

Stellar Numbers

Objective

The objective of this Internal Assessment is to investigate the geometric numbers which lead to special numbers. The simplest example of these is square numbers, but over the course of this investigation, both triangular numbers and stellar numbers will be looked at in greater depth. The investigation is conducted for the International Baccalaureate Mathematics Standard Level class and has the purpose of looking at this mathematical concept at a much higher level. The assessment can be broken down into two major sections.

1. Triangular Numbers

1. Triangular Numbers are investigated to begin this assessment in order to give a general background to the concept and idea of special numbers, as to set a background for the in-depth look at stellar numbers.

2. Stellar Numbers

1. Stellar Numbers are the investigation itself, and over the course of the problem, expressions for stellar numbers at different stages will be found, along with general statements for stellar numbers with different amounts of vertices, tests for the validity of these general statements, scopes and limitations of these general statements, and explanations of how arrivals at these general statements were made.

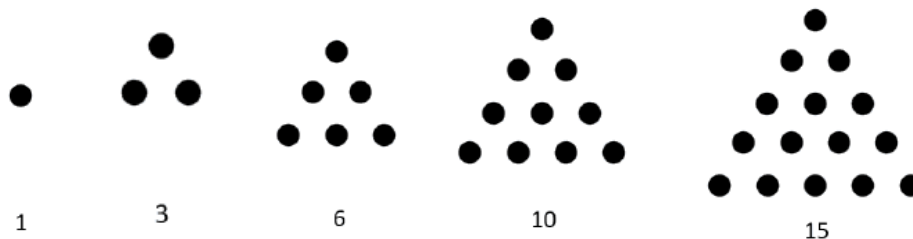
Materials

- The following materials were used over the course of this Internal Assessment
- Texas Instruments TI-83 Graphing Calculator
- Microsoft Word '98 and Microsoft Equation Editor

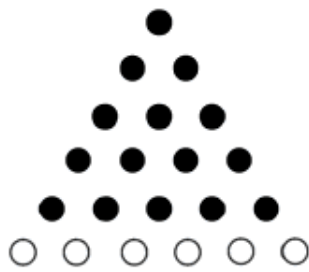
Question 1

1. Complete the triangular numbers sequence with three more terms.

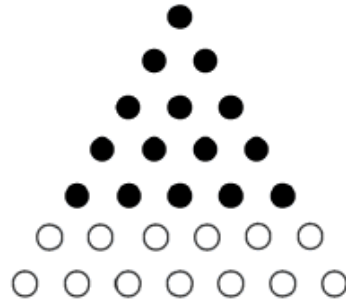
The given diagrams for the problem were S_1 through S_5 and are shown below.



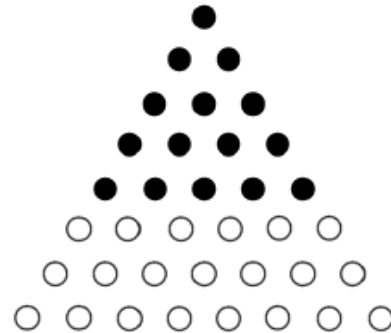
The next three terms needed to complete the diagram are S_6 through S_8 . These are illustrated in the pictures below. The white lines on bottom of each diagram are the dots that need to be added to go from one triangular number to the next.



21



28



36

Question 2

Find a general statement that represents the n^{th} triangular number in terms of n

When completing different trials in attempts to find the general term for Triangular Numbers, the following information was determined.

$$S_1=1$$

$$S_2=3$$

$$S_3=6$$

$$S_4=10$$

$$S_5=15$$

$$S_6=21$$

$$S_7=28$$

$$S_8=36$$

$$S_9=45$$

$$S_{10}=55$$

The first pattern that I noticed when looking at this information was that the order looks like this when broken down:

$$1, 1+2, 1+2+3, 1+2+3+4, 1+2+3+4+5, \dots, 1+2+3+4+5+6+7+8+9+10$$

The first attempt I made was saying the general term was simply $(n+1)$. After testing the validity of this statement by plugging in different numbers, it was very clear that this was incorrect.

The second attempt made was started off by determining that the first term could be set as equal to $\frac{N}{N+1}$. To continue on this trend, I set S_2 as equal to $\frac{N-1}{N+1}$ and S_3 as equal to $\frac{N-2}{N+1}$. Eventually, when I continued this trend all the way up until S_{10} , the equation that was determined is $N(N+1)$.

Essentially, the idea behind this attempt at the general statement can be backed by the statement that to get the sum of the first N positive integers, you have to divide by two.

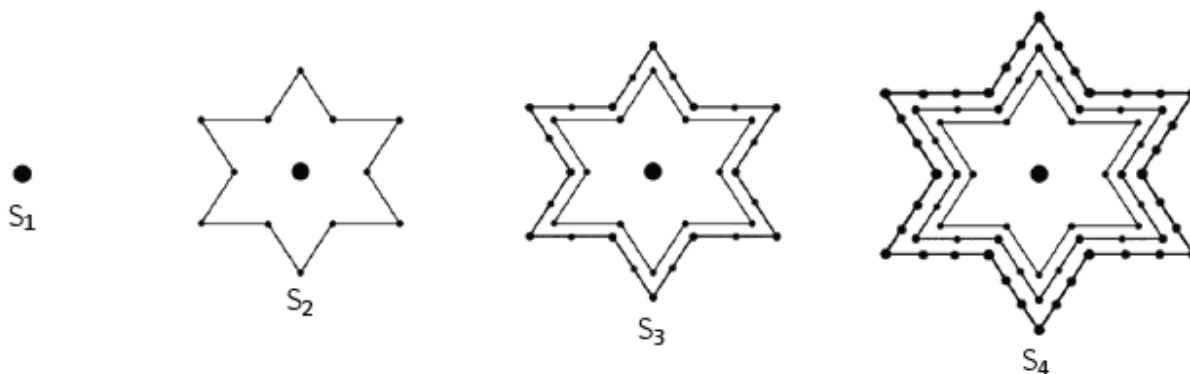
After this attempt, it was determined that the general statement was $\frac{N(N+1)}{2}$ which can also be equal to $\frac{1}{2}N(N+1)$. Meaning that:

The general statement for Triangular Numbers in Terms of N is $\frac{N(N+1)}{2}$.

Question 3

Consider stellar (star) shapes with p vertices, leading to p -stellar numbers. The first four representations for a star with six vertices are shown in the four stages S_1 - S_4 below. The 6-stellar number at each stage is the total number of dots in the diagram.

The diagrams that were given with the problem are displayed below.



The actual task of Question 3 is to find the number of dots (i.e. the stellar number) in each stage up to S_6 and to organize the data so that any patterns can be recognized and described.

After looking at the diagrams above the following information can be easily determined.

$$S_1=1$$

$$S_2=13$$

$$S_3=37$$

$$S_4=73$$

From this information, the pattern that I determined in that each term is $12n$ more than the previous term, in which n is equal to the term number of the previous term.

An example of this and the information determined from this can be seen below.

Ex. $S_2 = S_1 + 12n$ where $n=1$ because S_1 is the previous term.

Ex. $S_3 = S_2 + 12n$ where $n=2$ because S_2 is the previous term.

Term #	S_1	S_2	S_3	S_4	S_5	S_6	S_7
Stellar #	1	13	37	73	121	171	243

Question 4

Find an expression for the 6-stellar number at stage S_7

The expression which I used for the 6-stellar number at stage S_7 is carried down from the . previous information determined that each term is $12n$ more than the previous term, in which n is equal to the term number of the previous term.

According to this pattern, $S_7 = S_6 + 12n$ where $n=6$ because S_6 is the previous term.

While this is a recursive definition as suppose to a general term, it is an expression for the 6-stellar number at the stage S_7 .

Question 5

Find a general statement for the 6-stellar number at stage S_n is terms of n .

The first attempt I made at finding the general statement was one of the most basic ways I could think of. The equation in my first attempt was $12n+1$. When testing this statement, it worked for the first three terms, but ceased working when the pattern reached S_4 . The second attempt was the formula $12n-1$. I tried this because since $12n+1$ was close, it seemed logical. This attempt was not even close. My third and fourth attempts both shied further away from the correct answer. These attempts were $12(n+1)$ and $12(n-1)$. These formulas didn't work even for S_1 .

After these first four attempts, I examined the numbers with much closer detail and realized that the differences from term to term make a pattern. The first differences are 12, 24, 36, 48, and so on. The second differences, which can be found by taking the differences of the first difference is 12, 12, 12, 12, and so on. It is a fact in math, that when the second differences are constant, the equation has to be a quadratic. The general form for a quadratic is ax^2+bx+c .

Now that it has been determined that the equation is a quadratic, I set x equal to 1,2,3, etc. and set the equation as a whole equal to 1,13,37, etc. From there on out, I solved for a , b , and c respectively, and determined that the general formula for 6-stellar numbers at stage S_n in terms of n is $n(6(n-1))+1$, which simplified is equal to $6n^2-6n+1$.

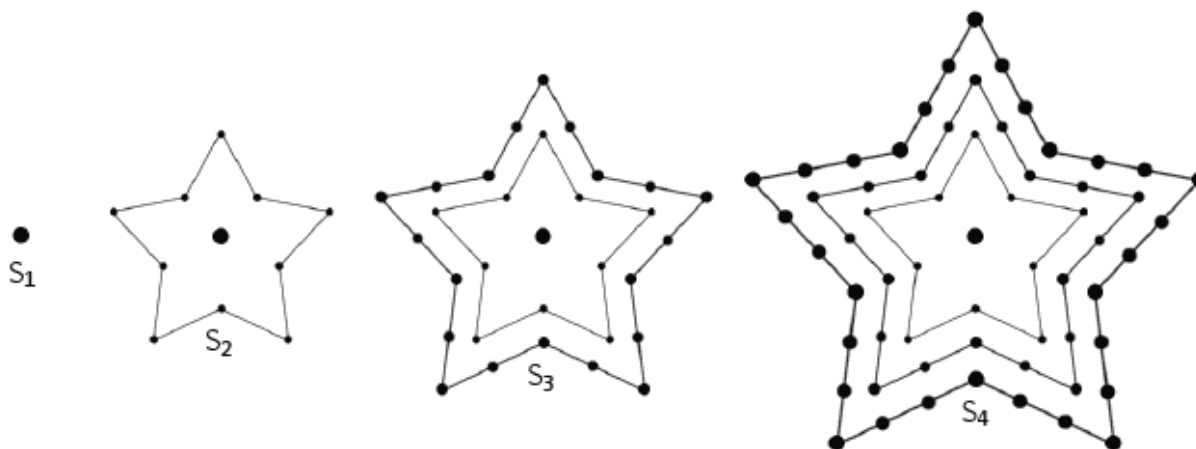
The general statement for 6-stellar numbers in terms of n is $6n^2-6n+1$

Question 6

Now repeat the steps above for other values of p

The values of p that I decided to find the general formulas for are 5 and 7, meaning that I found the general statements for 5-stellar numbers and 7-stellar numbers.

I started off this problem by diagramming out S_1 through S_4 (the first four 5-stellar numbers) as to have it as a visual means of reference to find patterns and other such important pieces of data.



I then looked at these drawings and added up the number of points in each star diagram to have a more organized piece of data to examine closely.

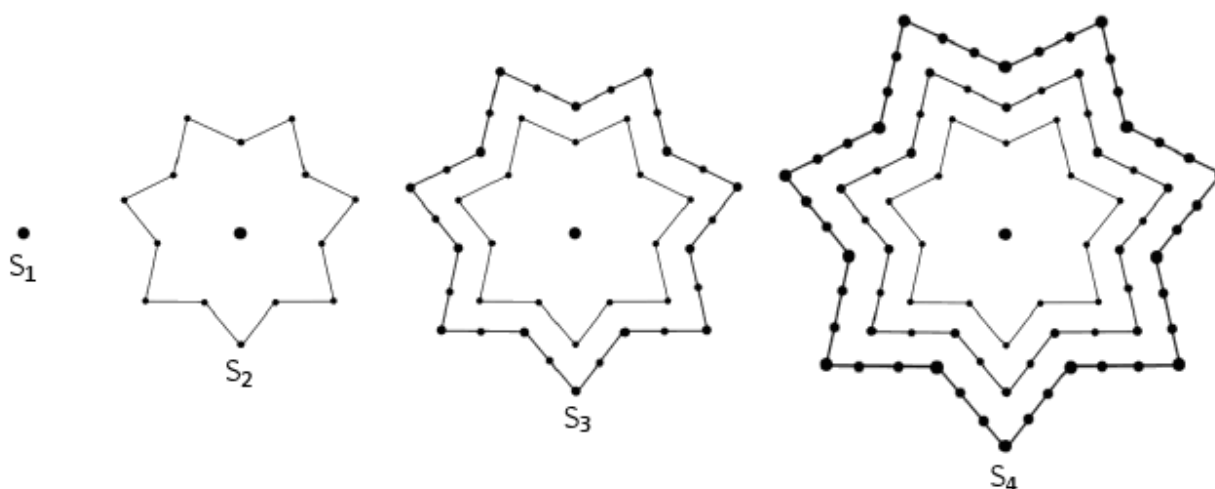
Term Number	S_1	S_2	S_3	S_4
Stellar Number	1	11	31	61

I realized that this information goes by close to the same pattern at the 6-stellar numbers do. While the first differences are 10, 20, 30, and so on with the same increasing increments of 10, the second differences are always 10, meaning that it is a quadratic. I noticed that the formula for 6-stellar numbers was $6n^2-6n+1$, and throughout that formula the number 6 is consistent, so I decided to try for 5-stellar numbers, that the formula would be $5n^2-5n+1$.

After plugging in numbers to this formula until up to S_8 and seeing that all of them worked, the validity of this formula had been tested and proven.

The general statement for 5-stellar numbers in terms of n is $5n^2 - 5n + 1$

For the 7-stellar numbers, I felt like I had already found somewhat of a general statement, but I diagramed out the first four 7-stellar numbers and made a chart just to have the data available and be able to confirm the trends that had been seen earlier in the investigation.



I then looked at these drawings and added up the number of points in each star diagram to have a more organized piece of data to examine closely.

Term Number	S_1	S_2	S_3	S_4
Stellar Number	1	15	43	85

Just like the two previous values of p that were examined in terms of stellar numbers, while the first differences of these stellar numbers go from 14 to 28 to 42 and on, but the second difference is consistently 14, meaning that it, just like the 6-stellar and 5-stellar numbers, is based upon a quadratic general statement. I went along with the pattern and tested the validity of my first guess for the general statement of 7-stellar numbers: $7n^2 - 7n + 1$. After plugging in numbers and testing it, I determined that the general statement for 7-stellar numbers is in fact $7n^2 - 7n + 1$.

The general statement for 7-stellar numbers in terms of n is $7n^2 - 7n + 1$

Question 7

Hence, produce the general statement, in terms of p and n , that generates the sequence of p -stellar numbers for any value of p at stage S_n

Going along with the pattern seen in the 5, 6, and 7-stellar numbers, I determined that in terms of p and n , the general statement for p -stellar numbers for any value of p at stage S_n is $pn^2 - pn + 1$.

The general statement for p -stellar numbers in terms of n is $pn^2 - pn + 1$

Question 8

Test the validity of this statement

To test the validity of this general statement, I plugged in the number 4 into all three of the pre-determined general statements for 5, 6, and 7-stellar numbers to make sure that they lined up correctly with the stellar numbers that had previously been determined.

5-Stellar Numbers

$$5n^2 - 5n + 1$$

Plug in 4

$$5(4)^2 - 5(4) + 1$$

$$80 - 20 + 1$$

$$61$$



Valid.

Term Number	S_1	S_2	S_3	S_4
Stellar Number	1	11	31	61

6-Stellar Numbers

$$6n^2 - 6n + 1$$

Plug in 4

$$6(4)^2 - 6(4) + 1$$

$$96 - 24 + 1$$

$$73$$



Valid.

Term #	S_1	S_2	S_3	S_4
Stellar #	1	13	37	73

7-Stellar Numbers

$$7n^2 - 7n + 1$$

Plug in 4

$$7(4)^2 - 7(4) + 1$$



Valid.

$$112 - 28 + 1$$

$$85$$

Term Number	S_1	S_2	S_3	S_4
Stellar Number	1	15	43	85

Question 9

Discuss the scope or limitations of the general statement

The scope of the general statement is that it applies to only stellar numbers, and not to any other polygonal shape. The main limitation of the general statement is that it applies only to whole integers and real numbers.

Question 10

Explain how you arrived at the general statement

I derived the general statement from the information I got from deciphering the individual statements for 5, 6, and 7-stellar numbers, finding the patterns in these, and going from there. The numerical coefficients are equal in each equation to the corresponding number of p that is representative of the stellar number.