Olivia Bloch Copenhagen International School 30.05.2011

The aim of this task is to investigate geometric shapes, which lead to special numbers. The simplest example of these are square numbers, such as 1, 4, 9, 16, which can be represented by squares of side 1, 2, 3, and 4.

Triangular numbers are defined as "the number of dots in an equilateral triangle uniformly filled with dots". The sequence of triangular numbers are derived from all natural numbers and zero, if the following number is always added to the previous as shown below, a triangular number will always be the outcome:

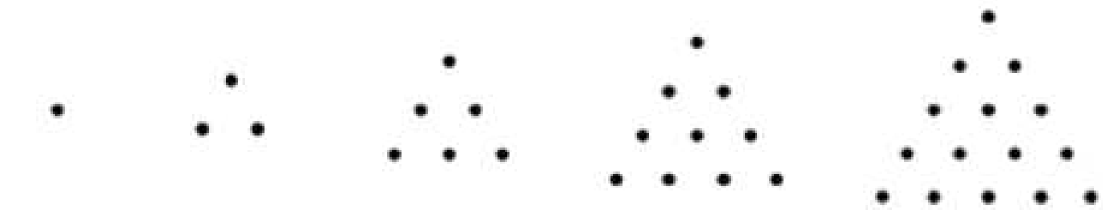
$1 = 1$
$2 + 1 = 3$
$3 + (2 + 1) = 6$
$4 + (1 + 2 + 3) = 10$
$5 + (1 + 2 + 3 + 4) = 15$

Moreover, triangular numbers can be seen in other mathematical theories, such as Pascal's triangle, as shown in the diagram below. The triangular numbers are found in the third diagonal, as highlighted in red.

1
1 1
1 2 **1**
1 **3** **3** 1
1 4 **6** 4 1
1 5 **10** **10** 5 1
1 6 **15** 20 **15** 6 1

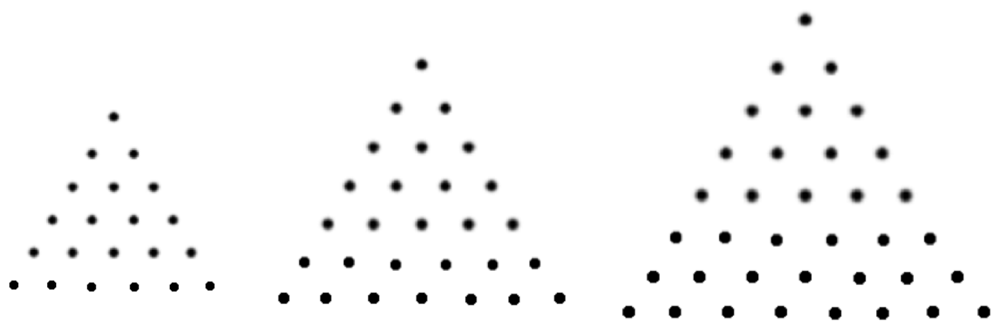
Diagram 0.1

The first diagrams to be considered show a triangular pattern of evenly spaced dots, and the number of dots within each diagram represents a triangular number.



1 Diagram 1 | 3 Diagram 2 | 6 Diagram 3 | 10 Diagram 4 | 15 Diagram 5

Thereafter, the sequence was to be developed into the next three terms as shown below.



21 Diagram 6 | 28 Diagram 7 | 36 Diagram 7.1

The information from the diagrams above is represented in the table below.

Term Number (n)	1	2	3	4	5	6	7	8
Triangular Number (T_n)	1	3	6	10	15	21	28	36

Establishing the following three terms in the sequence was done by simply drawing another horizontal row of dots to the previous equilateral and adding those dots to the previous count. However, following the method described earlier can also do this calculation, as shown in the illustration below.

T_1 $1 = 1$
T_2 $2 + 1 = 3$
T_3 $3 + (2 + 1) = 6$
T_4 $4 + (1 + 2 + 3) = 10$
T_5 $5 + (1 + 2 + 3 + 4) = 15$
T_6 $6 + (1 + 2 + 3 + 4 + 5) = 21$
T_7 $7 + (1 + 2 + 3 + 4 + 5 + 6) = 28$
T_8 $8 + (1 + 2 + 3 + 4 + 5 + 6 + 7) = 36$

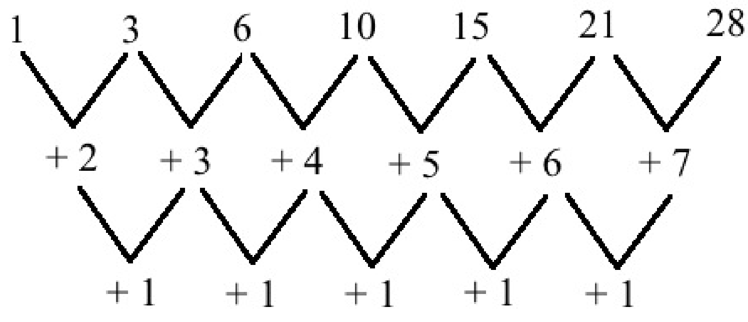


Diagram 8

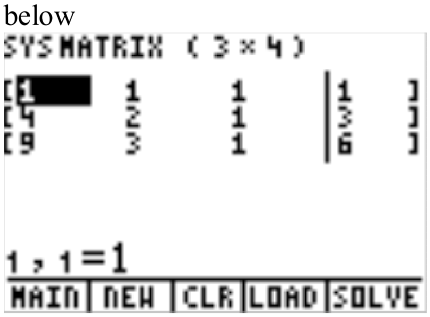
As seen in the diagram above, the second difference is the same between the terms, and the sequence is therefore quadratic. This means that the equation $T_n = an^2 + bn + c$ will be used when representing the data in a general formula. Since some of the values of T_n have already been established this makes it possible to work out the general formula. The first step is to substitute the established values into the three quadratic equations, as shown below:

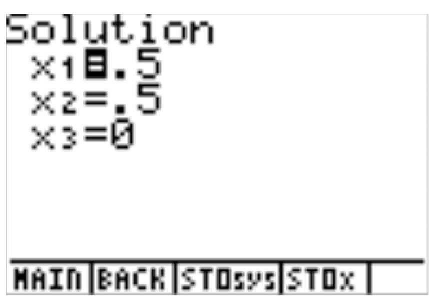
When n = 1, $T_n = 1$
$1 = a(1)^2 + b(1) + c$
$1 = 1a + 1b + c$
$1 = a + b + c$

When n = 2, $T_n = 3$
$3 = a(2)^2 + b(2) + c$
$3 = 4a + 2b + c$

When n = 3, $T_n = 6$
$6 = a(3)^2 + b(3) + c$
$6 = 9a + 3b + c$

Thereafter polysmlt on the graphic display calculator is used in order to retrieve the values of a, b, and c. The coefficients above are simply plugged into the calculator; this can be seen in the screenshots below





Hence the general formula is T_n = $.5n^2 + .5n$, which can also be stated as T_n =

To be certain that the general formula in in fact correct, it can also be worked out as shown below.

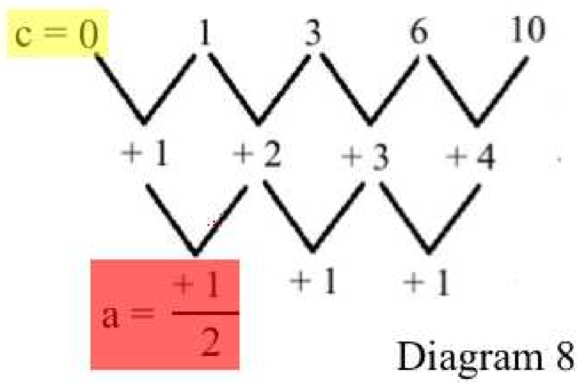


Diagram 8.1

The second difference of the sequence is divided by 2, since this is the value of a, thereafter the sequence is extended to T_0, as the number highlighted in red is c. To figure out the value of b, the values have been substituted into the quadratic formula as shown below, after which the equation has been solved using simple algebra.

$T_n = an^2 + bn + c$
$T_n = ½n^2 + bn + 0$
$T_1 = (½)(1)^2 + b(1)$
$1 = ½ + b$
$b = ½$

□ T_n =

As seen above, this method works out the same general formula, and this is tested below to assess the validity.

T_n = $T_3 = (3^2/2) + (3/2)$ $T_3 = (9/2) + (3/2)$ $T_3 = 12/2$ $T_3 = 6$	T_n = $T_5 = (5^2/2) + (5/2)$ $T_5 = (25/2) + (5/2)$ $T_5 = 30/2$ $T_5 = 15$	**T_n =** **$T_{12} = (12^2/2) + (12/2)$** **$T_{12} = (144/2) + (12/2)$** **$T_{12} = 156/2$** **$T_{12} = 78$**

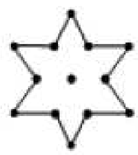
As seen above the general formula is indeed correct; however, this can also be determined by the method continued from page 2…

T_9 $9 + (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) = 45$
T_{10} $10 + (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 55$
T_{11} $11 + (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) = 66$
T_{12} **$12 + (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11) = 78$**

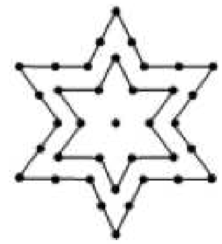
This furthermore proves the validity of the formula, as the 12^{th} term is unknown, and not used in order to determine the general formula. This means that the general formula does not just work for a limited number of values for n.

After establishing the general formula for the triangular numbers, stellar (star) shapes with p vertices leading to p-stellar numbers were to be considered. The diagrams below represent shapes with 6 vertices, and the pattern involved.

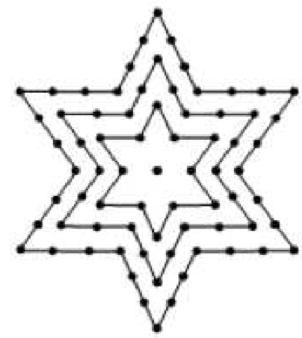
S_1
Diagram 8



S_2
Diagram 9

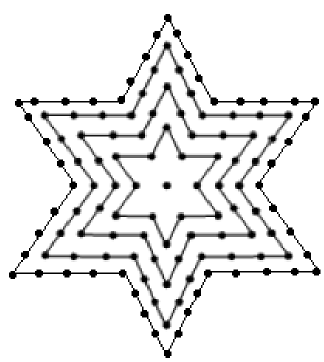


S_3
Diagram 10

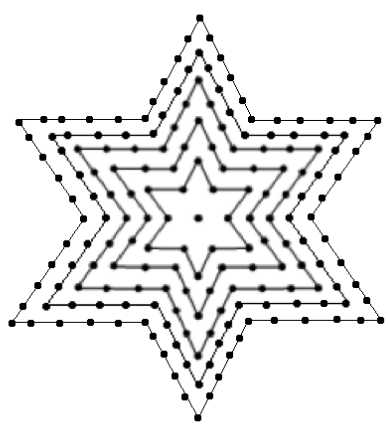


S_4
Diagram 11

After studying the relationship between these 4 diagrams, the next two terms in the sequence were added accordingly.



S_5
Diagram 12



S_5
Diagram 13

The 6-stellar number at each stage represents the number of dots in each of the diagram. This information is represented in the table on the next page. Instead of counting the dots in diagrams 12 and 13, I considered the pattern seen in the first four terms. Since the second difference (illustrated in diagram 14) is 12, 12 was added to 36, which would then be the first difference 48, 48 is then added to 73 as this will be the 5th term in the sequence. The same was then done with term 6.

Term Number (n)	1	2	3	4	5	6
Stellar Number (S_n)	1	13	37	73	121	181

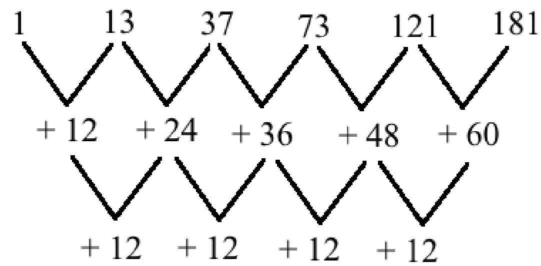


Diagram 14

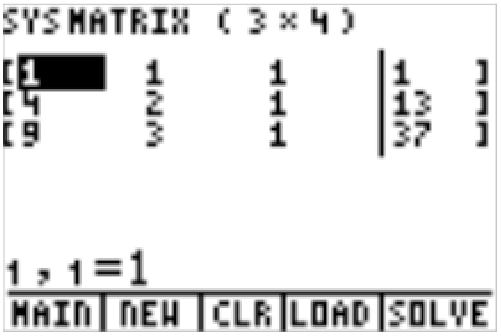
As seen in the diagram above, the second difference is the same between the terms, and the sequence is therefore quadratic. This means that the equation $S_n = an^2 + bn + c$ will be used when representing the data in a general formula. Since some of the values of S_n have already been established this makes it possible to work out the general formula. The first step is to substitute the established values into the three quadratic equations, as shown below:

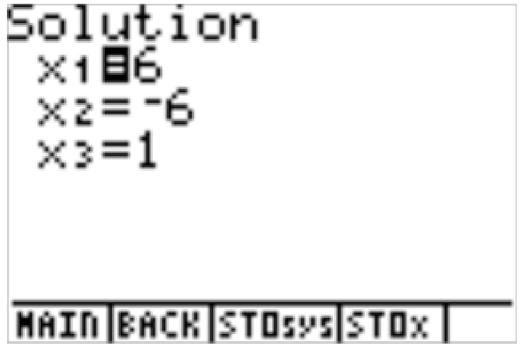
$S_n = an^2 + bn + c$
Therefore: quadratic
When n = 1, $S_n = 1$
$1 = a(1)^2 + b(1) + c$
$1 = 1a + 1b + c$
$1 = a + b + c$

When n = 2, $S_n = 13$
$13 = a(2)^2 + b(2) + c$
$13 = 4a + 2b + c$

When n = 3, $S_n = 6$
$37 = a(3)^2 + b(3) + c$
$37 = 9a + 3b + c$

Thereafter polysmlt on the graphic display calculator is used in order to retrieve the values of a, b, and c. The coefficients above are simply plugged into the calculator; this can be seen in the screenshots below





Hence the general formula for the sequence is $S_n = 6n^2 - 6n + 1$, to be certain that this is in fact correct, it can also be worked out as shown below.

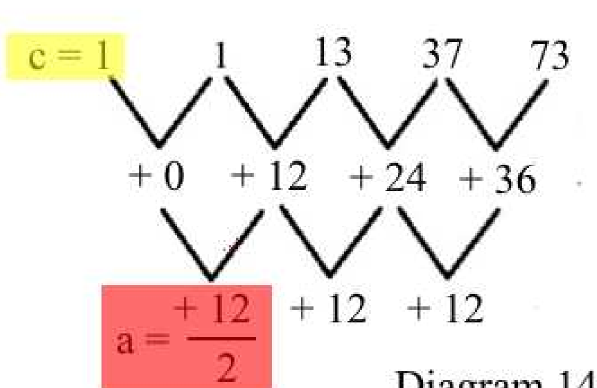


Diagram 14.1

The second difference of the sequence is divided by 2, since this is the value of a, thereafter the sequence is extended to T_0, as the number highlighted in red is c. To figure out the value of b, the values have been substituted into the quadratic formula as shown below, where after the equation has been solved using simple algebra.

$S_n = an^2 + bn + c$
$S_n = 6n^2 + bn + 1$
$S_1 = (6)(1)^2 + b(1) + 1$
$1 = 6 + b + 1$
$b = - 6$
□ $S_n = 6n^2 - 6n + 1$

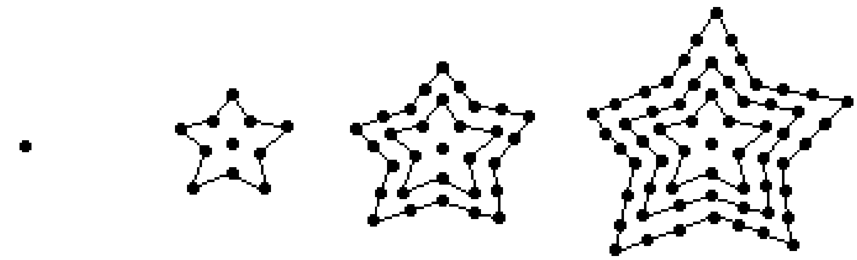
As seen above, this method works out the same general formula, and this is tested below to assess the validity.

$S_n = 6n^2 - 6n + 1$	$S_n = 6n^2 - 6n + 1$

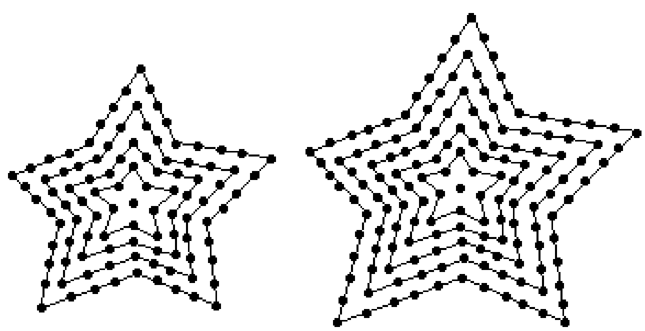
$S_3 = 6(3)^2 - 6(3) + 1$	$S_5 = 6(5)^2 - 6(5) + 1$
$S_3 = 6(9) - 18 + 1$	$S_5 = 6(25) - 30 + 1$
$S_3 = 54 - 17$	$S_5 = 150 - 29$
$S_3 = 37$	$S_5 = 121$

The results derived from the general formula are the same as worked out earlier when not applying any formula at all; hence, it is correct and can be applied to terms 3 and 5 and also other values of n.

The previous example was then repeated using other values of p; hence, the number of vertices is being changed from 6 to 5. The 5-stellar number at each stage represents the number of dots in each of the diagrams below.



V_1 Diagram 29 V_2 Diagram 30 V_3 Diagram 31 V_4 Diagram 32



V_5 Diagram 33 V_6 Diagram 34

The information in the diagrams above was collected and is represented in the table below.

Term Number (n)	1	2	3	4	5	6
Stellar Number (V_n)	1	11	31	61	101	151

This information is considered and the pattern in the sequence is represented below, and a quadratic pattern throughout the investigation has become very obvious.

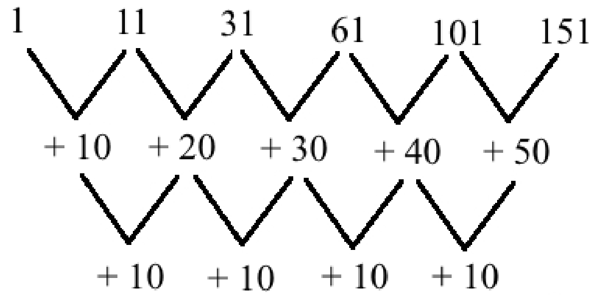


Diagram 35

As seen in the diagram above, the second difference is the same between the terms, and the sequence is therefore quadratic. This means that the equation $V_n = an^2 + bn + c$ will be used when representing the data in a general formula. Since some of the values of V_n have already been established this makes it possible to work out the general formula. The first step is to substitute the established values into the three quadratic equations, as shown below:

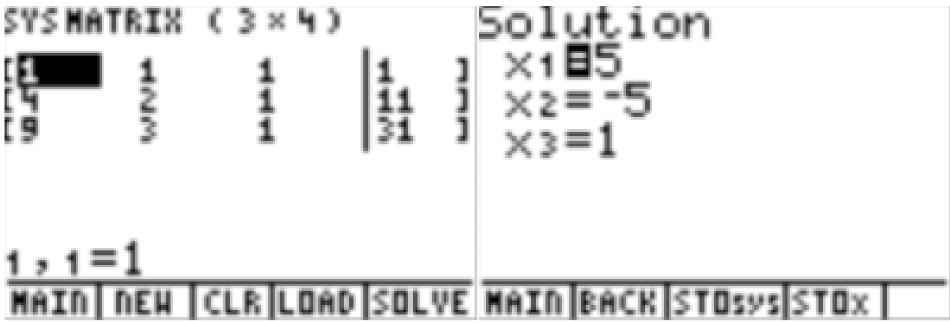
$R_n = an^2 + bn + c$
Therefore: quadratic
When n = 1, $V_n = 1$
$1 = a(1)^2 + b(1) + c$
$1 = 1a + 1b + c$
$1 = a + b + c$

When n = 2, $V_n = 13$
$11 = a(2)^2 + b(2) + c$
$11 = 4a + 2b + c$

When n = 3, $V_n = 6$
$31 = a(3)^2 + b(3) + c$
$31 = 9a + 3b + c$

As seen above, this method works out the same general formula, and this is tested below to assess the validity.

Thereafter polysmlt on the graphic display calculator is used in order to retrieve the values of a, b, and c. The coefficients above are simply plugged into the calculator; this can be seen in the screenshots below



Hence, the general formula for this sequence is $R_n = 5n^2 - 5n + 11$

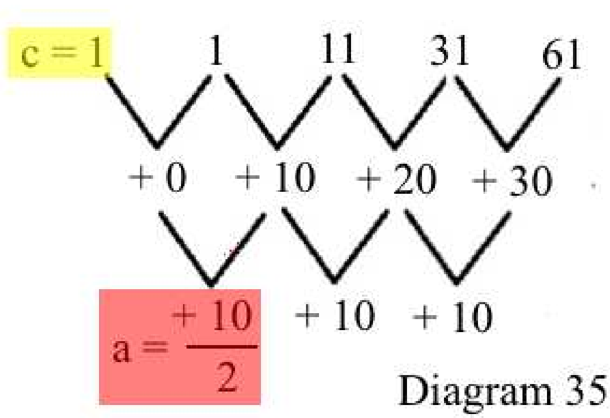


Diagram 35.1

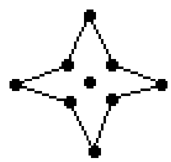
$V_n = an^2 + bn + c$
$V_n = 6n^2 + bn + 1$
$V_1 = (5)(1)^2 + b(1) + 1$
$1 = 5 + b + 1$
$b = - 5$
□ $S_n = 5n^2 - 5n + 1$

$V_n = 5n^2 - 5n + 1$	$V_n = 5n^2 - 5n + 1$
$V_3 = 5(3)^2 - 5(3) + 1$	$V_5 = 5(5)^2 - 5(5) + 1$
$V_3 = 5(9) - 15 + 1$	$V_5 = 5(25) - 25 + 1$
$V_3 = 45 - 14$	$V_5 = 125 - 24$
$V_3 = 31$	$V_5 = 101$

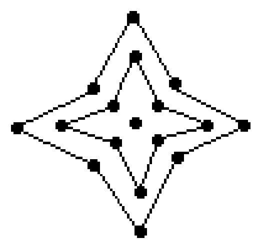
The results derived from the general formula are the same as worked out earlier when not applying any formula at all; hence, it is correct and can be applied to terms 3 and 5.

This was repeated using yet again another value for p, this time changing the p value from 5 to 4, resulting in the diagrams below.

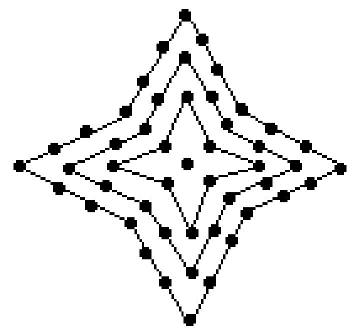
R_1
Diagram 22



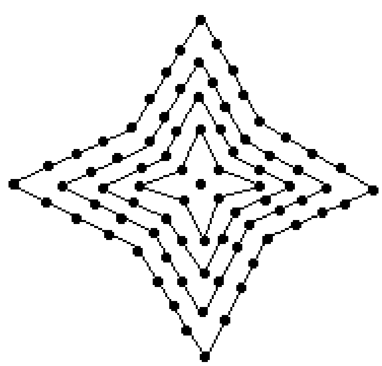
R_2
Diagram 23



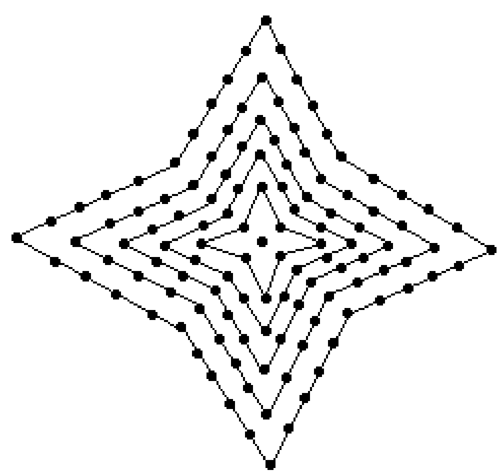
R_3
Diagram 24



R_4
Diagram 25



R_5
Diagram 26



R_6
Diagram 27

The information from the diagrams above was collected and is represented in the table below.

Term Number (n)	1	2	3	4	5	6
Stellar Number (R_n)	1	9	25	49	81	121

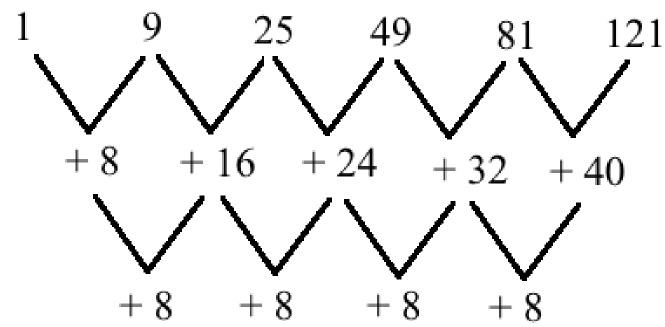


Diagram 28

As seen in the diagram above, the second difference is the same between the terms, and the sequence is therefore yet again quadratic. This means that the equation $R_n = an^2 + bn + c$ will be used when representing the data in a general formula. Since some of the values of R_n have already been established this makes it possible to work out the general formula. The first step is to substitute the established values into the three quadratic equations, as shown below:

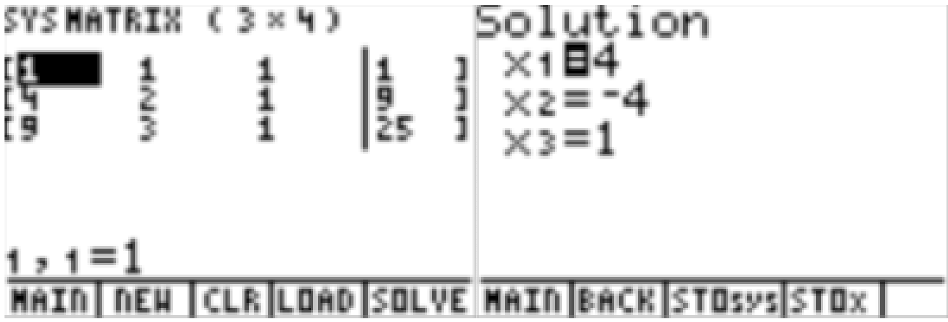
$R_n = an^2 + bn + c$
Therefore: quadratic
When n = 1, $R_n = 1$
$1 = a(1)^2 + b(1) + c$
$1 = 1a + 1b + c$
$1 = a + b + c$

When n = 2, $R_n = 13$
$9 = a(2)^2 + b(2) + c$
$9 = 4a + 2b + c$

When n = 3, $R_n = 6$
$25 = a(3)^2 + b(3) + c$
$25 = 9a + 3b + c$

As seen above, this method works out the same general formula, and this is tested below to assess the validity.

Thereafter polysmlt on the graphic display calculator is used in order to retrieve the values of a, b, and c. The coefficients above are simply plugged into the calculator; this can be seen in the screenshots below



This resulted in the general formula $R_n = 4n^2 - 4n + 1$

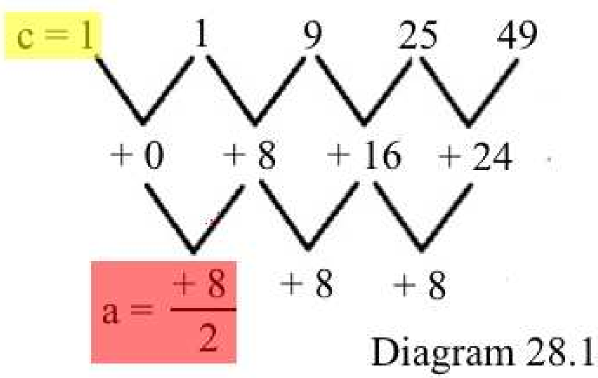


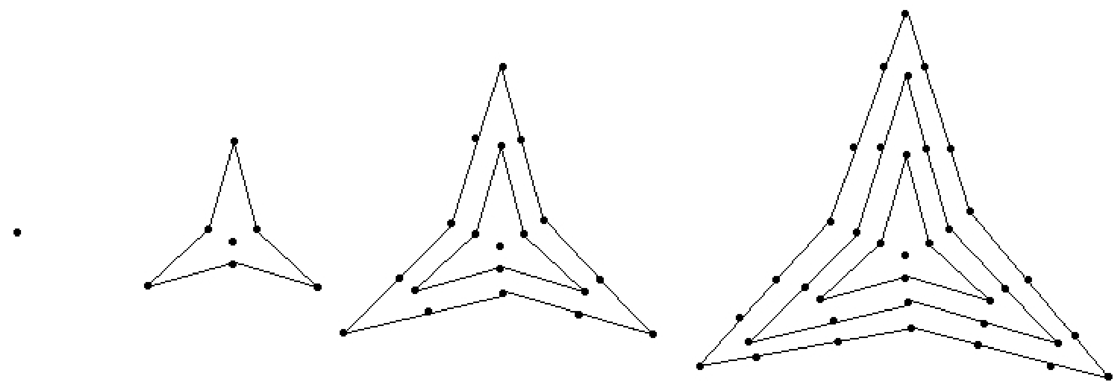
Diagram 28.1

$R_n = an^2 + bn + c$
$R_n = 4n^2 + bn + 1$
$R_1 = (4)(1)^2 + b(1) + 1$
$1 = 4 + b + 1$
$b = - 4$
□ $R_n = 4n^2 - 4n + 1$

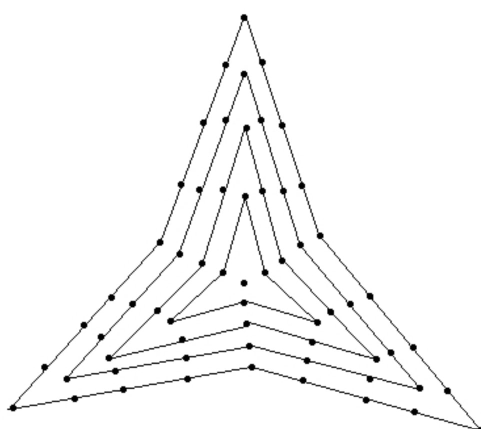
$V_n = 4n^2 - 4n + 1$ $V_3 = 4(3)^2 – 4(3) + 1$ $V_3 = 4(9) – 12 + 1$ $V_3 = 36 – 11$ $V_3 = 25$	$V_n = 4n^2 - 4n + 1$ $V_5 = 4(5)^2 – 4(5) + 1$ $V_5 = 4(25) – 20 + 1$ $V_5 = 100 – 19$ $V_5 = 81$

The results derived from the general formula are the same as worked out earlier when not applying any formula at all; hence, it is correct and can be applied to terms 3 and 5.

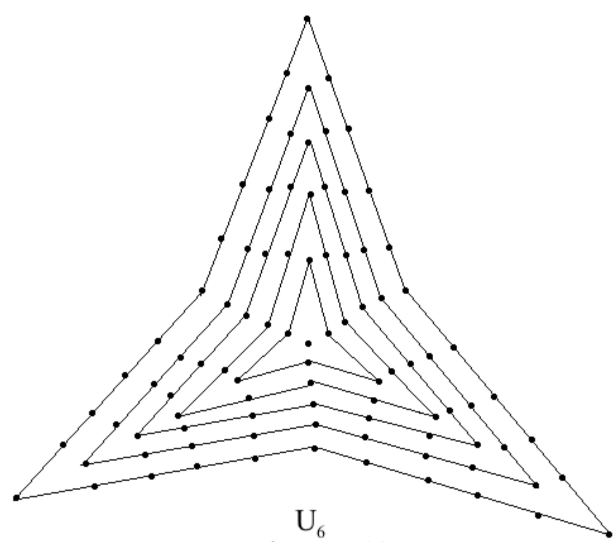
This was repeated using yet again another value for p, this time changing the p value from 4 to 3, resulting in the diagrams below.



U_1 Diagram 14 U_2 Diagram 15 U_3 Diagram 16 U_4 Diagram 17



U_5
Diagram 18



U_6
Diagram 19

The information from the diagrams above was collected and is represented in the table below.

Term Number (n)	1	2	3	4	5	6
Stellar Number (U_n)	1	7	19	37	61	91

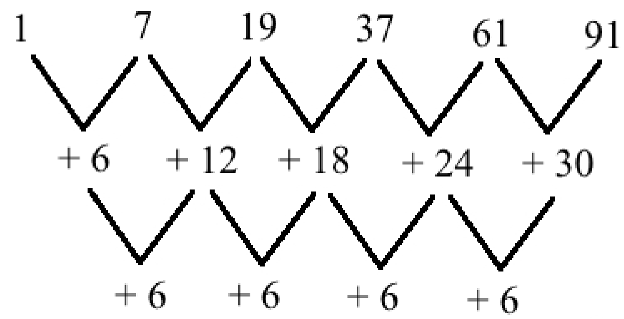


Diagram 20

As seen in the diagram above, the second difference is the same between the terms, and the sequence is therefore yet again quadratic. This means that the equation $U_n = an^2 + bn + c$ will be used when representing the data in a general formula. Since some of the values of U_n have already been established this makes it possible to work out the general formula. The first step is to substitute the established values into the three quadratic equations, as shown below:

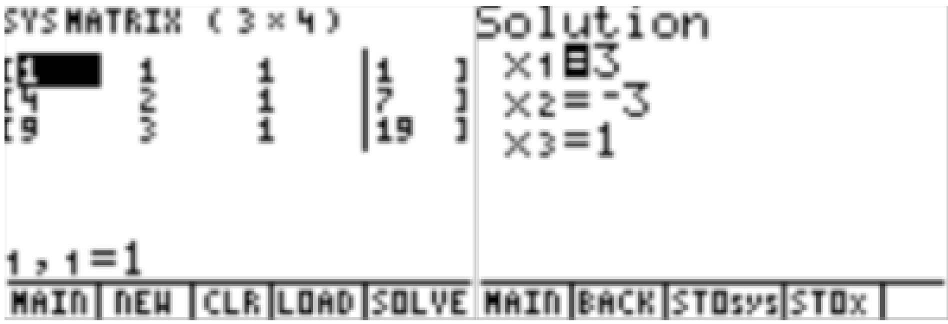
$U_n = an^2 + bn + c$
Therefore: quadratic
When n = 1, $U_n = 1$
$1 = a(1)^2 + b(1) + c$
$1 = 1a + 1b + c$
$1 = a + b + c$

When n = 2, $U_n = 13$
$7 = a(2)^2 + b(2) + c$
$7 = 4a + 2b + c$

When n = 3, $U_n = 6$
$19 = a(3)^2 + b(3) + c$
$19 = 9a + 3b + c$

As seen above, this method works out the same general formula, and this is tested below to assess the validity.

Thereafter polysmlt on the graphic display calculator is used in order to retrieve the values of a, b, and c. The coefficients above are simply plugged into the calculator; this can be seen in the screenshots below



Thus the general formula for this quadratic sequence $U_n = 3n^2 - 3n + 1$

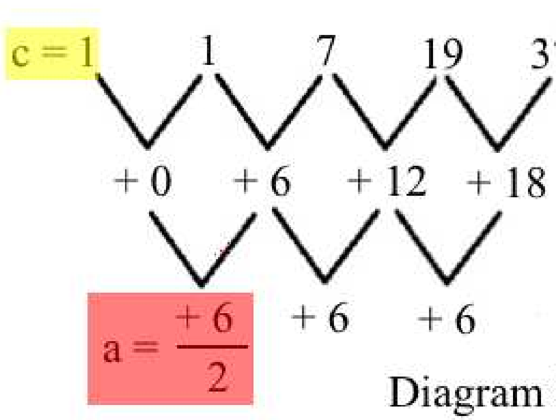


Diagram 20.1

$S_n = an^2 + bn + c$
$S_n = 3n^2 + bn + 1$
$S_1 = (3)(1)^2 + b(1) + 1$
$1 = 3 + b + 1$
$b = - 3$
 $S_n = 3n^2 - 3n + 1$

$U_n = 3n^2 - 3n + 1$	$U_n = 3n^2 - 3n + 1$
$U_3 = 3(3)^2 – 3(3) + 1$	$U_5 = 3(5)^2 – 3(5) + 1$
$U_3 = 3(9) – 9 + 1$	$U_5 = 3(25) – 15 + 1$
$U_3 = 27 – 8$	$U_5 = 75 – 14$
$U_3 = 19$	$U_5 = 61$

The results derived from the general formula are the same as worked out earlier when not applying any formula at all; hence, it is correct and can be applied to terms 3 and 5.

As mentioned earlier, a pattern can be detected when working with stellar numbers. It has become clear that in the quadratic equation $T_n = an^2 + bn + c$, a and b are the same coefficients; however, a is positive and b is negative. Furthermore the value of the coefficient is in both cases simply the p-value, for example when p=3 $S_n = 3n^2 - 3n + 1$, and when p=5 $S_n = 5n^2 - 5n + 1$. (I will now use S_n to define all the terms, as I am trying to produce the general statement that generates the sequence p-stellar numbers for any values of p at stage S_n.) It can thereby be claimed that when p=7 $S_n = 7n^2 - 7n + 1$, even though this has not been tested and drawn, but this seems to be a logical formula when studying previous results and applying mathematical reasoning. Therefore, it can be claimed that p replace a and b; hence, the general statement that generates the sequence p-stellar numbers for any values of p at stage S_n is $S_n = pn^2 - pn + 1$. The relationship between p and n can further be studied and tested in excel as seen below.

Term Number (n)	1	2	3
Stellar Number (S_n)	1	13	37

fx = E5*D5^2-(E5*D5)+1

C	D	E	F
	n	p	
	1	6	1
	2	6	13
	3	6	37

As seen on the left, the values in the table on the top correspond to the excel document on the bottom. I plugged in the general formula discussed above on the far right hand side, and the outcomes were as displayed on the left. Using other values of p and n, which would allow for a more detailed observation, extended this study, which can be seen below. To ensure that the formula in excel did indeed calculate the correct values, I set up the previous results up to compare, and they did correspond.

Term Number (n)	4	5	6
Stellar Number (V_n)	61	101	151

fx = E4*D4^2-(E4*D4)+1

C	D	E	F
	n	p	
	4	5	61
	5	5	101
	6	5	151

Term Number (n)	1	2	3
Stellar Number (R_n)	1	9	25

fx = E6*D6^2-(E6*D6)+1

C	D	E	F
	n	p	
	1	4	1
	2	4	9
	3	4	25

Term Number (n)	4	5	6
Stellar Number (U_n)	37	61	91

fx = E4*D4^2-(E4*D4)+1

C	D	E	F
	n	p	
	4	3	37
	5	3	61
	6	3	91

This investigation has considered geometric shapes in order to determine how many dots are in the different types of shapes, in this case from triangular to stellar, furthermore extending the investigatory work by changing the stellar number (the value of p). The limitations of this investigation is that the value of n must always be equal to or greater than 1, as it is impossible to have a negative triangle or stellar shape, as it is a physical entity, and it is therefore also impossible to achieve a negative value. In this instance, the values have to be whole numbers as all the dots are whole numbers, and by adding whole numbers together it is impossible to achieve the result of a decimal or a fraction that cannot be simplified. Additionally, all the values of n have not been tested, which means that we cannot be certain that it works for all the infinite values.