STANDARD
LEVEL
MATHEMATICS
TYPE I
COURSEWORK

Stellar Numbers

By | Sean Okundaye

Sean Okundaye

Contents

Sean Okundaye

Aim of the investigation: In this task I will consider geometric shapes which lead to special numbers. The simplest examples of these are square numbers, 1, 4, 9, 16, which can be represented by squares of side 1, 2, 3 and 4.

Triangular numbers (i.e. 1, 3, 6, 10...) are numbers that can be displayed in the form of a triangular pattern composed of evenly spaced dots. This can be seen below.
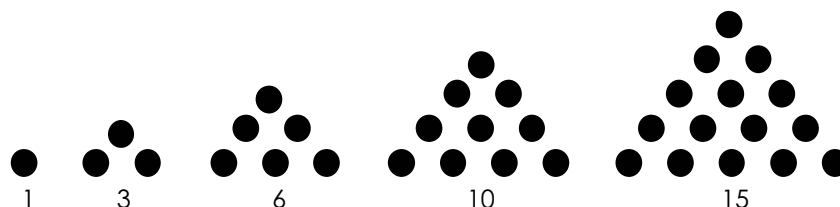
TASK 1: Complete the triangular number sequence with three more terms. Find a general statement that represents the nth triangular number in terms of n.

Given the fact that our current sequence is 1, 3, 6, 10, 15... we need to find the nth term.

Looking at the sequence, we see that there difference between each term increases by 1 so for example the difference between u_1 and u_2 is 2 and the difference between u_2 and u_3 is 3. Therefore given our current sequence, we can say that:

$$u_n = 1 + 2 + 3 + 4 + 5 \dots + (n - 1) + n$$

This can be simplified into the following equations:

$$\frac{n(n + 1)}{2} = \frac{n^2 + n}{2}$$

We need to find the next three terms and since we already have terms $u_1 - u_5$, the terms we need to find are u_6, u_7 and u_8. To find these terms, we need to input 6, 7 and 8 into the above equation. Therefore:

$$u_6 = \frac{6^2 + 6}{2} = 21 \quad u_7 = \frac{7^2 + 7}{2} = 28 \quad u_8 = \frac{8^2 + 8}{2} = 36$$

Now that we have found these terms, we can display them in the form of a triangular pattern which can be seen on the next page.

Sean Okundaye
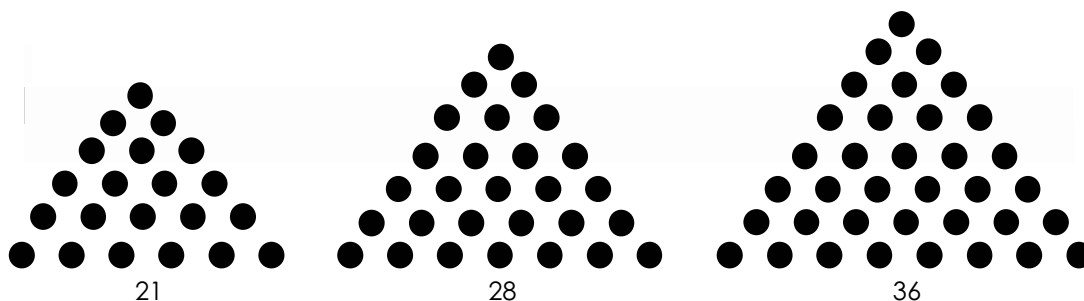
If we now turn our attention to stellar (star) shapes, these have 6 vertices which then lead onto "6-stellar numbers". If you look at the first four examples below, we can see stars with six vertices shown in stages $s_1 - s_4$.
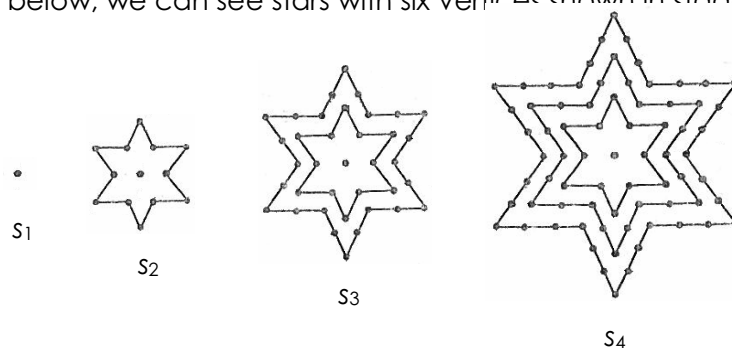

s_1

s_2

s_3

s_4

You can see that at each stage there are six vertices on the shape and so we can say that each shape has a "6-stellar number". This is the total number of dots in the above diagram and there is a way in which we can find the number of dots.

TASK 2: Find the number of dots (i.e. the stellar number) in each stage up to s_6. Organise the data so that you can recognise and describe any patterns.

Having been presented with the first four stages, I have been able to count the number of dots manually and have produced a sequence consisting of 1, 13, 37 and 73 dots from stages s_1 to s_4 respectively. The difference between each stage was 12, 24 and 36 respectively and the difference between each of these stages was 12. Therefore, we can find out the next two terms by continuing the sequence of adding 12 to the last stage. From there we can add the value to the last stage of the second value and complete our sequence up to s_6. This can be seen below.

<div align="center">

1   13   37   73   121   181

12   24   36   48   60

12   12   12   12

</div>

Sean Okundaye

Therefore can now produce two stellar shapes for s_5 and s_6 which can be seen on the next page.

| s_5 | s_6 |
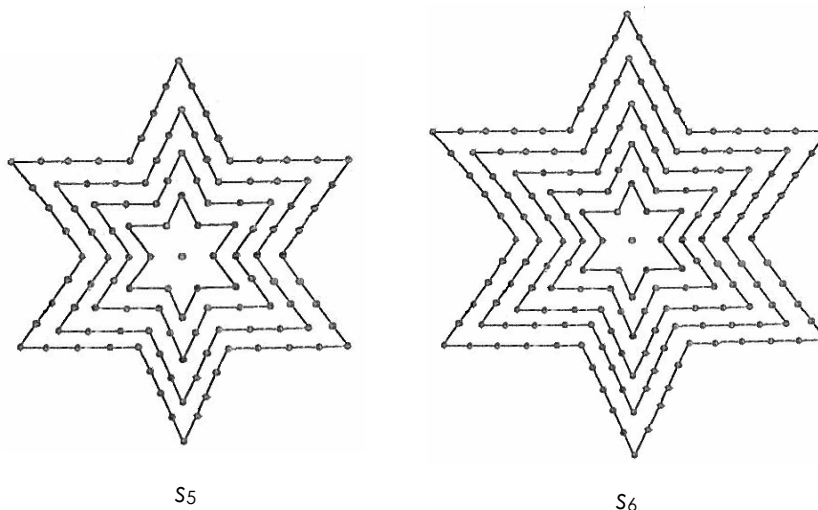
TASK 3: Find an expression for the 6-stellar number at stage s_7. Find a general statement for the 6-stellar number at stage s_n in terms of n.

Since we had to find the difference twice, the formula for the sequence is a quadratic equation, i.e.:

$$s_n = an^2 + bn + c$$

Therefore, an expression for the 6-stellar number at stage s_7 would be:

$$s_7 = a(7^2) + b(7) + c$$

$$\therefore s_7 = 49a + 7b + c$$

To be able to find a general statement for the 6-stellar number at stage s_n, we need to work out the value of a, b, and c. Using the first three terms as examples, we can say that:

$$s_1 = a(1^2) + b(1) + c \equiv 1 = a + b + c$$

$$s_2 = a(2^2) + b(2) + c \equiv 13 = 4a + 2b + c$$

$$s_3 = a(3^2) + b(3) + c \equiv 37 = 9a + 3b + c$$

We find the values of a, b, and c. We can start by subtracting s_2 from s_1:

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$$(4a + 2b + c) - (a + b + c) = 3a + b$$

$$3a + b = 13 - 1 = 12$$

We need to also subtract s_3 from s_2:

$$(9a + 3b + c) - (4a + 2b + c) = 5a + b$$

$$5a + b = 37 - 13 = 24$$

With the values we have obtained, we can now find out the value of a:

$$(5a + b) - (3a + b) = 24 - 12 = 12$$

$$\therefore a = \frac{12}{2} = 6$$

Now that we have found the value of c, one can find the value of a:

$$5a + b = 24$$

$$\therefore 5(6) + b = 24$$

$$b = -6$$

Since we now know the values of c and a, we can substitute these values into s_1:

$$1 = 6 - 6 + c$$

$$\therefore c = 1$$

Now that we now know all the values, we can now form the equation of the sequence which is as follows:

$$s_n = 6n^2 - 6n + 1$$

TASK 4: Repeat the previous tasks for other values of p. Hence, produce the general statement, in terms of p and n, that generates the sequence of p stellar numbers for any value of p at stage s_n.

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If one was to use a stellar shape with 5 vertices as an example, the 5-steller number for stages s_1 to s_4 would be as follows:

$$1 \qquad 11 \qquad 31 \qquad 61$$
$$10 \qquad 20 \qquad 30$$
$$10 \qquad 10$$

This can be seen in the stellar shapes below that are produced with a 5-stellar number.

S_1

S_2

S_3

S_4
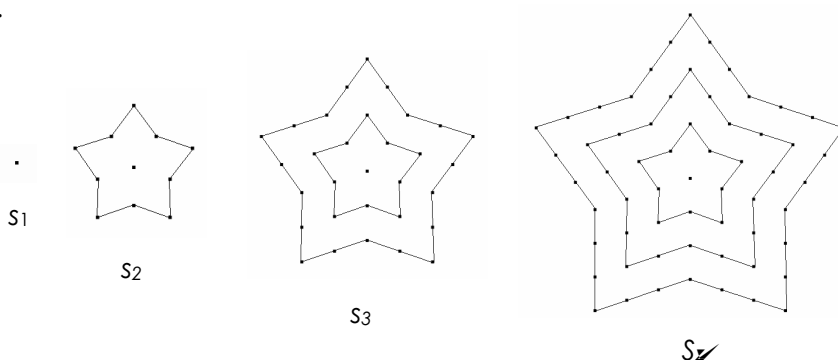
Like before, we will need to solve a quadratic equation to find a general statement for the stellar number at s_n, this time a 5-stellar number. We can use the first three stages:

$$s_1 = a(1^2) + b(1) + c \equiv 1 = a + b + c$$

$$s_2 = a(2^2) + b(2) + c \equiv 11 = 4a + 2b + c$$

$$s_3 = a(3^2) + b(3) + c \equiv 31 = 9a + 3b + c$$

To find the values of a, b, and c, we can subtract s_2 from s_1:

$$(4a + 2b + c) - (a + b + c) = 3a + b$$

$$3a + b = 11 - 1 = 10$$

We need to also subtract s_3 from s_2:

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$$(9a + 3b + c) - (4a + 2b + c) = 5a + b$$

$$5a + b = 31 - 11 = 20$$

With the values we have obtained, we can now find out the value of a:

$$(5a + b) - (3a + b) = 20 - 10 = 10$$

$$\therefore a = \frac{10}{2} = 5$$

Now that we have found the value of c, one can find the value of b:

$$5a + b = 20$$

$$\therefore 5(5) + b = 20$$

$$b = -5$$

Since we now know the values of c and b, we can substitute these values into s_1:
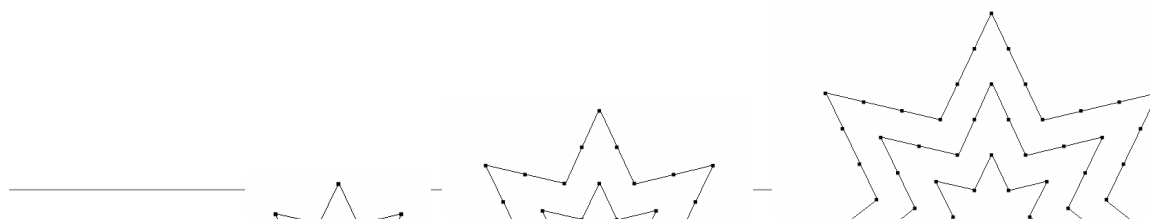
$$1 = 5 - 5 + c$$

$$\therefore c = 1$$

Therefore, we can say that with a 5-stellar number:

$$s_n = 5n^2 - 5n + 1$$

Now considering a stellar shape with 7 vertices and so a 7-stellar number, stages s_1 to s_4 are as follows:

<div>

1   15   43   6

14   28   42

14   14

</div>

This can be seen below in the form of stellar shapes:

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Once again we will need to solve a quadratic equation to find a general statement for the stellar number at s_n, this time a 7-stellar number. We will use the first three stages:

$$s_1 = a(1^2) + b(1) + c \equiv 1 = a + b + c$$

$$s_2 = a(2^2) + b(2) + c \equiv 11 = 4a + 2b + c$$

$$s_3 = a(3^2) + b(3) + c \equiv 31 = 9a + 3b + c$$

To find the values of a, b, and c, we can subtract s_2 from s_1:

$$(4a + 2b + c) - (a + b + c) = 3a + b$$

$$3a + b = 15 - 1 = 14$$

We need to also subtract s_3 from s_2:

$$(9a + 3b + c) - (4a + 2b + c) = 5a + b$$

$$5a + b = 43 - 15 = 28$$

With the values we have obtained, we can now find out the value of a:

$$(5a + b) - (3a + b) = 28 - 14 = 14$$

$$\therefore a = \frac{14}{2} = 7$$

Now that we have found the value of a one can find the value of b.

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$$5a + b = 28$$

$$\therefore 5(7) + b = 28$$

$$b = -7$$

Since we now know the values of a and b, we can substitute these values into s_1:

$$1 = 7 - 7 + c$$

$$\therefore c = 1$$

Therefore, we can say that with a 7-stellar number:

$$s_n = 7n^2 - 7n + 1$$

Looking at the general statements that have been produced, one can see that there is a trend. For each of the stellar numbers I have tested, the p value has been equal to the value of both a and b. Therefore, I have come to the conclusion that:

$$s_n = pn^2 - pn + 1$$

TASK 5: Test the validity of the general statement and discuss the scope or limitations of the general statement.

When testing, the validity I found that when I made the p value 2, a problem occurred:

$$s_n = 2n^2 - 2n + 1$$

$$s_1 = 2(1^2) - 2(1) + 1 = 1$$

$$s_2 = 2(2^2) - 2(2) + 1 = 5$$

$$s_3 = 2(3^2) - 2(3) + 1 = 13$$

$$s_4 = 2(4^2) - 2(4) + 1 = 27$$

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$$1 \quad 5 \quad 13 \quad 27$$
$$4 \quad 8 \quad 14$$
$$4 \quad 6$$
$$2$$

As can be seen from the above sequence, there are four levels. Therefore, the equation of the sequence should be cubic as opposed to quadratic and so this goes against the general statement. If the p value was 3, the same problem did not occur:

$$s_n = 3n^2 - 3n + 1$$

$$s_1 = 3(1^2) - 3(1) + 1 = 1$$

$$s_2 = 3(2^2) - 3(2) + 1 = 7$$

$$s_3 = 3(3^2) - 3(3) + 1 = 19$$

$$s_4 = 3(4^2) - 3(4) + 1 = 37$$

$$1 \quad 7 \quad 19 \quad 37$$
$$6 \quad 12 \quad 18$$
$$6 \quad 6$$

Therefore, one should probably also conclude that:

$$p \not< 3$$

I.e. p cannot be less than 3.

The limitation of my investigation is the fact that I did not test every single stellar number and therefore cannot be certain that every number that is not less than 3 will work. Furthermore, when working with values of p I only went up to as high as 4 and so I cannot be certain that my general statement is correct for numbers such as 80 or 200.

Sean Okundaye