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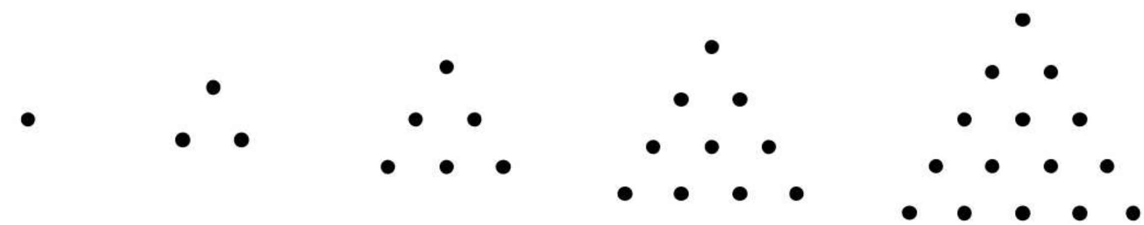
IB Mathematics Internal Assessment Portfolio Practice 1

Aim: The aim in this task is to consider geometric shapes, which lead to special numbers. The simplest examples of these are square numbers, 1, 4, 9, and 16, which can be represented by squares of side 1, 2, 3 and 4.

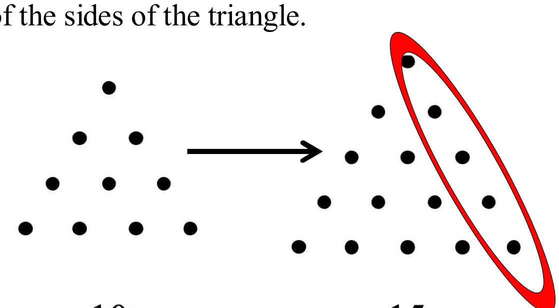
Introduction: In this study, we analyze geometrical shapes, which lead to special numbers. We consider various geometrical patterns of evenly spaced dots and derive general statements of the n^{th} special numbers in terms of n. Henceforth, all tables and mathematical formulas are generated by the Microsoft Word. All the stellar and triangular number visual representations are modulated using GeoGebra 4. The rest of the shapes are generated by "insert shape" function of Microsoft Word.

<u>Task 1</u>
The first task is to complete the triangular numbers sequence with three more terms.



1 3 6 10 15

As we can see, each successive triangle pattern numbered n (n is the stage number) can be derived from the previous one number ($n-1$) by simply adding n evenly spaced dots on either of the sides of the triangle.



10 15

In other words, the number of dots in the triangular number n equals to the number of dots in the triangular number ($n-1$) plus n. Lets denote this number by G_n

$$G_n = G_{n-1} + n$$

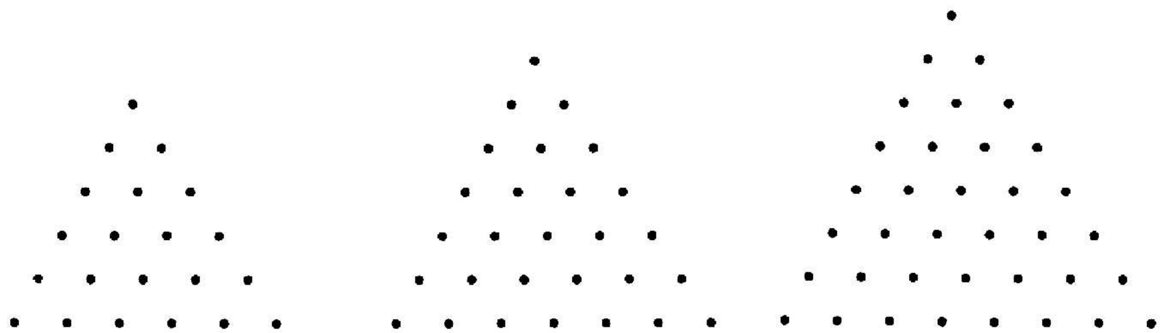
Using this formula, we can calculate the number of dots in the triangular number 6:
$15 + 6 = 21$
In the triangular number 7:
$21 + 7 = 28$
And in the triangular number 8:
$28 + 8 = 36$
and etc.

The following table can demonstrate this sequence:

Table 1.1

n - Number of the Triangular Pattern	G_n – n^{th} Triangular Number	$G_n - G_{n-1}$
1	1	
2	3	2
3	6	3
4	10	4
5	15	5
6	21	6
7	28	7
8	36	8

The last 3 triangular numbers can be graphically demonstrated as following.

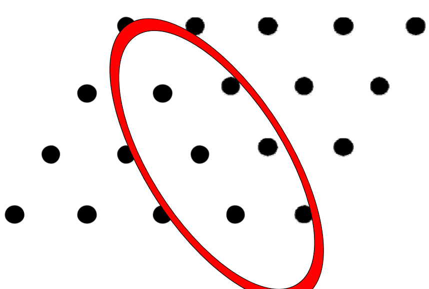


21 28 36

Using the expression $G_n = G_{n-1} + n$ one can continue the triangular number sequence indefinitely.

<u>Task 2</u>
Find a general statement that represents the n^{th} triangular number in terms of n.

In the previous task, we defined the $n^{\mathbf{th}}$ triangular number G_n in terms of the previous triangular number G_{n-1}
In this task we can derive a more general statement that represents the n^{th} triangular number only in terms of n. One can consider the rectangular structure that is comprised of two equivalent triangular structures. By merging two triangular shapes, one can produce a parallelogram shape



By further checking for the duplication of the diagonal, one can represent this structure as a parallelogram shape, where each side has n dots. Thus, the total number of dots of the two combined identical triangular structures equals to n^2 plus the additional number of dots in the extra diagonal n. In other words:

$$2\,G_n = n^2 + n$$

or

$$G_n = \frac{n(n+1)}{2}$$

Lets check this general statement with the following table:

Table 2.1

n	$G_n = G_n - 1 + n$	$G_n = \frac{n(n+1)}{2}$
1	1	1
2	1+2=3	$\frac{2 \times 3}{2} = 3$
3	3+3=6	$\frac{3 \times 4}{2} = 6$
4	6+4=10	$\frac{4 \times 5}{2} = 10$
5	10+5=15	$\frac{5 \times 6}{2} = 15$
6	15+6=21	$\frac{6 \times 7}{2} = 21$
7	21+7=28	$\frac{7 \times 8}{2} = 28$
8	28+8=36	$\frac{8 \times 9}{2} = 36$
9	36+9=45	$\frac{9 \times 10}{2} = 45$

One can confirm from this table that the general formula of the n^{th} triangular number in terms of n provides exactly the same results as the consecutive formula from Task 1. Finally, one can confirm that the two formulas are identical by the following calculation:

$$G_n = G_n - 1 + n$$

Substituting with the general statement:

$G_n = \frac{n(n+1)}{2}$ and $G_{n-1} = (n-1)\frac{n}{2}$

$$\therefore \frac{n(n+1)}{2} = (n-1)\frac{n}{2} + n$$

$$\therefore n(n+1) = (n-1)n + 2n$$

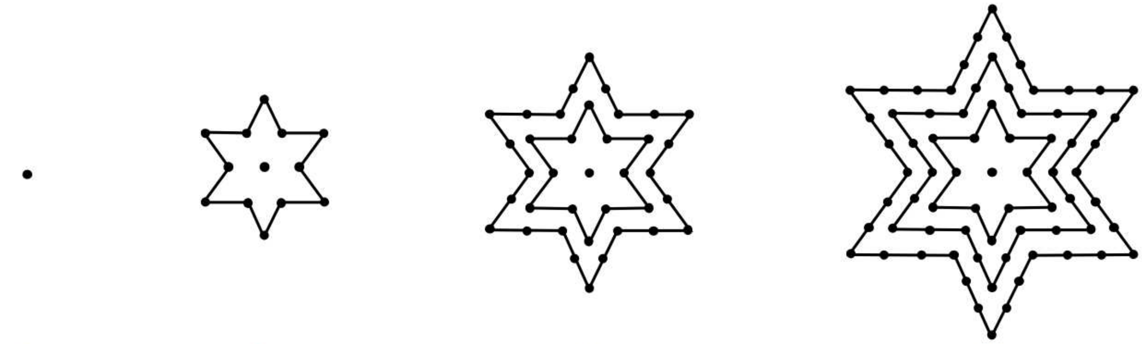
Opening the brackets, one can see that the two sides are identical; therefore the general formula for n is the same as the consecutive formula from Task 1.

Thus one can conclude that the general statement for the triangular numbers is as following:

$$G_n = \frac{n(n+1)}{2}$$

Task 3

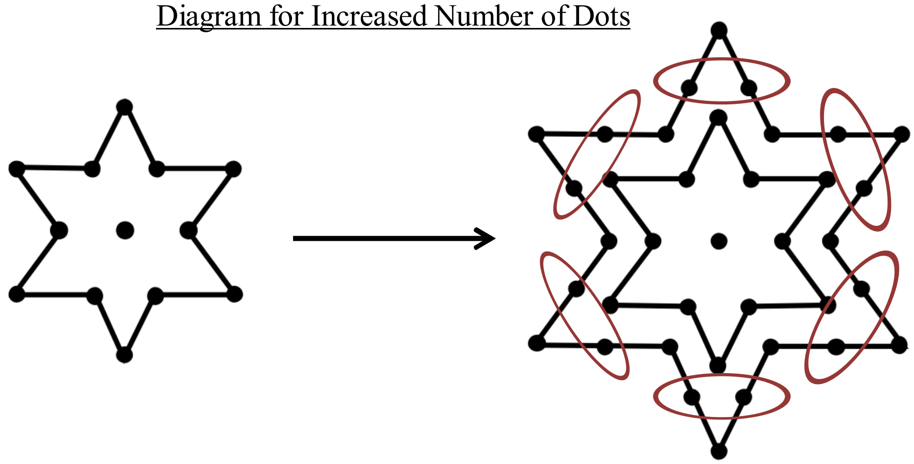
For the stellar shape with six vertices, find the number of dots (stellar number) in each stage up to S_6. Organize the data so that you can recognize and describe any patterns.



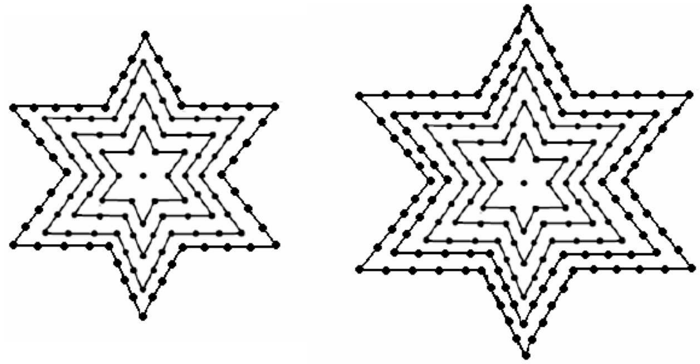
S_1 S_2 S_3 S_4

The first four representations for a star with six vertices (p=6) are shown in the four stages S_1 – S_4 above. From visual observations, one can notice, that each subsequent figure derives from the previous one by adding an outer star of a similar shape (with the same number of vertices). Thus, each figure at stage *n* consists of an *n* number of stars, placed inside one another similar to “Matryoshka dolls”. Consider the features of the consecutive stars. Each consecutive star is similar to the previous one with one notable exception: there are two more additional dots in each of the vertices.

<u>Diagram for Increased Number of Dots</u>



This diagram demonstrates the new dots added to the outer simple star at each consecutive stage (from S_2 to S_3 in this particular case). Using this visual representation, we can model the visual pattern of the stellar shapes for the stages S_5 and S_6.



To find the number of dots in S_5 ıge, one can simplify eac S_6 e as a combination of a number of simple stars with 6 s and the dot in the midd example: the shape S_3

consists of two simple stars and one dot in the middle; S_4 consists of three simple stars and the dot in the middle, etc. The total number of dots in any particular shape is the sum of the dots of comprising it simple stars, plus one dot.
Lets denote the number of dots of the outer simple star of the shape S_n as F_n ($F_1 = 0$).
For example: F_2 is to the number of dots in the outer star of S_2 and equals 12. The total number of dots is

$$F_2 + 1 = 12$$

We can also observe a simple sequence of the F_n. From the "Diagram for Increased Number of Dots" above, one can derive that:

$$F_n = F_{n-1} + 2*6$$

Thus, each consecutive outer simple star has 12 more dots than the previous one (two per each of the six vertices). We can summarize all these findings in the following table:

Table 3.1

n – the number of each consecutive stage	F_n – the number of dots in each consecutive outer simple star	The total number of dots equals to $\sum_{n=1}^{stage\ number} F_n + 1$
1	0	0+1=1
2	12	0+12+1=13
3	12+12=24	0+24+12+1=37
4	24+12=36	0+36+24+12+1=73
5	36+12=48	0+48+36+24+12+1=121
6	48+12=60	0+60+48+36+24+12+1=181

Lets denote by T_n the stellar number for the stage S_n. Judging from the table 3.1, we have the following result of the calculations:

Table 3.2

n – stage number	T_n – Stellar number (number of dots)
1	1
2	13
3	37
4	73
5	121
6	181

Task 4

Find an expression for the 6-stellar number at stage S_7.

Using the table 3.1 from the previous task, we can derive the number of dots in the outer simple star (F_7) at the stage S_7.

$$F_7 = F_6 + 12 = 60 + 12 = 72$$

Thus, F_7 equals to 72 dots.

Using the formula from the table 3.1 in the previous task, we can derive an **expression for the 6-stellar number at stage 7**:

$$T_7 = \sum_{n=1}^{n=7} F_n + 1$$

Where T_n is the stellar number (number of dots) and F_n is the number of dots in the respective simple stars.

From the table 3.1, we know the values from F_1 to F_7 to calculate T_7:

$$T_7 = 0 + 12 + 24 + 36 + 48 + 60 + 72 = 253$$

Thus, the number of dots in the 6-stellar number for the stage S_7 equals 253

Task 5

Find a general statement for the 6-stellar number at stage S_n in terms of n.

In order to find the general statement in terms of n, we have to generalize the formula for T_n – the stellar number.
To begin with, lets re-work the table 3.1 from Task 3.

Table 5.1

n – the number of each consecutive stage	F_n – the number of dots in each consecutive outer simple star	The total number of dots equals to $\sum_{n=1}^{stage\ number} F_n + 1$
1	0=12*0	12*0+1=1
2	12=12*1	12*(0+1)+1=13
3	12+12=24=12*2	12*(0+1+2)+1=37
4	24+12=36=12*3	12*(0+1+2+3)+1=73
5	36+12=48=12*4	12*(0+1+2+3+4)+1=121
6	48+12=60=12*5	12*(0+1+2+3+4+5)+1=181

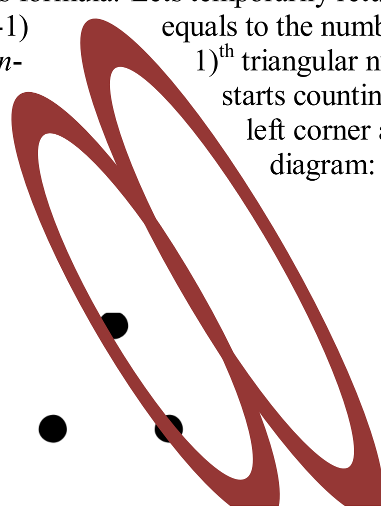
Observing the sequence from the last column, we can continue it to the stage S_7, for example:

$$12 * (0 + 1 + 2 + 3 + 4 + 5 + 6) + 1 = 253$$

or to any subsequent stage S_7:

$$12 * (0 + 1 + 2 \ldots + (n - 1)) + 1 = T_n$$

We will now attempt to simplify this formula. Lets temporarily return to Task 1. As one can notice, the sum of 1+2+3+4+…+(n-1) equals to the number of dots in the triangular pattern at the stage (n-1), or to the (n-1)th triangular number G_{n-1} . One can visually confirm this observation if he starts counting the dots of the triangular pattern diagram from the lower left corner and counts them layer by layer up to as depicted in the next diagram: 1+2+3+4+5…



We have already found the general statement for the n^{th} triangular number G_n in terms of n:

$$\frac{n(n+1)}{2}$$

If we look at the Task 2, we can now use this formula to calculate T_n:

$$T_n = 12 * \left(0+1+2+\ldots+(n-1)\right) + 1 = 12 \times \frac{(n-1)n}{2} + 1$$

Where, $\frac{(n-1)n}{2}$ represents $(n\text{-}1)^{th}$ triangular number G_{n-1}. Opening the brackets, we find the **general statement for the 6-stellar number at stage S_n in terms of n:**

$$\mathbf{T_n = 12 \times \frac{(n-1)n}{2} + 1 = 6n^2 - 6n + 1}$$

Let us verify this formula in the table below:

Table 5.2

n – Stage Number	The total number of dots equals to $\sum_{n=1}^{stage\ number} F_n + 1$	$T_n = 6n^2 - 6n + 1$
1	12*0+1=1	$6 \times 1 - 6 * 1 + 1 = 1$
2	12*(0+1)+1=13	$6 \times 4 - 6 * 2 + 1 = 13$
3	12*(0+1+2)+1=37	$6 \times 9 - 6 * 3 + 1 = 37$
4	12*(0+1+2+3)+1=73	$6 \times 16 - 6 * 4 + 1 = 73$
5	12*(0+1+2+3+4)+1=121	$6 \times 25 - 6 * 5 + 1 = 121$
6	12*(0+1+2+3+4+5)+1=181	$6 \times 36 - 6 * 6 + 1 = 181$
7	12*(0+1+2+3+4+5+6)+1=253	$6 \times 49 - 6 * 7 + 1 = 253$

A shown by the table, the two columns provide identical results thus supporting the validity of the general statement:

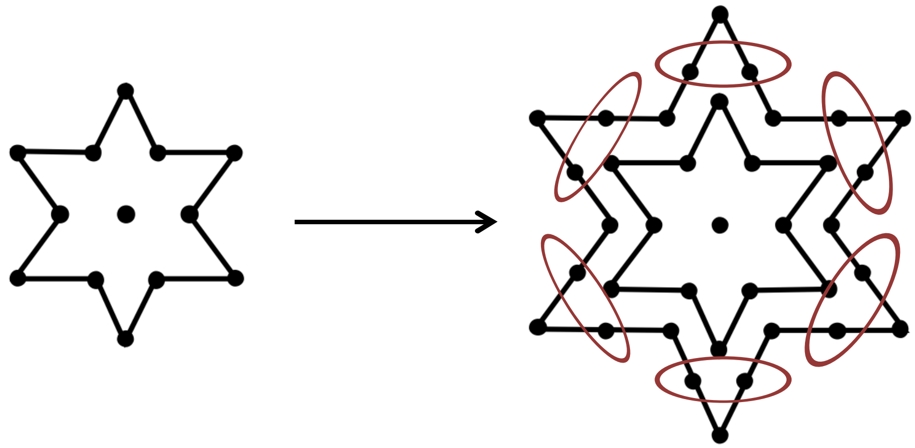
$$\mathbf{T_n = 6n^2 - 6n + 1}$$

Where T_n is the stellar number (total number of dots in the stellar shape at the stage S_n).

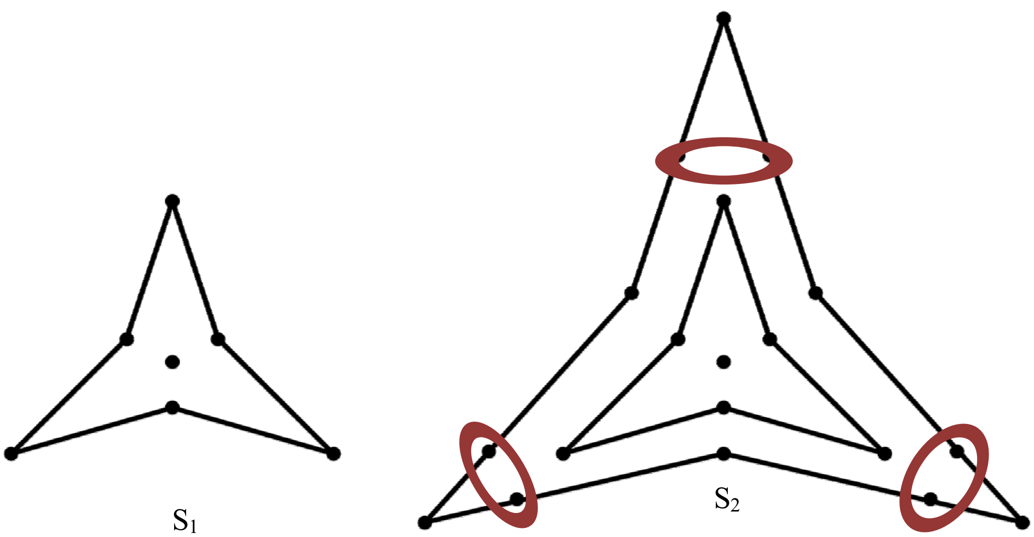
<u>Task 5</u>

Repeat the steps above for other values of p (number of vertices).

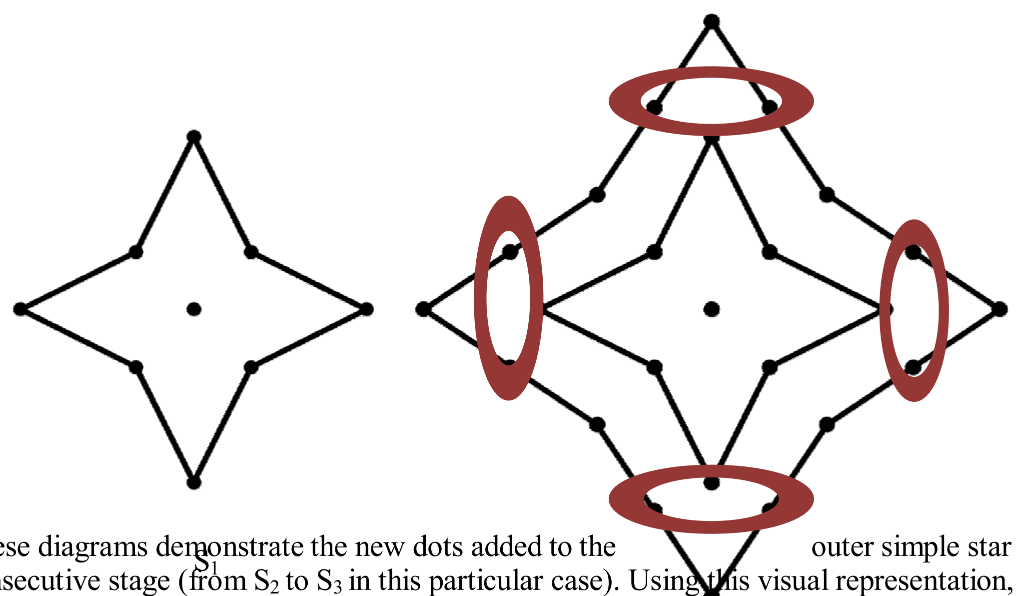
Lets consider the diagram for the increased number of dots from Task 3:



We can draw similar diagrams for stellar shapes with three or four vertices.



S_1 S_2



S_1 S_2

These diagrams demonstrate the new dots added to the outer simple star at each consecutive stage (from S_2 to S_3 in this particular case). Using this visual representation, we can model the logical pattern of the stellar shapes with three and four vertices: at each consecutive stage, the new outer simple star has two more dots per each vertices comparing to its predecessor. We can summarize this in the following table:

Table 6.1

n – Stage Number	F_n – Number of Dots in the Outer Simple Star with Three Vertices (p=3)	F_n – Number of Dots in the Outer Simple Star with Four Vertices (p=4)
1	0	0
2	0+2*3=6	0+2*4=8
3	6+2*3=12	8+2*4=16
4	12+2*3=18	16+2*4=24
5	18+2*3=24	24+2*4=32

Using the equivalent of the table 3.1, we can calculate the total stellar numbers for p=3 and p=4.

Table 6.2

n – the number of each consecutive stage	The total number of dots equals to $\sum_{n=1}^{stage\ number} F_n + 1$, p=3	The total number of dots equals to $\sum_{n=1}^{stage\ number} F_n + 1$, p=4
1	0+1=1	0+1=1
2	0+6+1=7	0+8+1=9
3	0+6+12+1=19	0+8+16+1=25
4	0+6+12+18+1=37	0+8+16+24+1=49
5	0+6+12+18+24+1=61	0+8+16+24+32+1=81

We can now see that there are two common denominators in the two columns - 6 in case p=3 and 8 in case p=4:

Table 6.3

n – the number of each consecutive stage	The total number of dots equals to $\sum_{n=1}^{stage\ number} F_n + 1$, p=3	The total number of dots equals to $\sum_{n=1}^{stage\ number} F_n + 1$, p=4
1	6*0+1=1	8*0+1=1
2	6*(0+1)+1=7	8*(0+1)+1=9
3	6*(0+1+2)+1=19	8*(0+1+2)+1=25
4	6*(0+1+2+3)+1=37	8*(0+1+2+3)+1=49
5	6*(0+1+2+3+4)+1=61	8*(0+1+2+3+4)+1=81

Using the analogous formula from Task 5 ($T_n = 6n^2 - 6n + 1$) and substituting 12 for 6 and 8 respectively, we can solve for the stellar numbers in case of p=3:

$$T_n = 3n^2 - 3n + 1$$

And for p=4:

$$T_n = 4n^2 - 4n + 1$$

Task 6

Hence, produce the general statement, in terms of p and n, that generates the sequence of p -stellar numbers for any value of p at stage S_n

By comparing the general statements for the *p*-stellar numbers:

p=3	*p*=4	*p*=6
$T_n = 3n^2 - 3n + 1$	$T_n = 4n^2 - 4n + 1$	$T_n = 6n^2 - 6n + 1$

We can conclude that the changing variable in these equations is p, so the final general statement in terms of *p* and *n* for all *p*-stellar numbers of any value of *p* at stage S_n is:

$$\mathbf{T_n = pn^2 - pn + 1}$$

or

$$\mathbf{T_n = pn(n - 1) + 1}$$

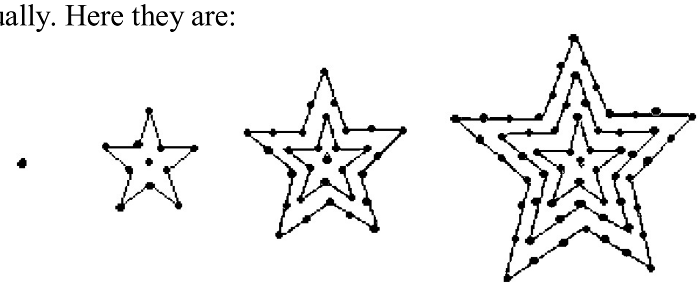
Where *p* is the number of vertices. Henceforth, we will have to test this general statement.

Task 7

Test the validity of the general statement.

Lets consider the stellar shapes with 5 vertices (*p*=5). We can calculate the number of dots (the stellar number) manually. Here they are:

T_n = total number of dots at the stage S_n



$T_1 = 1$ $T_2 = 11$ $T_3 = 31$ $T_4 = 61$

We can now verify the general statement in the following table:

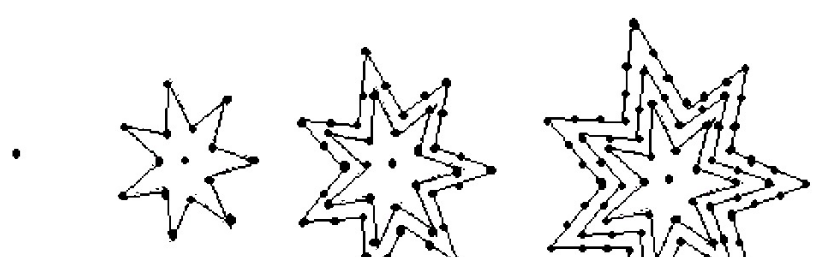
Table 7.1

n – Stage Number	T_n as calculated manually	$T_n = pn^2 - pn + 1$, *p*=5
1	1	$5 \times 1 - 5 * 1 + 1 = 1$
2	11	$5 \times 4 - 5 * 2 + 1 = 11$
3	31	$5 \times 9 - 5 * 3 + 1 = 31$
4	61	$5 \times 16 - 5 * 4 + 1 = 61$

As one can see, the numbers are identical, which supports the validity of the formula.

Another example to prove the general statement as a stellar with 7 vertices (*p*=7) is going to be attempted below:

T_n = total number of dots at the stage S_n



$T_1 = 1$ $T_2 = 15$ $T_3 = 43$ $T_4 = 85$

Table 7.2

n – Stage Number	T_n as calculated manually	$T_n = pn^2 - pn + 1$, p=5
1	1	$7*1-7*1+1=1$
2	15	$7*4-7*2+1=15$
3	43	$7*9-7*3+1=43$
4	85	$7*16-7*4+1=85$

Again, as one can notice, the numbers are identical, proving the general statement's authenticity.

Task 8

Discuss the scope and limitations of the general statemen t.

We have tested the general statement $T_n = pn^2 - pn + 1$ for a sufficient range of stellar structure with number of vertices from 3 to 7. Building the stellar structures with larger number of vertices is quite time consuming and thus further checks have to be done by a sophisticated computer program. However, the logic behind this general statement remains the same for any number of vertices, which leads us to the conclusion that the formula is correct. One has to use it properly, only for the values of n higher than 0 and for the values of p higher than 2, as one cannot imagine stellar shapes with one or two vertices. Technically it is possible to use any, even negative numbers, but we find no logical context for such calculations.

Task 9

Explain how you arrived at the general statement.

There were three key findings that helped me to arrive at the general statement. The first realization was that each new outer star has two more additional dots per each vertex. This allowed me to calculate the number of dots in any outer star using the recurrent formula.

Secondly, I noticed that each formula of the stellar number consists of one inalienable part: the sum of (0+1+2+3…+(n-1)).

Finally I noticed that this part above is in fact the triangular number from the Task 2.

Putting these three observations together, I found the formula, which was later confirmed by a decent number of verifications.