MARKHAM COLLEGE

STUDIIS ET REBUS HONESTIS

# IB Standard Mathematics Internal Assessment

Candidate Name: Alejandro Ortigas Vásquez
Candidate Number: -

Title: Stellar Numbers

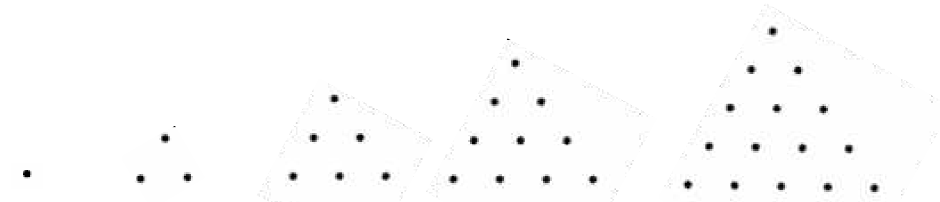
| D | | | DCP | | | CE | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | | | | | | | | |

## Aim:

To deduce the relationship found between the stellar numbers and the number of vertices in the stellar shapes that dictate their value.

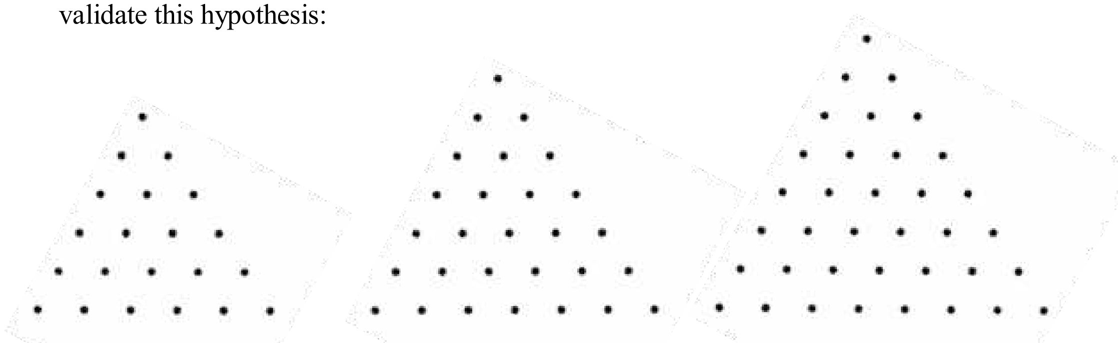
## Procedure:

When considering the stellar numbers, it is important to understand that their origin can be traced back to geometric shapes. Consider, for instance, triangular numbers (1, 3, 6…):



1    3    6    10    15

As we can see, we are able to obtain the same sequence from the triangular pattern of evenly spaced dots. We can also use these diagrams to expand the sequence and validate this hypothesis:

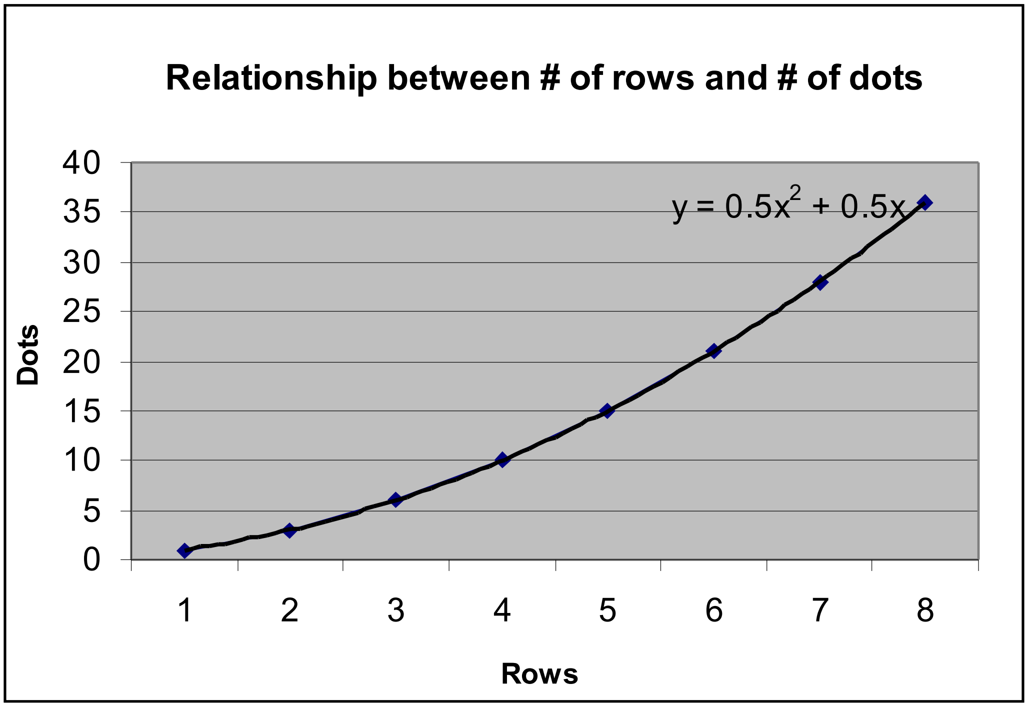


21    28    36

We can put this information in a table:

| Term | Dots |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 6 |
| 4 | 10 |
| 5 | 15 |
| 6 | 21 |
| 7 | 28 |
| 8 | 36 |

We can then find the relation between the values in the table by using the computer program Microsoft Excel and plotting them in a graph:




As seen, we can also use the Display Equation on Chart option to obtain the equation of the line and thus a general statement for any term of the triangular numbers. Let us consider the n number of rows, and so the general expression becomes:

$$\frac{n^2+n}{2}$$

We can test the validity of this statement by trying to arrive to the same expression without the use of technology and simply using mathematical reasoning. Let us look at the table again:

| Rows | Dots |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 6 |
| 4 | 10 |
| 5 | 15 |
| 6 | 21 |
| 7 | 28 |
| 8 | 36 |

In this table, however, instead of referring to simply ‘term’ we consider the number of rows, since that is a varying pattern that logically coincides with the concept of ‘term’.

From this we can get the general expression:

1. At first, the number of rows and dots do not seem to exhibit any discernible relation. However, if we double the value for the number of dots we obtain the following table:

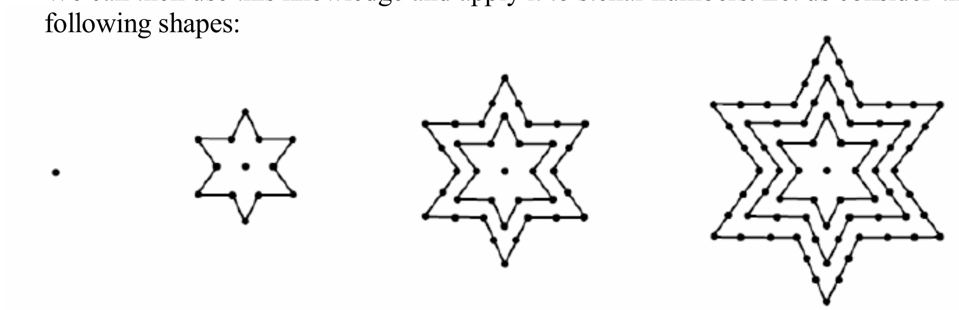
| Rows | Dots |
|---|---|
| 1 | 2 |
| 2 | 6 |
| 3 | 12 |
| 4 | 20 |
| 5 | 30 |
| 6 | 42 |
| 7 | 56 |
| 8 | 72 |

2. We then notice that the number of rows becomes a multiple of the number of dots. Starting with one, we can deduce the expression d = n + 1, where 'n' is the number of rows and 'd' the number of dots.
3. However, we then see that the expression is untrue for 2, since $d_2 = 2 + 1 = 3$, when the value in the table is 6. The expression can easily be fixed, nonetheless, by turning it into d = n (n+1), or $d = n^2 + n$.
4. After making sure that the expression is still true to previous values, we test with the next ones.
5. We then discover that the expression is true for all present values of n.
6. Since we had doubled the value of dots to obtain the values in this second table, the general expression thus becomes:

$$\mathbf{d = \frac{n^2+n}{2}}$$

This is concordant with the expression we obtained from the graph by using Microsoft Excel.
Thus, this proves that using diagrams in the form of geometric shapes can be helpful when trying to decipher the general statement of certain sequences of special numbers.

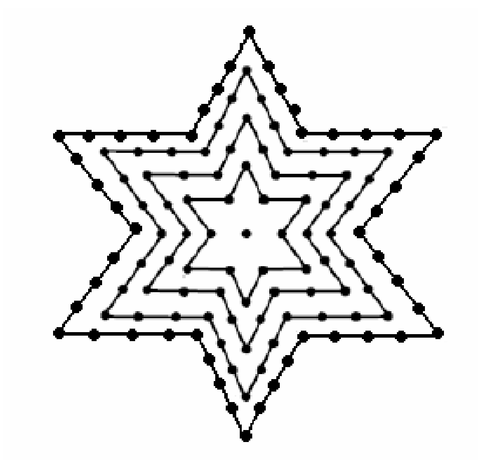
We can then use this knowledge and apply it to stellar numbers. Let us consider the following shapes:



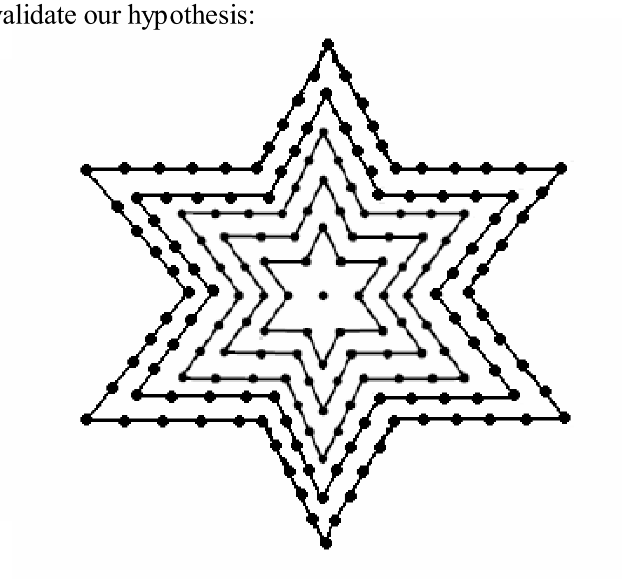
1 2 3 4

These are stellar (star) shapes that are used to find the stellar numbers. However, since the number of vertices in each star can vary, we call these numbers the p-stellar numbers, where p is the number of vertices in the shapes. Thus, in this particular case, we will try to find an expression for the 6-stellar numbers. To do this, we will use the same method used for finding the triangular numbers.

Since we will consider that we can obtain the 6-stellar numbers from the diagram, let us expand the number of shapes so as to validate our hypothesis:



5

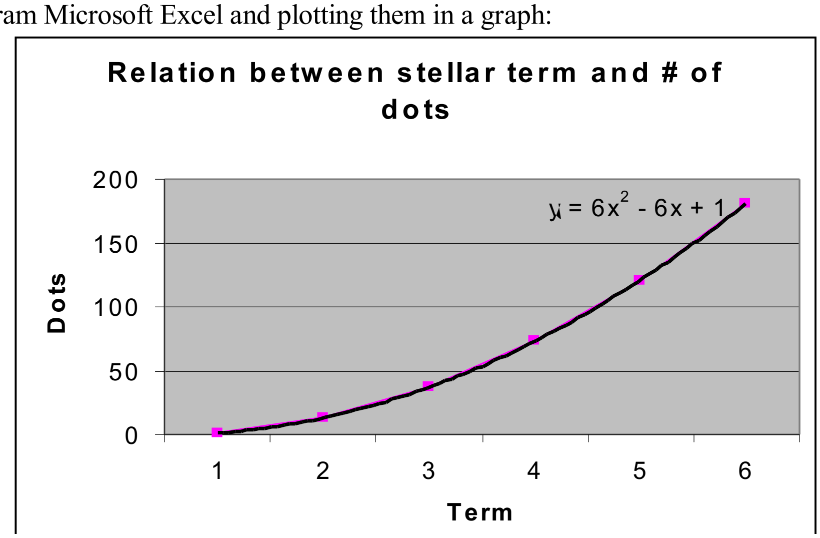


6

Now, let us put this information in a table:

| Term | Dots |
|---|---|
| 1 | 1 |
| 2 | 13 |
| 3 | 37 |
| 4 | 73 |
| 5 | 121 |
| 6 | 181 |

We can then find the relation between the values in the table by using the computer program Microsoft Excel and plotting them in a graph:



Once again, we can use the Display Equation on Chart option to obtain the equation of the line and thus a general statement for any term of the 6-stellar numbers. Let us consider the $n^{th}$ term, and so the general expression becomes:

$$\mathbf{d = 1 + 6(n^2\text{-}n)}$$

We can also test the validity of this statement by trying to arrive to the same expression without the use of technology and simply using mathematical reasoning. Let us consider the following table:

| Term | Dots |
|---|---|
| 1 | 1 |
| 2 | 13 |
| 3 | 37 |
| 4 | 73 |
| 5 | 121 |
| 6 | 181 |

From this we can obtain the general expression:

1. We notice that the current values for d, the number of dots, seem to have to no apparent relation with their corresponding term.
2. However, by looking at the diagrams we are able to decipher that the dot in the middle of the stellar shapes is constant between the different terms. Thus, if we remove the constant we obtain the following table:

| Term | Dots |
|---|---|
| 1 | 0 |
| 2 | 12 |
| 3 | 36 |
| 4 | 72 |
| 5 | 120 |
| 6 | 180 |

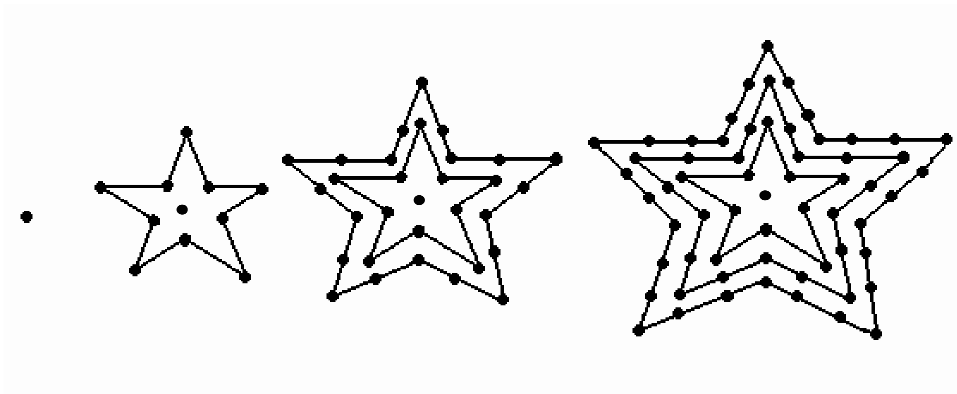
3. The values for the number of dots then become multiples of 12.
4. By using this information, we can take 12 to be the base value and alter it to fit the values shown for the number of dots. From this we get that, for the first term, it is 12 x 0, for the second term it is 12 x 1, fo r the third term it is 12 x3 and so on.
5. We notice that 12 in each case is being multiplied by the triangular number of the previous term as the stellar number. Thus, the general expression till this point becomes $6\ ((n\text{-}1)^2 + (n\text{-}1))$, which is the same as saying $6(n^2 - n)$.
6. If we remember to add the 1 that we had eliminated as a constant, we can obtain the general statement:

$$\mathbf{d = 1 + 6(n^2\text{-}n)}$$

This is concordant with the expression we obtained from the graph by using Microsoft Excel. Thus, the statement is correct.

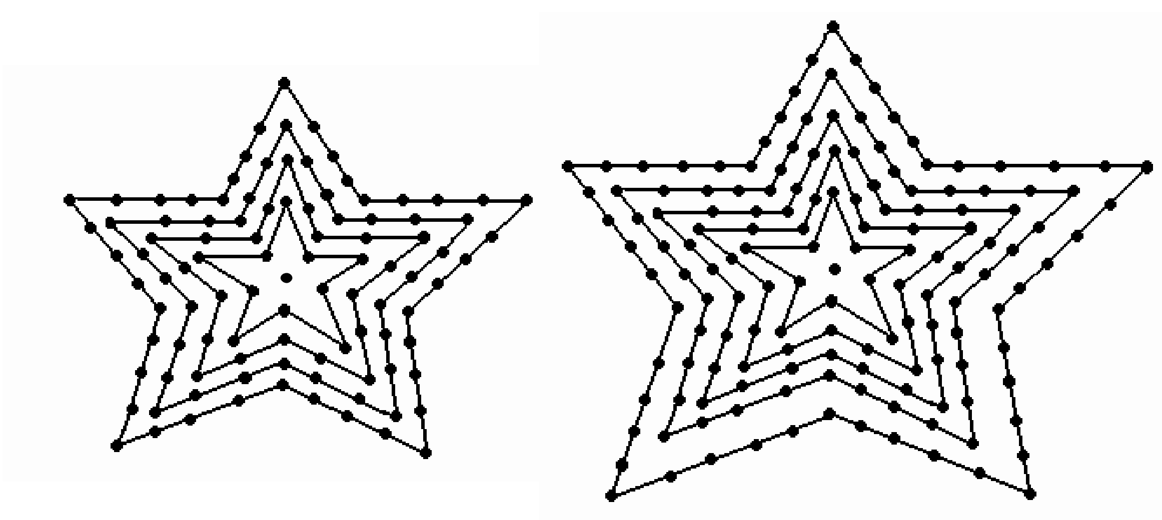
Now, as we mentioned, there is really an infinite number of vertices that a stellar shape can have. This could mean that the expression for one set of stellar numbers is not the same for others. We can test this assertion by using the same methods we have been using.

Here is an example where we conduct the same process but using the 5-stellar number instead. This implies the use of the following stellar diagrams, each made up of stellar shapes with 5 vertices:



1 2 3 4

Since we are considering that we can obtain the 5-stellar numbers from the diagram, let us expand the number of shapes so as to validate our hypothesis:

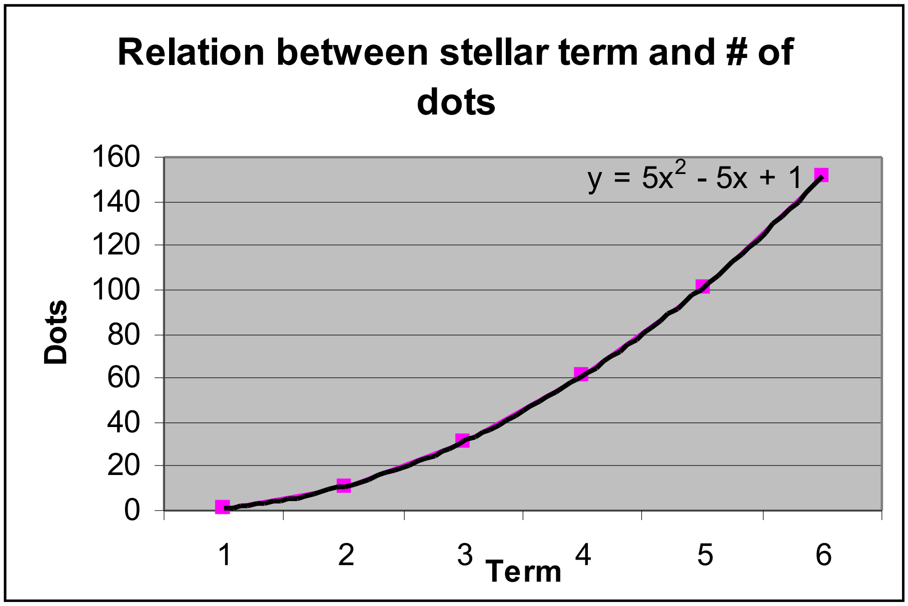


5 6

Now, let us put this information in a table:

| Term | Dots |
|---|---|
| 1 | 1 |
| 2 | 11 |
| 3 | 31 |
| 4 | 61 |
| 5 | 101 |
| 6 | 151 |

We can then find the relation between the values in the table by using the computer program Microsoft Excel and plotting them in a graph:




Once again, we can use the Display Equation on Chart option to obtain the equation of the line and thus a general statement for any term of the 5-stellar numbers. Let us consider the $n^{th}$ term, and so the general expression becomes:

$$\mathbf{d = 1 + 5(n^2 - n)}$$

We can also test the validity of this statement by trying to arrive to the same expression without the use of technology and simply using mathematical reasoning. Let us consider the following table:

| Term | Dots |
|---|---|
| 1 | 1 |
| 2 | 11 |
| 3 | 31 |
| 4 | 61 |
| 5 | 101 |
| 6 | 151 |

From this we can obtain the general expression:

1. We notice that the current values for d, the number of dots, seem to have to no apparent relation with their corresponding term.
2. However, by looking at the diagrams we are able to decipher that the dot in the middle of the stellar shapes is constant between the different terms. Thus, if we remove the constant we obtain the following table:

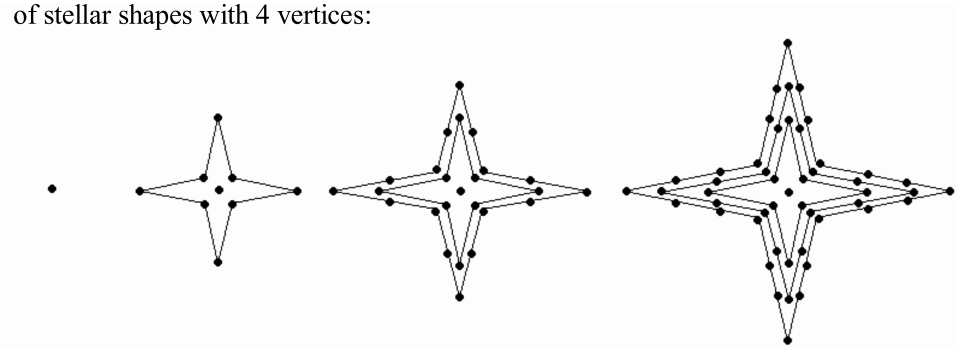
| Term | Dots |
|---|---|
| 1 | 0 |
| 2 | 10 |
| 3 | 30 |
| 4 | 60 |
| 5 | 100 |
| 6 | 150 |

3. The values for the number of dots then become multiples of 10.
4. By using this information, we can take 10 to be the base value and alter it to fit the values shown for the number of dots. From this we get that, for the first term, it is 10 x 0, for the second term it is 10 x 1, for the third term it is 10 x3 and so on.
5. We notice that 10 in each case is being multiplied by the triangular number of the previous term as the stellar number. Thus, the general expression till this point becomes $5((n-1)^2 + (n-1))$, which is the same as saying $5(n^2 - n)$.
6. If we remember to add the 1 that we had eliminated as a constant, we can obtain the general statement:

$$\mathbf{d = 1 + 5(n^2 - n)}$$

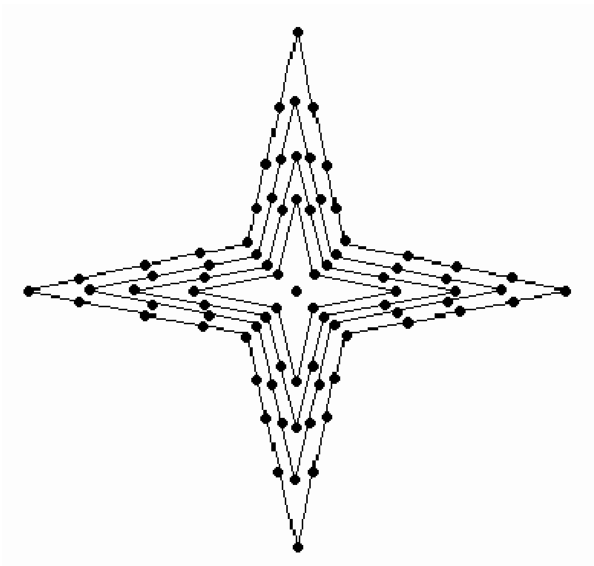
This is concordant with the expression we obtained from the graph by using Microsoft Excel. Thus, the statement is correct.

We can use the same criteria for 3-stellar numbers and 4-stellar numbers. However, since we have already verified the validity of the technology used by repeating the process with reasoning, there will be no need to do this again and thus only technology will be used.
Starting with 4-stellar numbers by using the following stellar diagrams each made up of stellar shapes with 4 vertices:

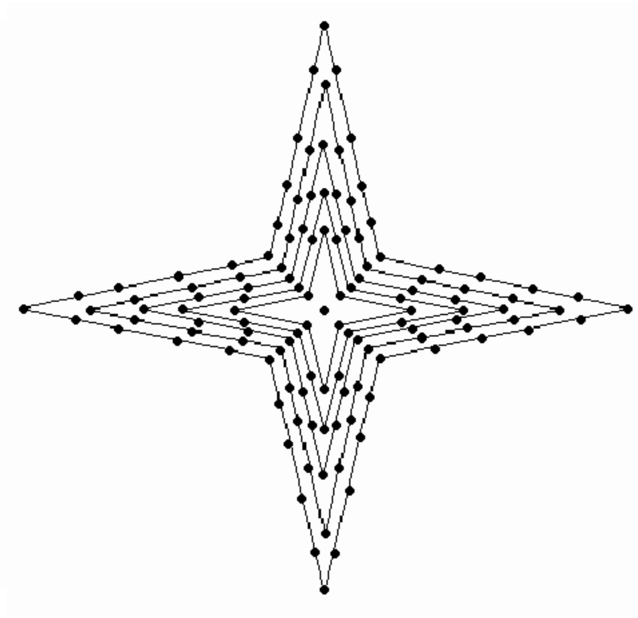


1 2 3 4

Since we are considering that we can obtain the 4-stellar numbers from the diagram, let us expand the number of shapes so as to validate our hypothesis:



5

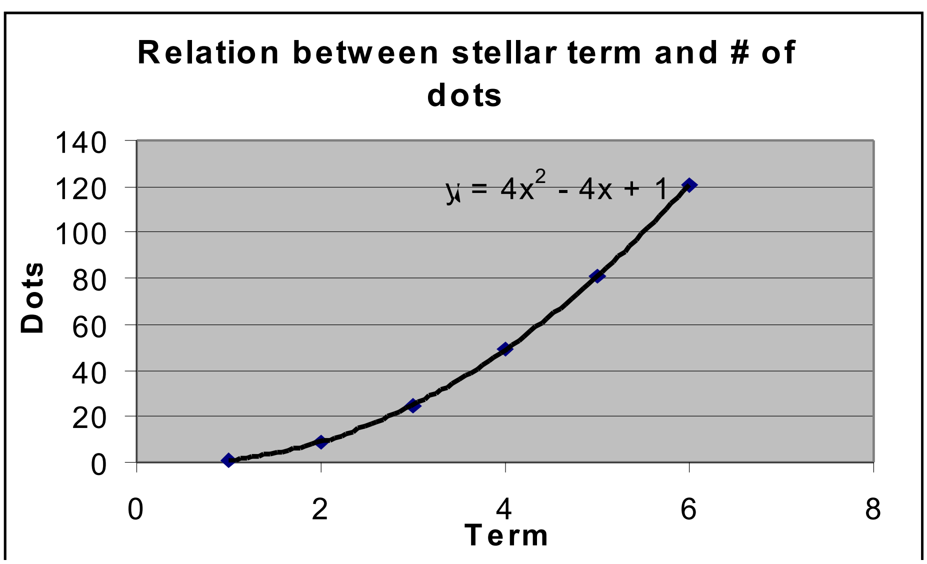


6

Now, let us put this information in a table:

| Term | Dots |
|---|---|
| 1 | 1 |
| 2 | 9 |
| 3 | 25 |
| 4 | 49 |
| 5 | 81 |
| 6 | 121 |

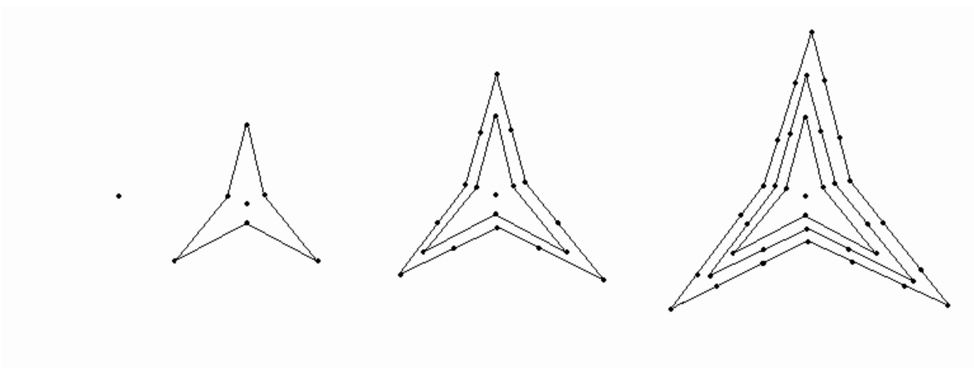
We can then find the relation between the values in the table by using the computer program Microsoft Excel and plotting them in a graph:



Once again, we can use the Display Equation on Chart option to obtain the equation of the line and thus a general statement for any term of the 4-stellar numbers. Let us consider the $n^{th}$ term, and so the general expression becomes:

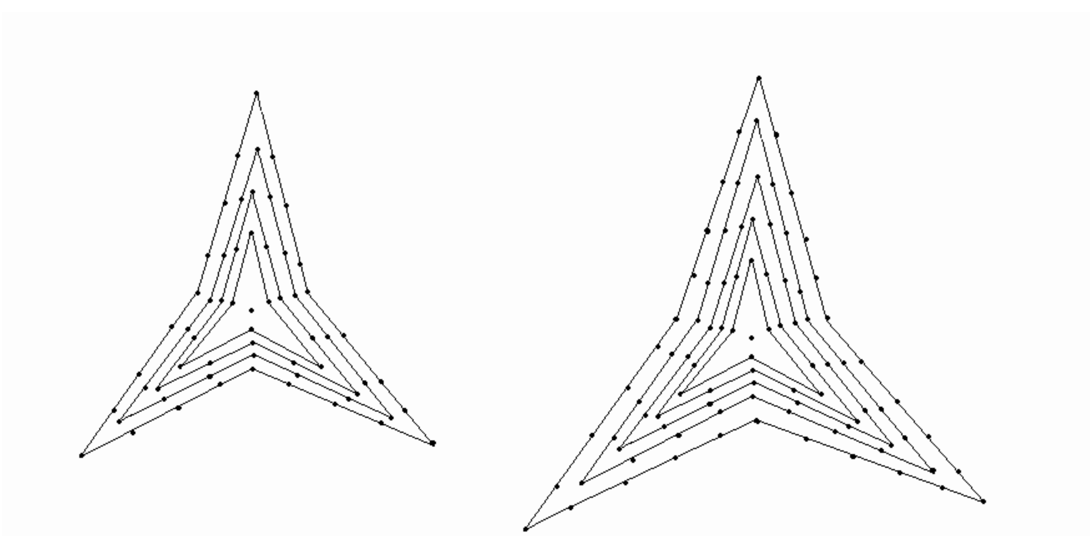
$$\mathbf{d = 1 + 4(n^2 - n)}$$

And as we do the same for the 3-stellar numbers by using the following stellar diagrams each made up of stellar shapes with 3 vertices:



1 2 3 4

Since we are considering that we can obtain the 4-stellar numbers from the diagram, let us expand the number of shapes so as to validate our hypothesis:

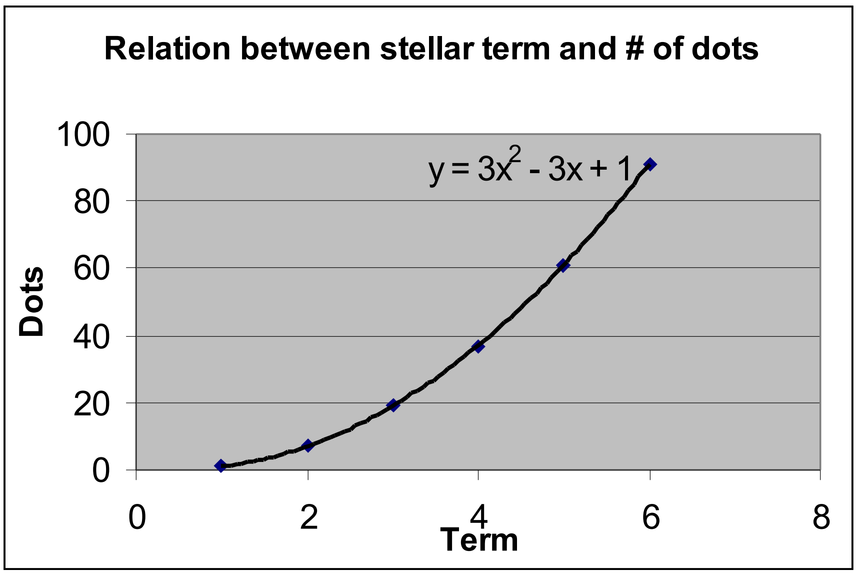


5 6

Now, let us put this information in a table:

| Term | Dots |
|---|---|
| 1 | 1 |
| 2 | 7 |
| 3 | 19 |
| 4 | 37 |
| 5 | 61 |
| 6 | 91 |

We can then find the relation between the values in the table by using the computer program Microsoft Excel and plotting them in a graph:




Once again, we can use the Display Equation on Chart option to obtain the equation of the line and thus a general statement for any term of the 3-stellar numbers. Let us consider the $n^{th}$ term, and so the general expression becomes:

$$\mathbf{d = 1 + 3(n^2 - n)}$$

Finally, from this we can obtain an easily discernable pattern. We notice how the basic expression $1 + (n^2 – n)$ is kept throughout all of the stellar numbers show with the only difference being the number that multiples $(n^2 – n)$, which corresponds to the stellar number. Thus, if we generalize the formula, we notice that the multiplying factor is the same as the p value in the p-stellar number (or the number of vertices in the stellar shapes). Thus, the general expression becomes:

$$\mathbf{d = 1 + p\,(n^2 – n)}$$

## Evaluation:

The general statement was tested with a decent range of values for p and n and it provided the correct answer for each of them. For this experiment, this is enough to validate the statement as it is far too time consuming to simply sketch a diagram for every term of any p-stellar number or when there are far too many vertices in the stellar shape. The program Microsoft Excel was used to illustrate the relation between two or more variables through graphs. It also provided a means of easily obtaining the equation of a trend in the form of the Display Equation on Chart option. This was crucial when validating the results.

## Conclusion:

The general statement can be used in almost any situation since 'n' is a value decided by the individual, in the sense that when asking "how many d dots are there in the diagram of the $n^{th}$ term of the p-stellar numbers?" the 'd' value acts as a dependent variable while the 'n' and 'p' values act as independent.
The limitations of this experiment include that the most reliable way to validate the patterns is to draw the stellar shape diagrams. However, drawing a lot of them out would be far too time consuming. Additionally, counting dots is a rather inaccurate method of obtaining values for d which could, in the end, potentially alter the results. This investigation was carried out by analyzing three different variables and trying to understand the mathematical relationship between them. Before starting, an analysis of other types of special numbers and their origin from geometric shapes was conducted so as to validate the method. The statement was found after first observing the effect of an increase in term in the number of dots present in the stellar shape and then doing the same for the number of vertices. Finally, an analysis of the variation in the general expression of the original p-stellar sequence allowed a pattern to be found that helped form an equation that included them both.