} \end{array}\right\} \quad \left.\begin{array}{l} y = -\frac{1}{a}x + a \\ y = bx - \frac{1}{b} \end{array}\right\}$$

As the value of a goes from → ∞ zero the slope increases from zero → + ∞, while the y-intercept is always negative and it decreases in absolute value. The x-intercept is equal to a^2 and decreases as the value of a decreases. When a becomes positive, the slopes of the lines are negative. Actually the slopes increase from - ∞ zero as the a goes from zero + ∞.

We observe a similar pattern in the other family of equations. We notice that we get the same line for $a = -\frac{1}{2}$ as for b = 2; for $a = -\frac{1}{5}$ as for b = 5; etc. IN other words we get the exact line whenever $b = -\frac{1}{a}$.

The y-intercept of the family of curves goes from - ∞ + ∞ while the x-intercept is always positive.

The solution of the 2x2 general system of equations is:

$$\left.\begin{array}{l} x + ay = a^2 \\ bx - y = \frac{1}{b} \end{array}\right\} \quad \left.\begin{array}{l} y = -\frac{1}{a}x + a \\ y = bx - \frac{1}{b} \end{array}\right\}$$

$$-\frac{1}{a}x + a = bx - \frac{1}{b} \Rightarrow \left(b + \frac{1}{a}\right)x = a + \frac{1}{b} \Rightarrow x = \frac{\frac{ab+1}{b}}{\frac{ab+1}{a}} \Rightarrow x = \frac{a}{b}$$

$$y = bx - \frac{1}{b} \Rightarrow y = b\left(\frac{a}{b}\right) - \frac{1}{b} \Rightarrow y = a - \frac{1}{b} \Rightarrow y = \frac{ab-1}{b}$$

Both equations are the same, when $b = -\frac{1}{a}$. This implies that $a = -\frac{1}{b}$.

Therefore, the first equation, y= - $\frac{1}{a}x$ + a, becomes y = bx – $\frac{1}{b}$.