Maths Internal Assessment

Type 1 – Mathematical Investigation

Mathematics Standard Level

Stellar Numbers

September 2011

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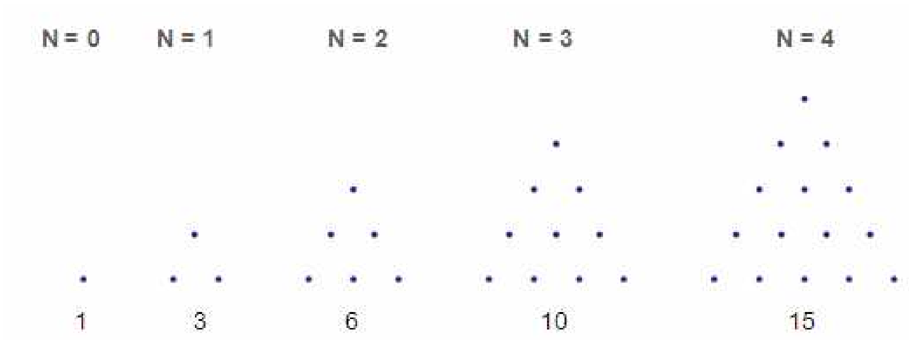
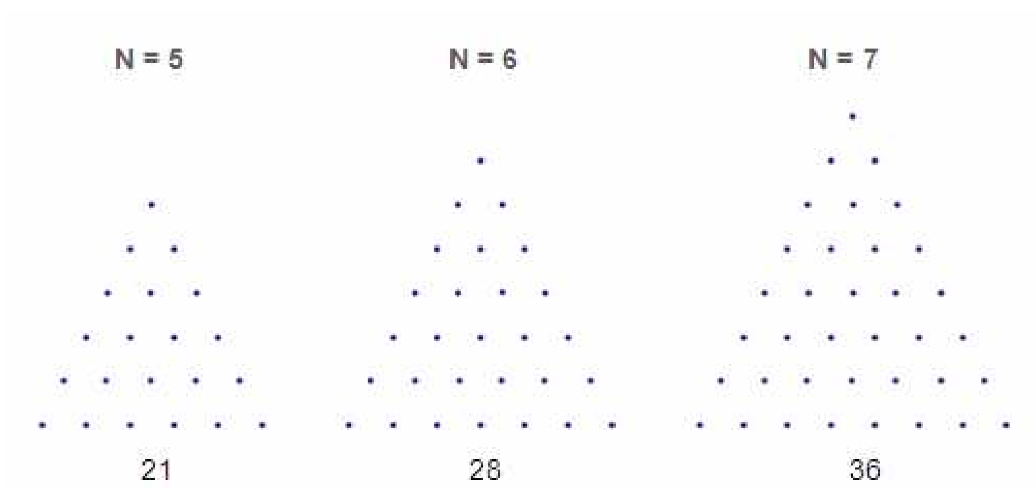
Aim – In this task geometric shapes which lead to special numbers will be considered.

For example the easiest of these are square numbers which can be represented by squares of side 1, 2, 3 and 4.

1. *The following diagrams show a triangular pattern of evenly spaced dots. The numbers of dots in each diagram are examples of triangular numbers. Complete the triangular numbers sequence with three more terms. Find a general statement that represents the n^{th} triangular number in terms of n.*[1]





Finding a general statement:

N*ow that three more terms have been drawn, the general statement can be found for the sequence;* 1, 3, 6, 10, 15, 21, 28, 36.

[1] All the questions in dark blue are from the Oporto British School Maths internal Assessment handout 2011

To do this, the constant difference in the sequence will need to be found, as shown below. This is needed to determine the type of equation (linear, quadratic, cubic etc...) [2]

Number of term (n)	0	1	2	3	4	5	6	7
Sequence	1	3	6	10	15	21	28	36
First Difference	2	3	4	5	6	7	8	
Second Difference	1	1	1	1	1	1		

The standard rules to find the general statement were researched and the following the method was put into practice for all the shapes in this portfolio.

If the second difference is a constant, the formula for the nth term contains n^2 as in a quadratic formula i.e. $ax^2 + bx + c$.

The value of 'a' is half the constant difference. In this example $a=\frac{1}{2}$

Hence, that the first part of the formula is $\frac{1}{2}n^2$. To find the rest of the formula, the differences between the values in the sequence and the values of $\frac{1}{2}n^2$ will need to be calculated.

Number of term (n)	0	1	2	3	4	5	6	7
Sequence	1	3	6	10	15	21	28	36
$\frac{1}{2}n^2$	0	0.5	2	4.5	8	12.5	18	24.5
Difference between Sequence and $\frac{1}{2}n^2$	1	2.5	4	5.5	7	8.5	10	11.5
Second difference	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	

[2] Steps followed by "The nth term of quadratics" at http://www.pearsonpublishing.co.uk/education/samples/S_492153.pdf

This second difference illustrates the value for ‘b’ which is equal to $\frac{3}{2}$.

However the value of ‘c’ has not yet been determined. It was calculated using an example:

Using n=2:

$\frac{1}{2}n^2 + \frac{3}{2}n + c = 6$
$\frac{1}{2}(2)^2 + \frac{3}{2}(2) + c = 6$
$5 + c=6$
$c=1$

From the example we can verify that ‘c’ must be equal to 1 to reach the desired figure.

To check that these are the correct values two more examples were used:

→ Using n=5
$\frac{1}{2}n^2 + \frac{3}{2}n + 1 = 21$
$\frac{1}{2}(5)^2 + \frac{3}{2}(5) + 1 = 21$

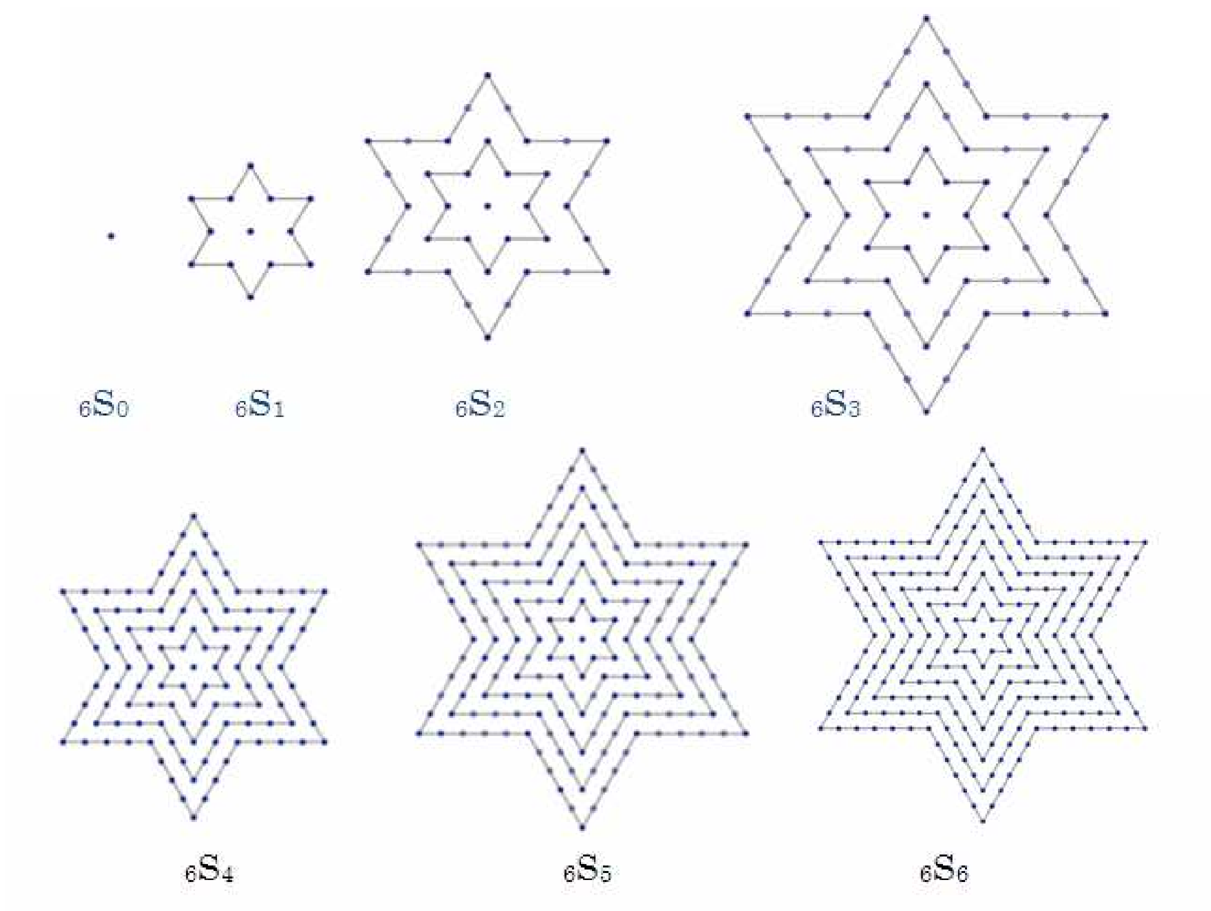
→ Using n=7:
$\frac{1}{2}n^2 + \frac{3}{2}n + 1 = 36$
$\frac{1}{2}(7)^2 + \frac{3}{2}(7) + 1 = 36$

With proof that the formula works it is concluded that the general statement for this pattern is:

$$\frac{1}{2}n^2 + \frac{3}{2}n + 1$$

2. *Consider stellar (star) shapes with p vertices, leading to p-stellar numbers. The first four representations for a star with six vertices are shown in the four stages S_1-S_4. The 6-stellar number at each stage is the total number of dots in the diagram.*

3. *Find the number of dots (i.e. the stellar number) in each stage up to S_6. Organize the data so that you can recognize and describe any patterns.*

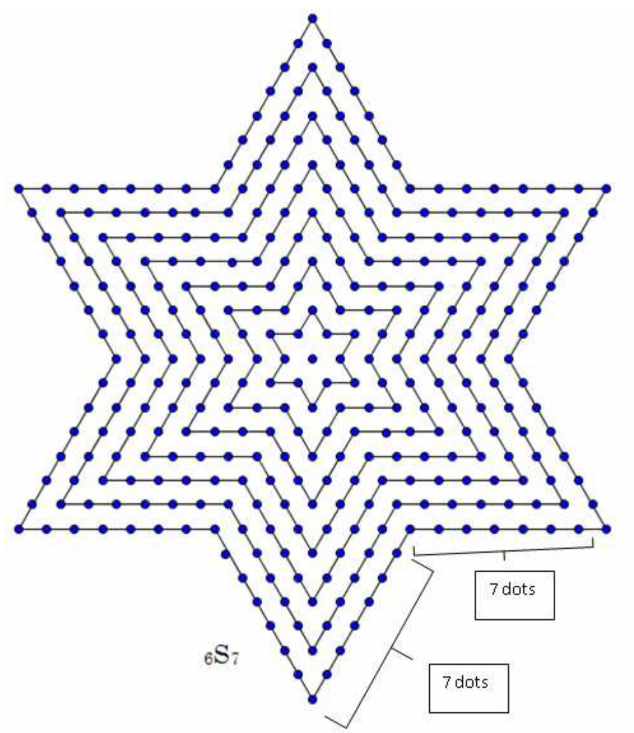


Stage Number	Number of Dots	*Notes and observations*
$_6S_0$	1	None
$_6S_1$	13	*Adding* 12 *to previous*
$_6S_2$	37	*Adding* 12x2 *to previous*
$_6S_3$	73	*Adding* 12x3 *to previous*
$_6S_4$	121	*Adding* 12x4 *to previous*
$_6S_5$	181	*Adding* 12x5 *to previous*
$_6S_6$	253	*Adding* 12x*6* *to previous*

4. *Find an expression for 6-stellar number at stage S_7*

By *observing the pattern in the table* I *can use it to calculate the number of dots at stage 7.* If I *add 12x6 to the previous number* (253) I *can calculate the number of dots in stage 7.*

$_6S_7 = 253 + (12x6)$
$= 337$



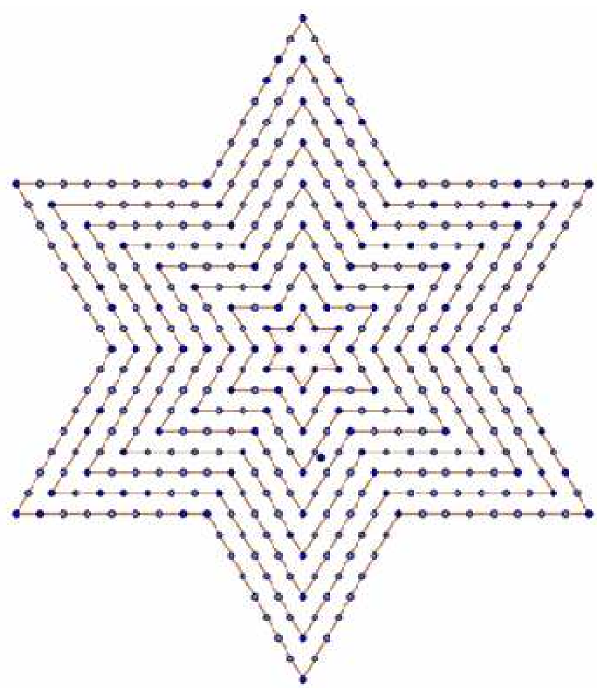
A *relationship was established in the sequence.* This *was determined by observing the sides of the shape and analyzing the table.* There *is a link between the number of sides and the number of dots on the side.* There *are* 12 *sides, each with 7 dots respectively.* In *fact, the number of dots in the next stage is equal to the number of dots in the previous stage, plus the multiplication of* 12 *and the term number.*

I.e.: $_6S_n = {_6S_{n-1}} + 12n$

To *prove my equation* I *will use an existing example from above and then prove it with one stage further.*

For Stage 3:
$_6S_n = {}_6S_{n-1} + 12n$
$_6S_{4=}\ {}_6S_3 + (12x3)$
$= 37 + 36$
$=73$

For stage 8:
$_6S_n = {}_6S_{n-1} + 12n$
$_6S_8 = {}_6S_7 + (12x7)$
$=337 + 84$
$=433$



$_6S_8$

With this formula proven I can reach a statement for stage 7:

$_6S_7 = {}_6S_{7\text{-}1} + 12(7)$

5. *Find a general statement for the 6-stellar at stage S_n in terms of n*

Finding a general statement:

Now that I have more terms and now that they have been drawn I can now find a general statement for the sequence, using the previous methods.

Again I will need to find the constant difference in the sequence in order to establish the type of equation.

$_pS_n$	$_6S_0$	$_6S_1$	$_6S_2$	$_6S_3$	$_6S_4$	$_6S_5$	$_6S_6$
Sequence	1	13	37	73	121	181	253
First Difference	12	24	36	48	60	72	
Second Difference		12	12	12	12	12	

Again the second difference is the constant therefore the formula for the nth term contains n^2 as in the quadratic equation: $ax^2 + bx + c$

The value of 'a' is half the constant difference. In this example a= $\frac{12}{2}$ = 6

Now that I know that the first part of the formula is $6n^2$ I can proceed to find the values of 'b' and 'c' just as in the first step.

$_pS_n$	$_6S_0$	$_6S_1$	$_6S_2$	$_6S_3$	$_6S_4$	$_6S_5$	$_6S_6$
Sequence	1	13	37	73	121	181	253
$6n^2$	0	6	24	54	96	150	216
Difference between Sequence and $0.5n^2$	5	11	17	23	29	35	41
Second difference	6	6	6	6	6	6	

This second difference tells me the value for 'b' which is equal to 6.

Yet again we still have not determined what the value for 'c' is. So an example should help me find out

Using n=4:

$6n^2 + 6n + c = 121$
$6(4)^2 + 6(4) + c = 121$
$120 + c = 121$
$c = 1$

To check that these are the correct values I used two more examples:

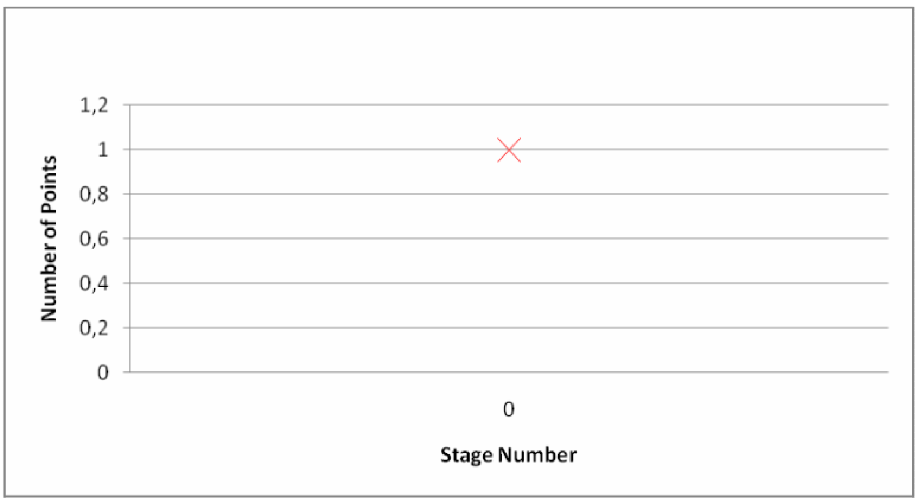
→ Using n=2
$6n^2 + 6n + c = 121$·
$6(2)^2 + 6(2) + 1 = 121$

→ Using n=6:
$6n^2 + 6n + 1 = 121$
$6(6)^2 + 6(6) + 1 = 121$

With enough proof that the general statement works I conclude that it is:

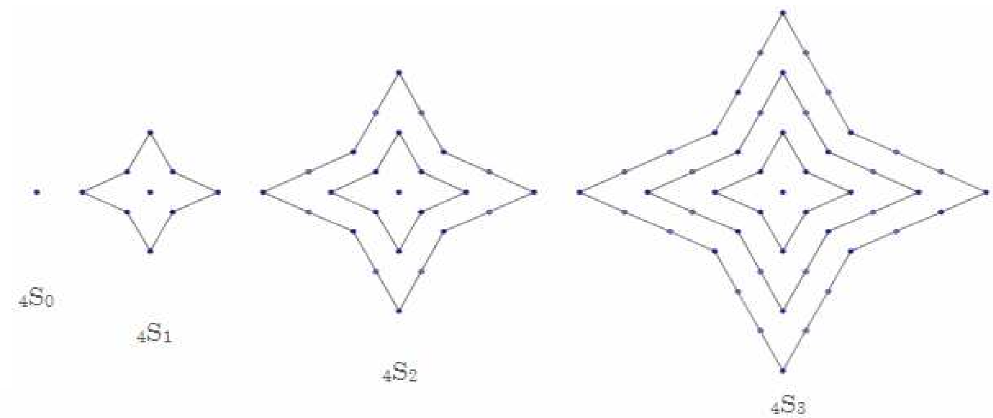
$6n^2 + 6n + 1$

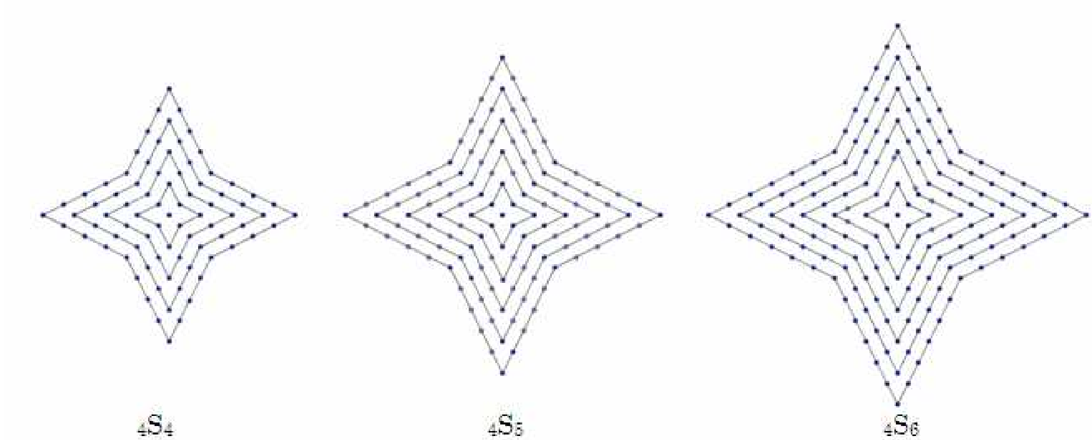
As this is a quadratic equation, a graph was plotted to demonstrate how it expanded



6. Now repeat the steps for other values of p

Firstly, a shape with a p value of 4 will be considered:





Stage Number	Number of Points	Notes and Observations
$_4S_0$	1	None
$_4S_1$	9	Adding 8 to previous
$_4S_2$	25	Adding 8x2 to previous
$_4S_3$	49	Adding 8x3 to previous
$_4S_4$	81	Adding 8x4 to previous
$_4S_5$	121	Adding 8x5 to previous
$_4S_6$	169	Adding 8x6 to previous
		The numbers are all the squares of odd integers. (1^2, 3^2, 5^2, 7^2, 9^2, 11^2 and 13^2)

Again, another pattern related to the pattern in step 4 has been detected. The relationship discovered is that the number of dots in the next stage is equal to the number of dots in the previous stage plus the term of the current stage multiplied by 8.

I.e.: ${}_4S_n = {}_4S_{n-1} + 8n$

To prove this equation an existing example from above will be used.

For Stage 3:
${}_4S_n = {}_4S_{n-1} + 8n$
${}_4S_3 = {}_4S_2 + (8x3)$
$= 25 + 24$
$= 49$

For Stage 5:
${}_4S_n = {}_4S_{n-1} + 8n$
${}_4S_5 = {}_8S_4 + (8x5)$
$= 81 + 40$
$=121$

Finding a general statement:

Now that I have more terms and now that they have been drawn I can now find a general statement for the sequence, using the previous methods.

${}_pS_n$	${}_4S_0$	${}_4S_1$	${}_4S_2$	${}_4S_3$	${}_4S_4$	${}_4S_5$	${}_4S_6$
Sequence	1	9	25	49	81	121	169
First Difference		8	16	24	32	40	48
Second Difference			8	8	8	8	8

Again the second difference becomes constant, therefore the formula for the nth term contains n^2 as in the quadratic equation: $ax^2 + bx + c$.

The value of 'a' is half the constant difference. In this example $a= \frac{8}{2} =4$

Now that I know that the first part of the formula is $4n^2$ I can proceed to find the values of 'b' and 'c'.

$_pS_n$	$_4S_0$	$_4S_1$	$_4S_2$	$_4S_3$	$_4S_4$	$_4S_5$	$_4S_6$
Sequence	1	9	25	49	81	121	169
$4n^2$	0	4	16	36	64	100	144
Difference between Sequence and n^2	1	5	9	13	17	21	25
Second difference	4	4	4	4	4	4	

This second difference tells me the value for 'b' which is equal to 4. To find the value of 'c' I will use the previous methods:

Using n=2:

$4n^2 +4n + c = 25$
$4(2)^2 +4(2) + c = 25$
$24 + c = 25$
$c = 1$

To check that these are the correct values I used two more examples:

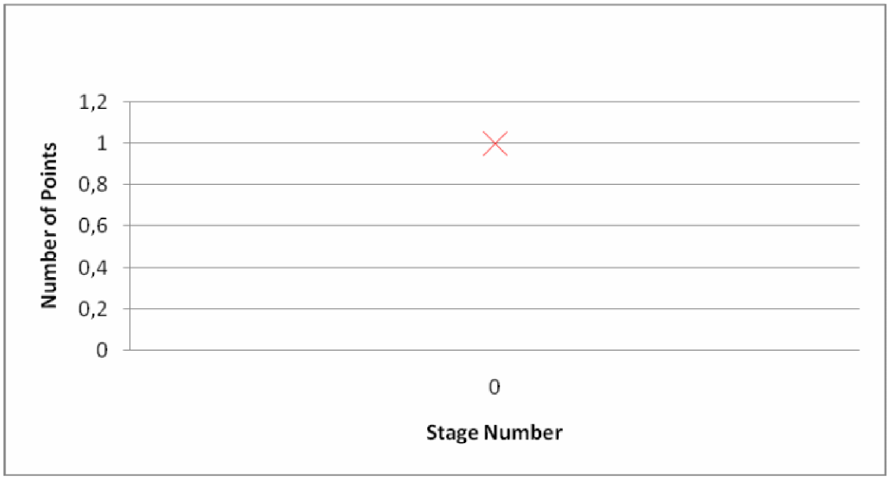
Using n=6
$4n^2 +4n + 1 = 169$
$4(6)^2 +4(6) + 1 = 169$

Using n=5:
$4n^2 +4n + 1 = 121$
$4(5)^2 +4(5) + 1 = 121$

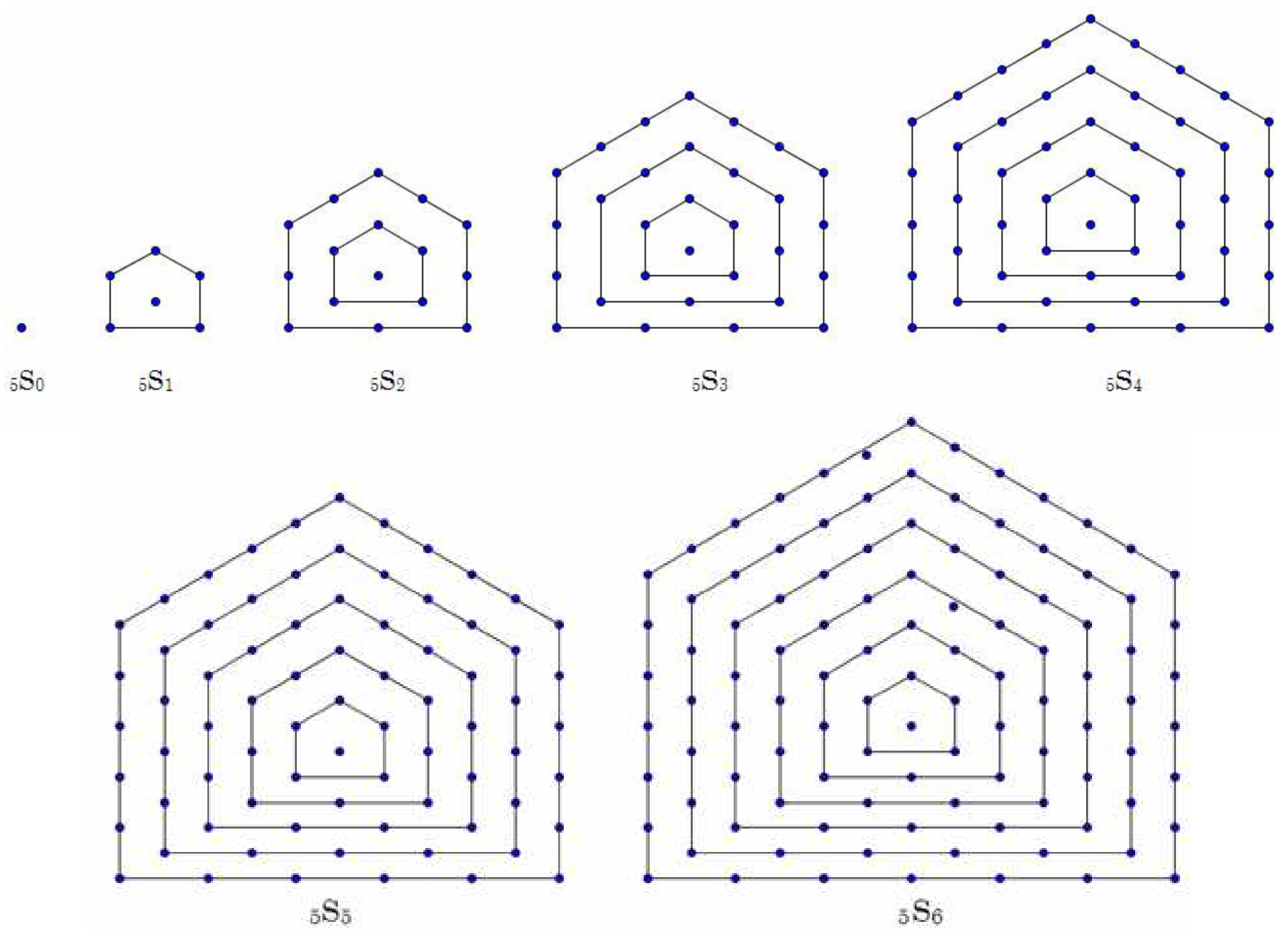
Therefore the general statement for this shape is:

$4n^2 +4n + 1$

As this is a quadratic equation, a graph was plotted to demonstrate how it expanded



This time, a shape with a p value of 5 (i.e. a 5-vertices shape), will be shown:



Stage Number	Number of Dots	Notes and observations
${}_5S_0$	1	None
${}_5S_1$	6	Adding 5 to previous
${}_5S_2$	16	Adding 5x2 to previous
${}_5S_3$	31	Adding 5x3 to previous
${}_5S_4$	51	Adding 5x4 to previous
${}_5S_5$	76	Adding 5x5 to previous
${}_5S_6$	106	Adding 5x6 to previous

Again, another 'side' association related. The relationship discovered is that the number of dots in the next stage is equal to the number of dots in the previous stage plus the term of the current stage multiplied by 5.

I.e.: ${}_5S_n = {}_5S_{n-1} + 5n$

For Stage 2:
${}_5S_n = {}_5S_{n-1} + 5n$
${}_5S_2 = {}_5S_1 + (5x2)$
$= 6 + 10$
$=16$

For stage 4:
${}_5S_n = {}_5S_{n-1} + 5n$
${}_5S_4 = {}_5S_3 + (5x4)$
$=31 + 20$
$=51$

Finding a general statement:

Now that I have more terms and now that they have been drawn I can now find a general statement for the sequence, using the previous methods.

${}_pS_n$	${}_5S_0$	${}_5S_1$	${}_5S_2$	${}_5S_3$	${}_5S_4$	${}_5S_5$	${}_5S_6$
Sequence	1	6	16	31	51	76	106
First Difference	5	10	15	20	25	30	
Second Difference	5	5	5	5	5		

Again the second difference is the constant therefore the formula for the nth term contains n^2 as in the quadratic equation: $ax^2 + bx + c$

The value of 'a' is half the constant difference. In this example a= $\frac{5}{2}$

Now that I know that the first part of the formula is $\frac{5}{2}n^2$ I can proceed to find the values of 'b' and 'c'.

${}_pS_n$	${}_5S_0$	${}_5S_1$	${}_5S_2$	${}_5S_3$	${}_5S_4$	${}_5S_5$	${}_5S_6$
Sequence	1	6	16	31	51	76	106
$\frac{5}{2}n^2$	0	2.5	10	22.5	40	62.5	90
Difference between Sequence and $\frac{5}{2}n^2$	1	3.5	6	8.5	11	13.5	16
Second difference	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	

This second difference tells me the value for 'b' which is equal to $\frac{5}{2}$. To find the value of 'c' I will use the previous methods:

Using n=4:

$\frac{5}{2}n^2 + \frac{5}{2}n + c = 51$

$\frac{5}{2}(4)^2 + \frac{5}{2}(4) + c = 51$

$50 + c = 51$

c = 1

To check that these are the correct values, two more examples were used:

Using n=1:

$\frac{5}{2} n^2 + \frac{5}{2} n + 1 = 6$

$\frac{5}{2} (1)^2 + \frac{5}{2} (1) + 1 = 6$

Using n=6:

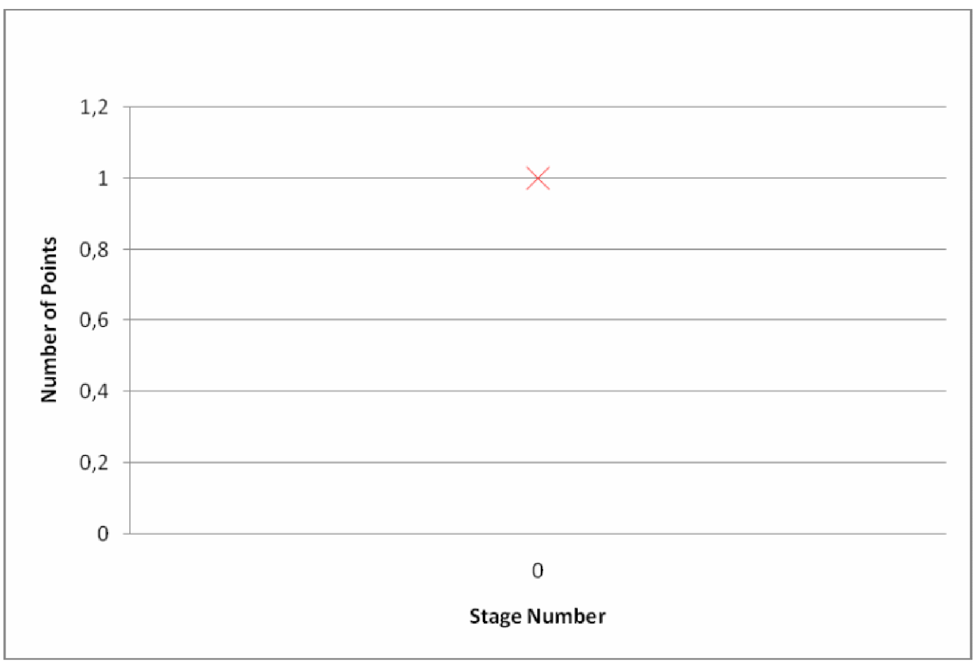
$\frac{5}{2} n^2 + \frac{5}{2} n + 1 = 106$

$\frac{5}{2} (6)^2 + \frac{5}{2} (6) + 1 = 106$

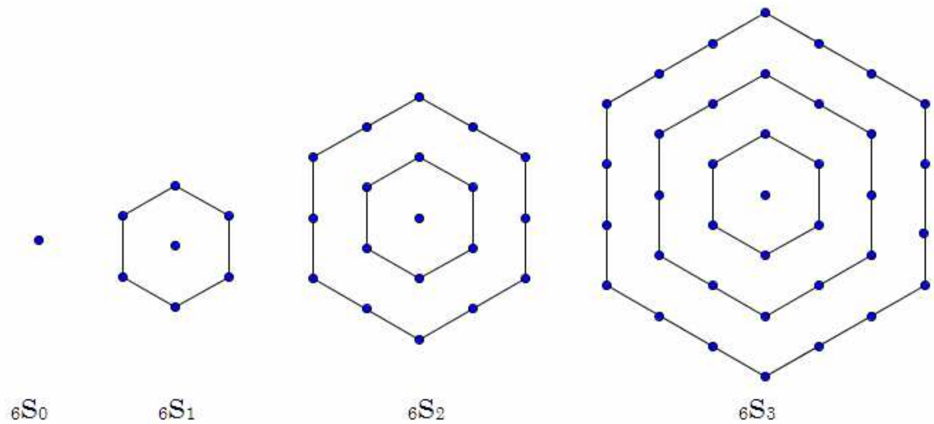
Therefore the general statement for this shape is:

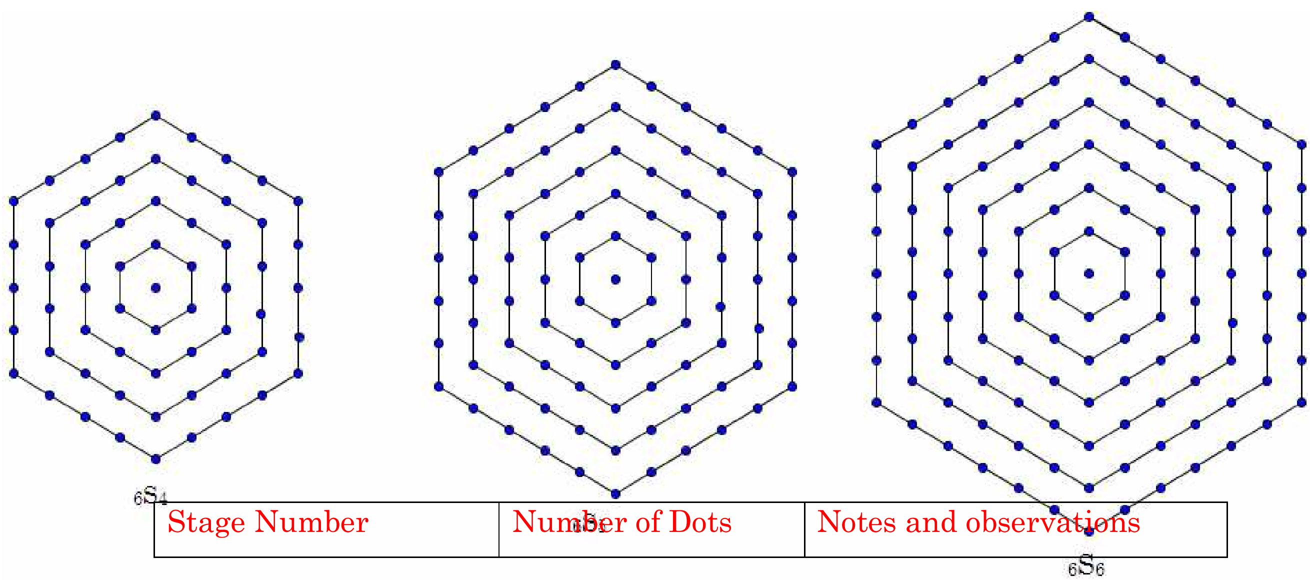
$\frac{5}{2} n^2 + \frac{5}{2} n + 1$

As this is a quadratic equation, a graph was plotted to demonstrate how it expanded



Now, a *p* value of 6 will be used based on a regular hexagon:





Stage Number	Number of Dots	Notes and observations

${}_6S_0$	1	None
${}_6S_1$	7	Adding 6 to previous
${}_6S_2$	19	Adding 6x2 to previous
${}_6S_3$	37	Adding 6x3 to previous
${}_6S_4$	61	Adding 6x4 to previous
${}_6S_5$	91	Adding 6x5 to previous
${}_6S_6$	127	Adding 6x6 to previous

Again, another pattern related to the pattern in other shapes has been detected. The relationship discovered is that the number of dots in the next stage is equal to the number of dots in the previo us stage plus the term of the current stage multiplied by 6.

I.e.: ${}_6S_n = {}_6S_{n-1} + 6n$

To prove this equation an existing example from above will be used.
For Stage 2:
${}_6S_n = {}_6S_{n-1} + 6n$
${}_6S_2 = {}_6S_1 + (6x2)$
$= 7 + 12$
$=19$

For Stage 4:
${}_6S_n = {}_6S_{n-1} + 6n$
${}_6S_4 = {}_6S_3 + (6x4)$
$= 37 + 24$
$=61$

Finding a general statement:

Now that I have more terms and now that they have been drawn I can now find a general statement for the sequence, using the previous methods.

${}_pS_n$	${}_6S_0$	${}_6S_1$	${}_6S_2$	${}_6S_3$	${}_6S_4$	${}_6S_5$	${}_6S_6$
Sequence	1	7	19	37	61	91	127
First	6	12	18	24	30	36	

Difference	
Second Difference	6 6 6 6 6

Again the second difference is the constant therefore the formula for the nth term contains n^2 as in the quadratic equation: $ax^2 + bx + c$

The value of 'a' is half the constant difference. In this example a= $\frac{6}{2} = 3$

Now that I know that the first part of the formula is $3n^2$ I can proceed to find the values of 'b' and 'c'.

${}_pS_n$	${}_6S_0$	${}_6S_1$	${}_6S_2$	${}_6S_3$	${}_6S_4$	${}_6S_5$	${}_6S_6$
Sequence	1	7	19	37	61	91	127
$3n^2$	0	3	12	27	48	75	108
Difference between Sequence and n^2	1	4	7	10	13	16	19
Second difference	3 3 3 3 3 3						

This second difference tells me the value for 'b' which is equal to 3. To find the value of 'c' I will use the previous methods:

Using n=4

$3n^2 + 3n + c = 61$
$3(4)^2 + 3(4) + c = 61$
$60 + c = 61$
$c = 1$

To check that these are the correct values, two more examples were used:

Using n=6:

$3n^2 + 3n + 1 = 127$
$3(6)^2 + 3(6) + 1 = 127$

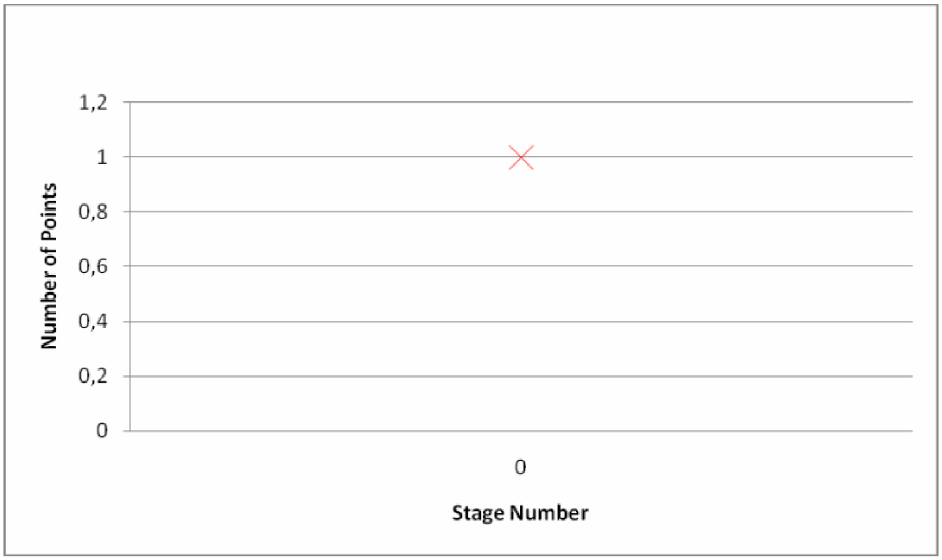
Using n=2:

$3n^2 + 3n + c = 19$
$3(2)^2 + 3(2) + 1 = 19$

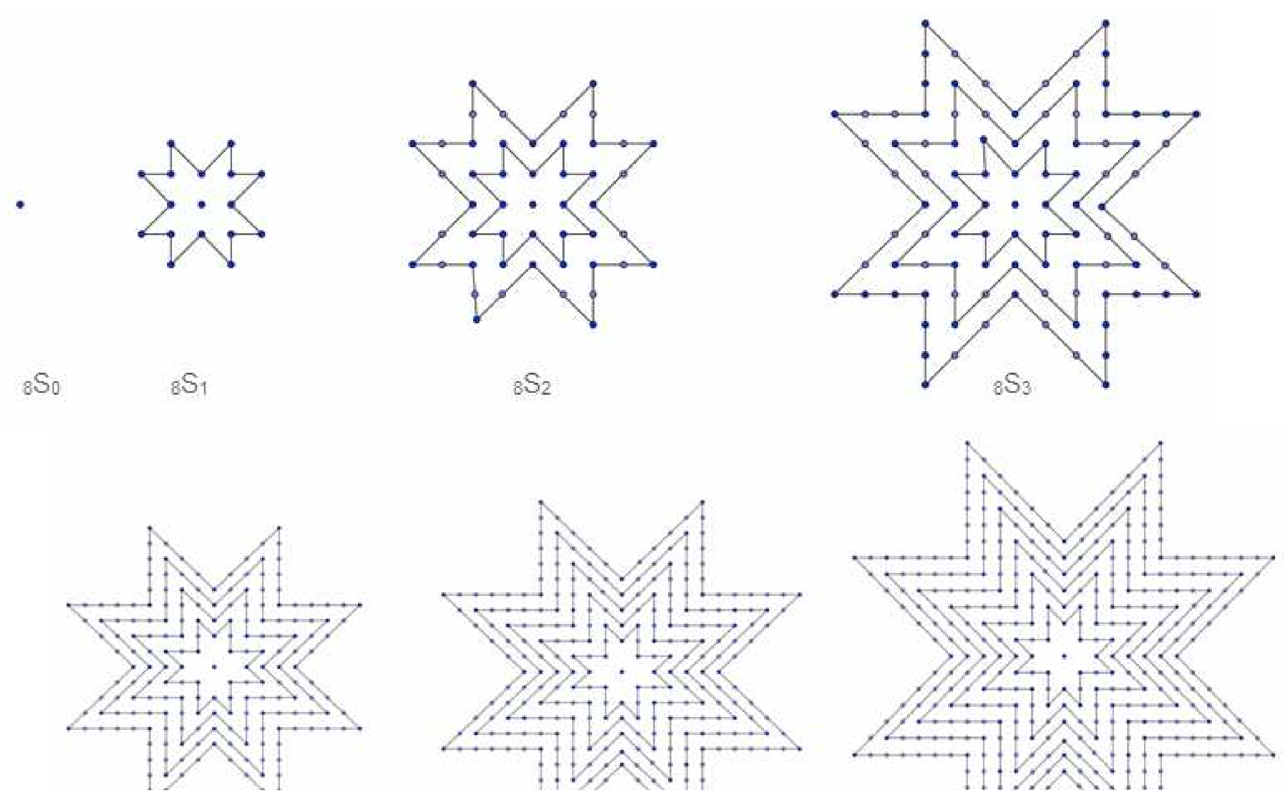
Therefore the general statement for this shape is:

$3n^2 + 3n + 1$

As this is a quadratic equation, a graph was plotted to demonstrate how it expanded:



Finally, a p value of 8 will be analysed:



Stage Number	Number of Dots	Notes and observations
$_8S_0$	1	None
$_8S_1$	17	Adding 16 to previous
$_8S_2$	49	Adding 16x2 to previous
$_8S_3$	97	Adding 16x3 to previous
$_8S_4$	161	Adding 16x4 to previous
$_8S_5$	241	Adding 16x5 to previous
$_8S_6$	337	Adding 16x6 to previous

Once again, another pattern was observed as the stage numbers developed. This time, the connection discovered is that the number of dots in the next stage is equal to the number of dots in the previous stage plus the term of the current stage multiplied by 16.

I.e.: $_8S_n = {_8S_{n-1}} + 16n$

For Stage 3:
$_8S_n = {_8S_{n-1}} + 16n$
$_8S_3 = {_6S_2} + (16x3)$
= 49 + 48
=97

For Stage 6:
$_8S_n = {_8S_{n-1}} + 16n$
$_8S_6 = {_8S_5} + (16x6)$
= 241 + 96
=337

Finding a general statement:

Now that I have more terms and now that they have been drawn I can now find a general statement for the sequence, using the previous methods.

$_pS_n$	$_8S_0$	$_8S_1$	$_8S_2$	$_8S_3$	$_8S_4$	$_8S_5$	$_8S_6$
Sequence	1	17	49	97	161	241	337
First Difference		16	32	48	64	80	9 6
Second Difference			16	16	16	16	1 6

Again the second difference is the constant therefore the formula for the nth term contains n^2 as in the quadratic equation: $ax^2 + bx + c$

The value of 'a' is half the constant difference. In this example a= $\frac{16}{2}$ = 8

Now that I know that the first part of the formula is $8n^2$ I can proceed to find the values of 'b' and 'c'.

$_pS_n$	$_8S_0$	$_8S_1$	$_8S_2$	$_8S_3$	$_8S_4$	$_8S_5$	$_8S_6$
Sequence	1	17	49	97	161	241	337
$8n^2$	0	8	32	72	128	200	288
Difference between Sequence and n^2	1	9	17	25	33	41	49
Second difference		8	8	8	8	8	8

This second difference tells me the value for 'b' which is equal to 8. To find the value of 'c' I will use the previous methods:

Using n=4

$8n^2 + 8n + c = 161$
$8(4)^2 + 8(4) + c = 161$
$160 + c = 161$
$c = 1$

To check that these are the correct values, two more examples were used:

Using n=2:

$8n^2 + 8n + 1 = 49$
$8(4)^2 + 8(4) + 1 = 49$

Using n=3:

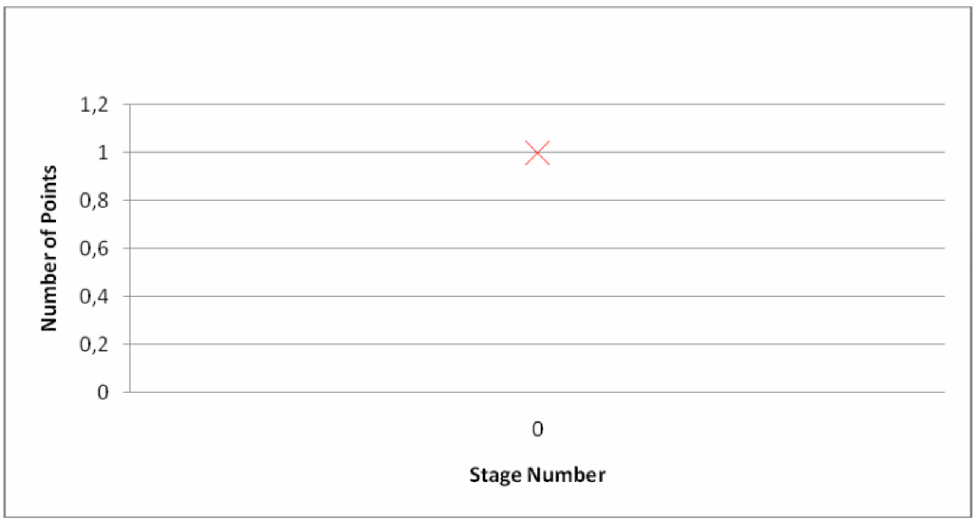
$8n^2 + 8n + 1 = 97$

$8(3)^2 + 8(3) + 1 = 97$

Therefore the general statement for this shape is:

$8n^2 + 8n + 1$

As this is a quadratic equation, a graph was plotted to demonstrate how it expanded:



7. *Hence, produce the general statement, in terms p and n, that generates the sequence of p-stellar numbers for any value of p at stage S_n.*

Polygon	${}_pS_n$	a	b	c
1	${}_3S_n$	$\frac{1}{2}$	$\frac{3}{2}$	1
2	${}_6S_n$	6	6	1
3	${}_4S_n$	4	4	1
4	${}_5S_n$	$\frac{5}{2}$	$\frac{5}{2}$	1
5	${}_6S_n$	3	3	1
6	${}_8S_n$	8	8	1

When analyzing all the examples given, it is clear that there is no regular general pattern for the values of a, b and c that relate to the values of p and n. For polygons 2, 3 and 6 it seems that the values of a and b are the equal to the values of p. However, for polygons 4 and 5 it appears the values of a and b are half the value of p.

Nevertheless, I noticed that some shapes (such as polygon 2, 3 and 6) can be considered to have double their number of vertices, if we include the points that go in i.e. the concave lines. This arouses the question: what are vertices? A vertex should be considered "the common endpoint of two or more rays or line segments (…) Vertex typically means a corner or a point where lines meet." [3] If we took this and followed it exactly then polygon 2 would have 12 vertices, instead of 6, polygon 3 would have 8, instead of 4, and polygon 6 would have 16 vertices, instead of 8. Using this these terms I can now find a general statement:

Polygon	$_pS_n$	a	b	c	General Statement
1	$_3S_n$	$\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}n^2 + \frac{3}{2}n + 1$
2	$_{12}S_n$	6	6	1	$6n^2 + 6n + 1$
3	$_8S_n$	4	4	1	$4n^2 + 4n + 1$
4	$_5S_n$	$\frac{5}{2}$	$\frac{5}{2}$	1	$\frac{5}{2}n^2 + \frac{5}{2}n + 1$
5	$_6S_n$	3	3	1	$3n^2 + 3n + 1$
6	$_{16}S_n$	8	8	1	$8n^2 + 8n + 1$

Green indicates the modified terms according to the definition of a vertex.

It is plainly indicated that the ones that changed were the stellar (star) shapes as they are the ones with concave and convex lines.

It can clearly be seen that in almost all the cases 'a' and 'b' are half the values of p. Therefore for these types of shapes I can conclude that a = $\frac{p}{2}$ and that b =$\frac{p}{2}$. All the way through, c is constantly equal to 1. All the general statements that were derived were a quadratic one, consequently, so is the

[3]. Definition from http://www.mathopenref.com/vertex.html

general rule, hence the general rule for the shapes is:

$$_{p}S_{n} = \frac{p}{2}n^{2} + \frac{p}{2}n + 1$$

Another extra statement that can be formulated relates to the 'side' equations found out.

Polygon	$_{p}S_{n}$	Side Equation
1	$_{3}S_{n}$	N/A
2	$_{12}S_{n}$	$_{12}S_{n} = {}_{12}S_{n-1} + 12n$
3	$_{8}S_{n}$	$_{8}S_{n} = {}_{8}S_{n-1} + 8n$
4	$_{5}S_{n}$	$_{5}S_{n} = {}_{5}S_{n-1} + 5n$
5	$_{6}S_{n}$	$_{6}S_{n} = {}_{6}S_{n-1} + 6n$
6	$_{16}S_{n}$	$_{16}S_{n} = {}_{16}S_{n-1} + 16n$

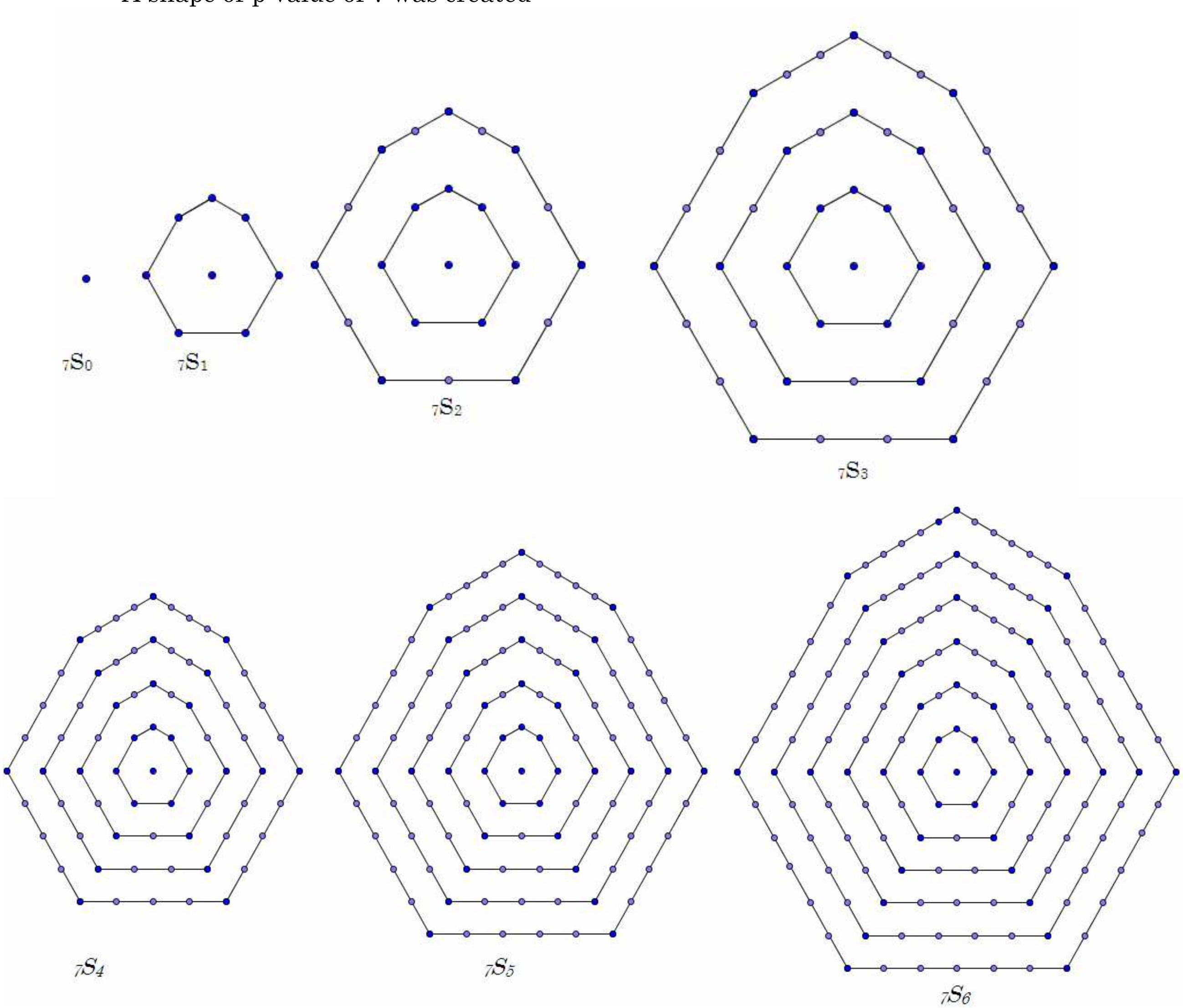
If we take the exact definition of a vertex as before we can consider another expression to find the nth term:

$$_{p}S_{n} = {}_{p}S_{n-1} + pn$$

Polygon 1 does not fit into any of the rules and this will be discussed in step 9.

8. Test the validity of the statement.

Let's firstly consider this general rule above and confirm it.
A shape of p value of 7 was created:



Once again, another pattern was observed as the stage numbers developed. This time, the link that was found was that the number of dots in the next stage is equal to the number of dots in the previous stage plus the term of the current stage multiplied by 7

I.e.: ${}_7S_n = {}_7S_{n-1} + 7n$

For Stage 5:
${}_7S_n = {}_7S_{n-1} + 7n$
${}_7S_5 = {}_7S_4 + (7x5)$
$= 71 + 35$
$=106$

For Stage 6:
${}_7S_n = {}_7S_{n-1} + 7n$
${}_7S_4 = {}_7S_5 + (7x6)$
$= 106 + 42$
$=148$

<u>Finding a general statement:</u>

Now that I have more terms and now that they have been drawn I can now find a general statement for the sequence, using the previous methods.

${}_pS_n$	${}_7S_0$	${}_7S_1$	${}_7S_2$	${}_7S_3$	${}_7S_4$	${}_7S_5$	${}_7S_6$
Sequence	1	8	22	43	71	106	148
First Difference	7	14	21	28	35	42	
Second Difference		7	7	7	7	7	

Stage Number	Number of Dots	Notes and observations
${}_7S_0$	1	None
${}_7S_1$	8	Adding 7 to previous
${}_7S_2$	22	Adding 7x2 to previous
${}_7S_3$	43	Adding 7x3 to previous
${}_7S_4$	71	Adding 7x4 to previous
${}_7S_5$	106	Adding 7x5 to previous
${}_7S_6$	148	Adding 7x6 to previous

Again the second difference is the constant therefore the formula for the nth term contains n^2 as in the quadratic equation: $ax^2 + bx + c$

The value of 'a' is half the constant difference. In this example a= $\frac{7}{2}$

Now that I know that the first part of the formula is $\frac{7}{2}n^2$ I can proceed to find the values of 'b' and 'c'.

$_pS_n$	$_7S_0$	$_7S_1$	$_7S_2$	$_7S_3$	$_7S_4$	$_7S_5$	$_7S_6$
Sequence	1	8	22	43	71	106	148
$\frac{7}{2}n^2$	0	3.5	14	31.5	56	87.5	126
Difference between Sequence and $\frac{5}{2}n^2$	1	4.5	8	11.5	15	18.5	22
Second difference	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{7}{2}$	

This second difference tells me the value for 'b' which is equal to $\frac{7}{2}$. To find the value of 'c' I will use the previous methods:

Using n=4:

$\frac{7}{2}n^2 + \frac{7}{2}n + c = 71$

$\frac{7}{2}(4)^2 + \frac{7}{2}(4) + c = 71$

$70 + c = 71$

$c = 1$

To check that these are the correct value, two more examples were used:

Using n=1:

$\frac{7}{2}n^2 + \frac{7}{2}n + 1 = 8$

$\frac{7}{2}(1)^2 + \frac{7}{2}(1) + c = 8$

Using n=6:

$\frac{7}{2}n^2 + \frac{7}{2}n + 1 = 148$

$\frac{7}{2}(6)^2 + \frac{7}{2}(6) + c = 148$

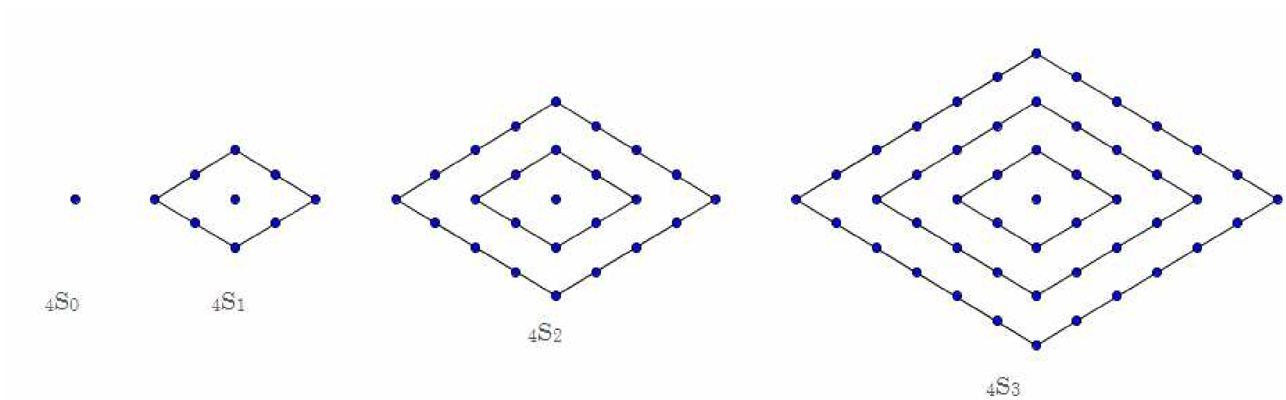
Therefore the general statement for this shape is:

$$\frac{7}{2}n^2 + \frac{7}{2}n + 1$$

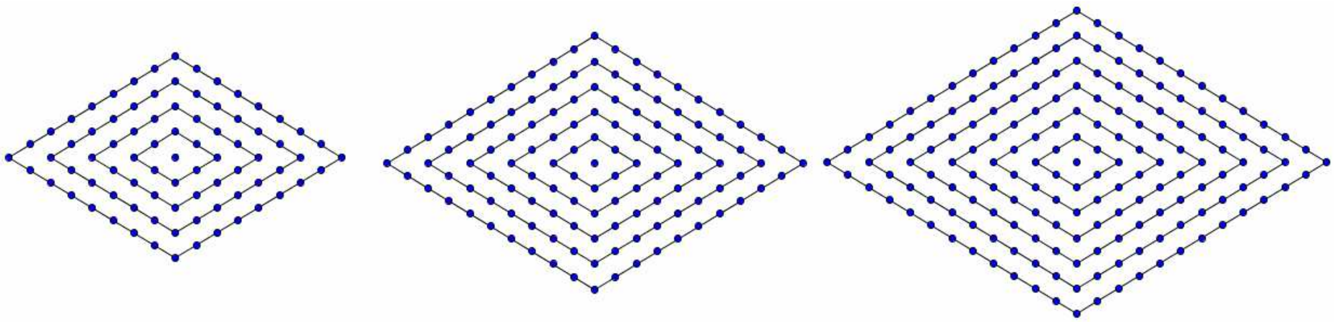
This proves that my general equation works. But, even though the general statement has been verified with 5 different shapes and an extra one, it does not apply to all shapes. For example for polygon 1 the rule does not apply, therefore there must be exceptions to the rule which derive other general equations. Let's explore those exceptions.

The first thing investigated was how the rule changed when the line length increased by a factor more than 1.

Let's consider the following shape with a p value of 4:



$_4S_3$	49	Adding 8x3 to previous
$_4S_4$	81	Adding 8x4 to previous
$_4S_5$	121	Adding 8x5 to previous
$_4S_6$	169	Adding 8x6 to previous



$_4S_4$ $_4S_5$ $_4S_6$

We can clearly observe that there is a difference in this shape. The subsequent stage increases the line length by 2 points instead of 1. Thus, the general statement was calculated for this shape to see if there was a difference between it and the one set in step 7.

Once again, another pattern was observed as the stage numbers developed. Using the second rule in step 7 ($_pS_n = {}_pS_{n-1} + pn$) this formula should be:

I.e.: $_4S_n = {}_4S_{n-1} + 8n$

Notice that this time we are multiplying the current term by 8. In the last shapes it would be equal to p but this time we are multiplying by 2p therefore we can see the effect of changing the length of the line by 2 on this formula.

<u>Finding a general statement:</u>

Now that I have more terms and now that they have been drawn I can now find a general statement for the sequence, using the previous methods.

$_pS_n$	$_4S_0$	$_4S_1$	$_4S_2$	$_4S_3$	$_4S_4$	$_4S_5$	$_4S_6$

Sequence	1	9	25	49	81	121	169
First Difference	8	16	24	32	40	48	
Second Difference	8	8	8	8	8		

Again, the second difference is the constant therefore the formula for the nth term contains n^2 as in the quadratic equation: $ax^2 + bx + c$

The value of 'a' is half the constant difference. In this example a= $\frac{8}{2} = 4$

Now that the first part of the formula is $4n^2$ the values of 'b' and 'c' can be found.

$_pS_n$	$_4S_0$	$_4S_1$	$_4S_2$	$_4S_3$	$_4S_4$	$_4S_5$	$_4S_6$
Sequence	1	9	25	49	81	121	169
$4n^2$	0	4	16	36	64	100	144
Difference between Sequence and n^2	1	5	9	13	17	21	25
Second difference	4	4	4	4	4	4	

This second difference tells me the value for ' b' which is equal to 4. To find the value of 'c' I will use the previous methods:

Using n=2

$4n^2 + 4n + c = 25$
$4(2)^2 + 4(2) + c = 25$
$24 + c = 25$
$c = 1$

To check that these are the correct values, two more examples were used:

Using n=3:

$4n^2 + 4n + 1 = 49$
$4(2)^2 + 4(2) + 1 = 49$

Using n=4:

$4n^2 + 4n + 1 = 81$
$4(4)^2 + 4(4) + 1 = 81$

According to the general rule formulated in question 7, the general statement should be $2n^2+2n+1$.

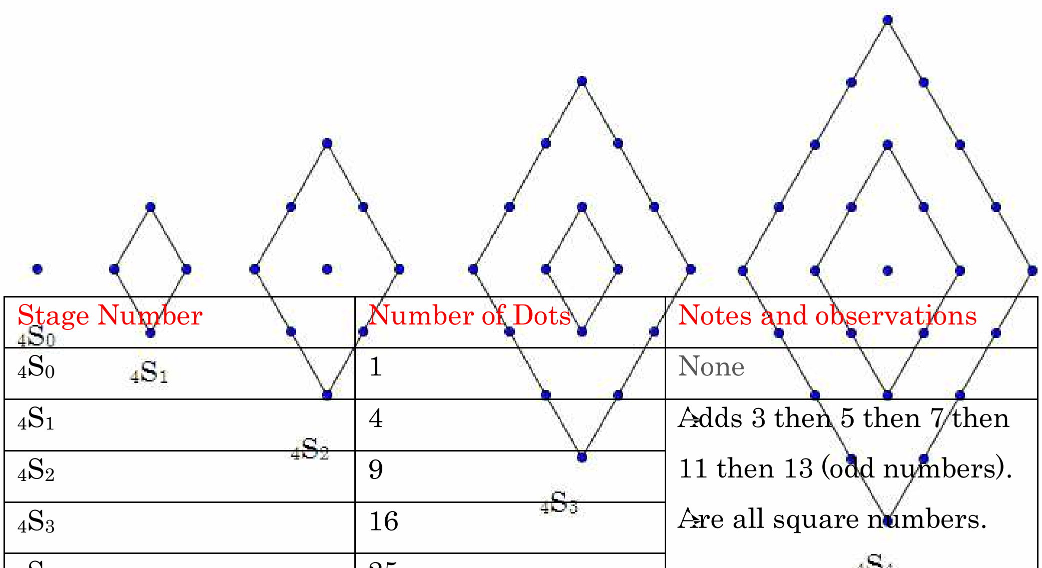
Yet, the general statement for this shape is:

$4n^2 + 4n + 1$

Therefore an increase in line length along the sequence by more than one point will prove to generate a new formula where a and b are both equal to p.

Another factor was considered: what if the number of dots in the centre of the shape varied? An extra shape was created to find out what could be the effect on the general rule:

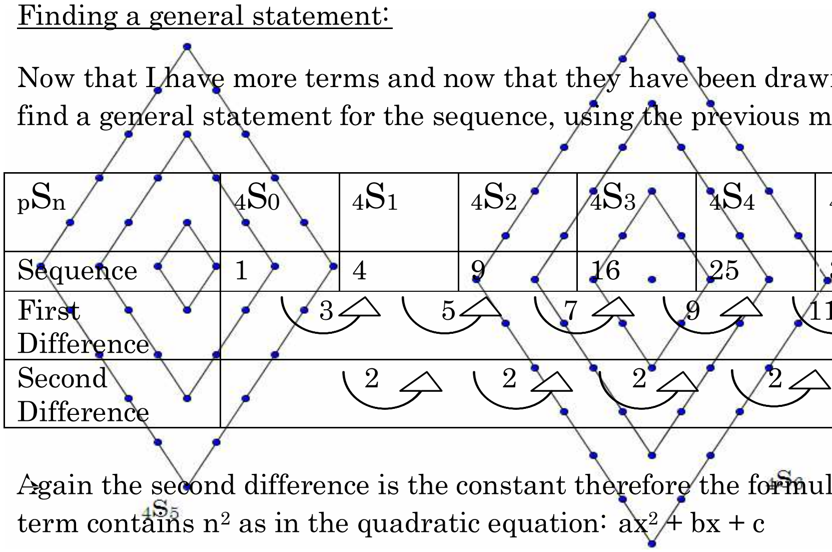
Considering another p value of 4:



Stage Number	Number of Dots	Notes and observations
$_4S_0$	1	None
$_4S_1$	4	Adds 3 then 5 then 7 then
$_4S_2$	9	11 then 13 (odd numbers).
$_4S_3$	16	Are all square numbers.
$_4S_4$	25	
$_4S_5$	36	
$_4S_6$	49	

It was observed that this time a 'side' formula could not be generated as clearly for this shape as with the other ones.

<u>Finding a general statement:</u>



Now that I have more terms and now that they have been drawn I can now find a general statement for the sequence, using the previous methods.

$_pS_n$	$_4S_0$	$_4S_1$	$_4S_2$	$_4S_3$	$_4S_4$	$_4S_5$	$_4S_6$
Sequence	1	4	9	16	25	36	49
First Difference		3	5	7	9	11	13
Second Difference			2	2	2	2	2

Again the second difference is the constant therefore the formula for the nth term contains n^2 as in the quadratic equation: $ax^2 + bx + c$

The value of 'a' is half the constant difference. In this example a=1

Now that I know that the first part of the formula is n^2 I can proceed to find the values of 'b' and 'c'.

$_pS_n$	$_4S_0$	$_4S_1$	$_4S_2$	$_4S_3$	$_4S_4$	$_4S_5$	$_6S_6$
Sequence	1	4	9	16	25	36	49
n^2	0	1	4	9	16	25	49
Difference between Sequence and n^2	1	3	5	7	9	11	13

Second difference	2 2 2 2 2 2

This second difference tells me the value for 'b' which is equal to 2.

To find the value of 'c' I will use the previous methods:

Using n=5:

$n^2 + 2n + c = 36$
$(5)^2 + 2(5) + c = 36$
$35 + c = 36$
$c = 1$

To check that these are the correct values I used two more examples:

→ Using n=1
$n^2 + 2n + c = 4$
$(1)^2 + 2(1) + 1 = 4$

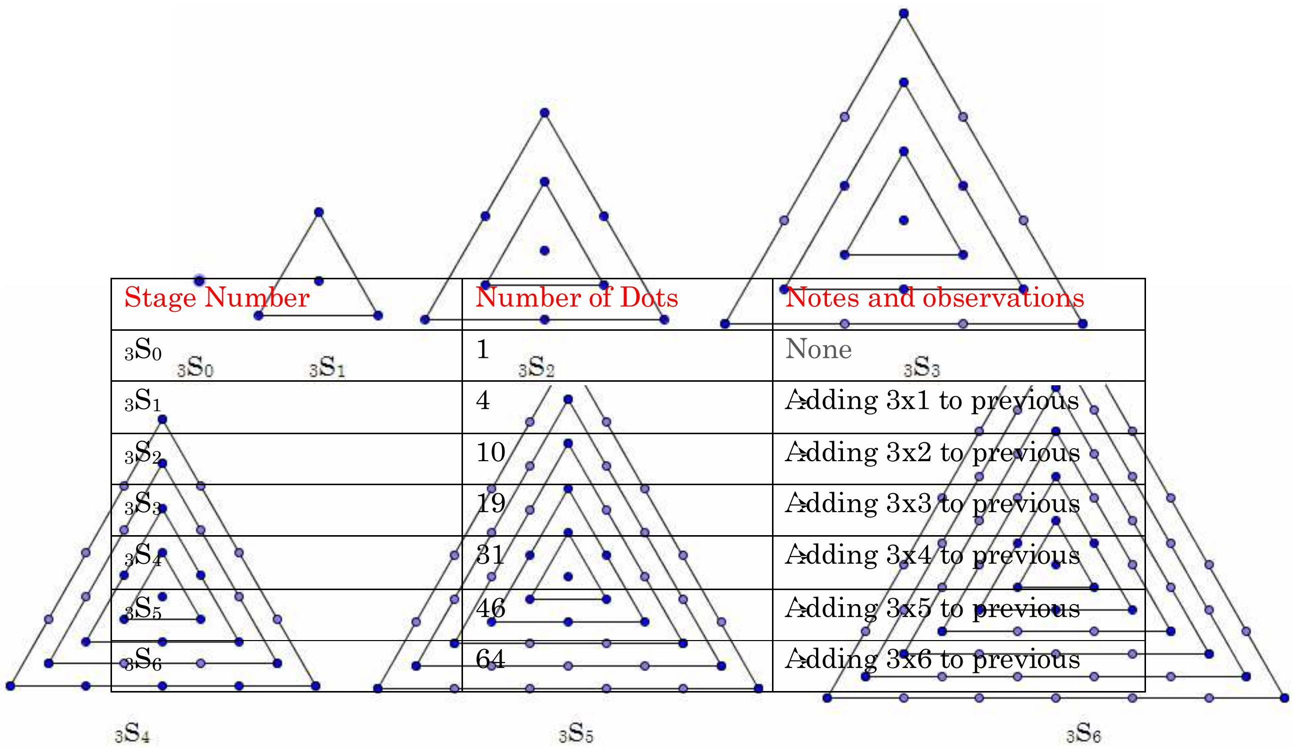
→ Using n=3:
$n^2 + 2n + c = 16$
$(3)^2 + 2(3) + 1 = 36$

With enough proof that the general statement works I conclude that it is:

$n^2 + 2n + 1$

According to the general rule formulated in question 7, the general statement for this case should be $2n^2+2n+1$. However this time the central dot did not stay consistent. A central dot did not exist when n would equal an odd number. There seems to be a pattern with odd numbers in this shape since on its first table this was underlined as well. Not having a consistent central dot also shows that a 'side' formula cannot be developed l ike the other polygons. Another issue applied here is that the in the sequence, the next level does not enclose the previous shape, in other words, the previous shape does not fit into the next shape like all the other.

Lastly, a polygon similar to polygon 1 was drawn, but, since polygon had no consistent central dot, one was drawn that always had a dot in the middle. Therefore a shape with a p value of 3 was drawn:



Stage Number	Number of Dots	Notes and observations
$_3S_0$	1	None
$_3S_1$	4	Adding 3x1 to previous
$_3S_2$	10	Adding 3x2 to previous
$_3S_3$	19	Adding 3x3 to previous
$_3S_4$	31	Adding 3x4 to previous
$_3S_5$	46	Adding 3x5 to previous
$_3S_6$	64	Adding 3x6 to previous

Once again, another pattern was observed as the stage numbers developed. Using the second rule in step 7 ($_pS_n = {_pS_{n-1}} + pn$) this formula should be:

I.e.: $_3S_n = {_3S_{n-1}} + 3n$
Let's verify this:

For Stage 3:
$_3S_n = {_3S_{n-1}} + 3n$
$_3S_3 = {_3S_2} + (3x3)$
$= 10 + 9$
$=19$

For Stage 6:

${}_3S_n = {}_3S_{n-1} + 3n$

${}_3S_6 = {}_3S_5 + (3x6)$

$= 46 + 18$

$=64$

<u>Finding a general statement:</u>

Now that I have more terms and now that they have been drawn I can now find a general statement for the sequence, using the previous methods.

${}_pS_n$	${}_3S_0$	${}_3S_1$	${}_3S_2$	${}_3S_3$	${}_3S_4$	${}_3S_5$	${}_3S_6$
Sequence	1	4	10	19	31	46	64
First Difference		3	6	9	12	15	18
Second Difference		3	3	3	3	3	

Again the second difference is the constant therefore the formula for the nth term contains n^2 as in the quadratic equation: $ax^2 + bx + c$

The value of 'a' is half the constant difference. In this example a= $\frac{3}{2}$

Now that I know that the first part of the formula is $\frac{3}{2}n^2$ I can proceed to find the values of 'b' and 'c'.

${}_pS_n$	${}_7S_0$	${}_7S_1$	${}_7S_2$	${}_7S_3$	${}_7S_4$	${}_7S_5$	${}_7S_6$
Sequence	1	4	10	19	31	46	64
$\frac{3}{2}n^2$	0	1.5	6	13.5	24	37.5	54
Difference between Sequence and $\frac{5}{2}n^2$	1	2.5	4	5.5	7	8.5	10
Second difference	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	

This second difference tells me the value for 'b' which is equal to $\frac{3}{2}$. To find the value of 'c' I will use the previous methods:

Using n=2:

$\frac{3}{2}n^2 + \frac{3}{2}n + c = 10$
$\frac{3}{2}(2)^2 + \frac{3}{2}(2) + c = 10$
$9 + c = 10$
$c = 1$

To check that these are the correct values, two more examples were used:

Using n=3:

$\frac{3}{2}n^2 + \frac{3}{2}n + 1 = 19$
$\frac{3}{2}(3)^2 + \frac{3}{2}(3) + 1 = 19$

Using n=5:

$\frac{3}{2}n^2 + \frac{3}{2}n + 1 = 46$
$\frac{3}{2}(5)^2 + \frac{3}{2}(5) + 1 = 46$

Therefore the general statement for this shape is:

$$\frac{3}{2}n^2 + \frac{3}{2}n + 1$$

This time, the rule generated in step 7 does apply to a shape with 3 vertices. This evidently shows the effect of having a constant central dot. Changing this factor seems to have changed the value of b, given that in polygon 1 it was $\frac{1}{2}$.

9. *Discuss the scopes or limitations of the general statement.*

When *reviewing the general statement*, *some exceptions can obviously* be *found*. Many *factors affect* values *in the general statement of the shapes.* Increasing *the length line* by *more than* 1, *having a number other than one as central* dots, *having altering central* dots, *having none at all* and *seeing* whether the shapes fit on top of each other are all aspects that are needed to be taken into account when formulating the overall expression. Therefore this statement will only work for shapes that have a constant central point, increase the length line by 1, and the previous shape must enclose inside the next one.

Accepting that all points at which lines meet as vertices is also important for the matter, as a stellar shape is assumed to be one that has the same number of concave and convex points. Another limitation is that the values in the sequence, the values of p and the values of n all have to be positive, as a negative value in each of these would be impossible to draw.

10. *Explain how you arrived at the general statement.*

Initially I *attempted to see whether there was a relationship* between *the number of dots and the values of* 'a' *and* 'b'. Seen *there was none*, I *decided to* create *a table so that the ideas were more organized.* I *then, started to see the* connection.

At first I *did not* consider *the* concave *points in some of the polygons to* be *vertices; if we were strictly speaking they are p-pointed stars.* However *when* I considered *these to* be *vertices and altered my general table* I *could see the immediate relationship* between *the values of* 'a' *and* 'b' *and the value of p.* Since 'a' *was equal to* 'b' *in most* cases *it was simple to* find *the relationship.*

As *a final point* c *was* continuously 1 *therefore* I believe *it to* be 1 for *the general expression derived.*