IB 2 Mathematical Portfolio:
Stellar Numbers

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Introduction:

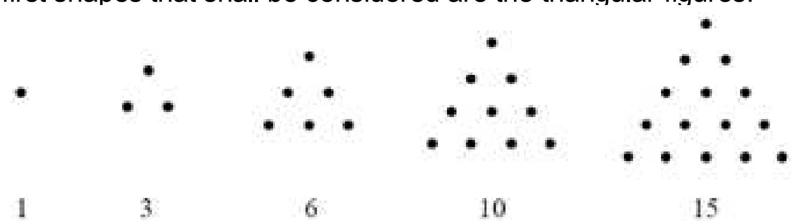
Certain geometric shapes can yield special types of numbers. The simplest examples of these numbers would square numbers 1, 4, 9, 16, which are the squares of the valu es 1, 2, 3 and 4 as demonstrated by this table and by the formula $n = m^2$. Henceforth, all tables generated are from the Numbers program by the macintosh company and all mathematical notation is from the program Mathtype 6.0. Any graph s displayed will be generated from the graph program “Graph 4.3.”

n	m^2	$S_{n+1} - S_n = D_n$	$D_{n+1} - D_n = E_n$
1	1	3	2
2	4	5	2
3	9	7	
4	16		

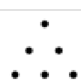
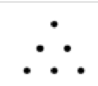
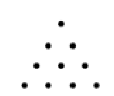
In this investigation the following geometric shapes shall be considered for investigation in order to determine how many dots are in ea ch type of shape from triangular figures to stellar (star) figures. With the ultimate goal being a general encompassing statement in order to determine the number of dots in any star with p - vertices in any n-stage.

<u>Triangular Numbers (Total number of dots in a triangle)</u>

The first shapes that shall be considered are the triangular figures:



When the values of the triangles (the number of dots) are input into a table:

Stage of triangle	**Image of Triangular shape**	n	S_n	**Counted number of dots**	$S_{n+1} - S_n = D_n$	$D_{n+1} - D_n = E_n$
S_1		1	1	1	2	1
S_2		2	3	3	3	1
S_3		3	6	6	4	1
S_4		4	10	10	5	
S_5		5	15	15		

<u>The variables will be defined the same for tables to do with triangluar numbers:</u>

-n will be defined as the stage number of the triangle
-S_n as the nth stage of the triangle

The variables will be defined the same for all tables hereafter:
-D_n as the difference between the two terms of the stages S_{n+1} and S_n
-E_n as the difference between the two terms of $D_{n+1} - D_n$

From the table it can be determined that there is no common difference (D_n) in the sequence of terms (S_n) meaning, that an arithmetic sequence formula cannot model the values produced by subsequent triangles, as if it could then the difference between each term would be constant e.g. (n+1,n+2,n+3...n etc.) and neither can it be geometric as there is no common ratio for the values. On the other hand the difference between the two terms $D_{n+1} - D_n$ the value E_n is constant as it is: (n+1),(n+2),(n+3)...n. The pattern that S_n follows is represented in this chart:

Stage of triangle	n	S_n
S_1	1	1
S_2	2	1+2
S_3	3	1+2+3
S_4	4	1+2+3+4
S_5	5	1+2+3+4+5
S_n	n	1+2+3+...+(n-2)+(n-1)+n

This pattern is clearly different as each subsequent value added to the next term is +1 greater than the value of the term before. Hence from this the next 3 triangle stages may be derived. In the 6th, 7th and 8th stages, the number of dots become respectively: 21, 28, 36. As demonstrated by this table:

Stage of triangle	n	S_n	Counted number of dots	$S_{n+1} - S_n = D_n$	$D_{n+1} - D_n = E_n$
S_6	6	21	21	7	1
S_7	7	28	28	8	1
S_8	8	36	36		

Thus, from the above further data, a general statement may be formed from finding S_n, which will now be F(n) to the nth term.

$$F(n) = 1 + 2 + 3 + ... + (n-2) + (n-1) + n$$

$+$

$$F(n) = n + (n-1) + (n-2)... + 3 + 2 + 1$$

n will be defined as the stage number of the triangle
F(n) will be defined as the function of the nth triangle

In the above the equation: $F(n) = 1 + 2 + 3 + ... + (n-2) + (n-1) + n$ appears twice once as normally viewed and the other time as $F(n) = n + (n-1) + (n-2)... + 3 + 2 + 1$ the reason is that they are being added together and the reason for them being added together is that to get the equation there must be a visualization, that the triangle is half of another triangle thus forming a square.

$$F(n) + F(n) = (1 + 2 + 3 + ... + (n-2) + (n-1) + n) +$$

$$n + (n-1) + (n-2)... + 3 + 2 + 1$$

$$\therefore 2F(n) = (n+1) + (n+2-1) + ... + (n-2+1) + (n+1)$$

hence the resulting number of dots would be half the number of the formed squares dots, which also means that the dots of two triangles is equal to one square.
Then,

$$2F(n) = (n+1) + (n+2-1) + ... + (n-2+1) + (n+1)$$

$$2F(n) = n(n+1)$$

$$\therefore F(n) = \frac{n(n+1)}{2}$$

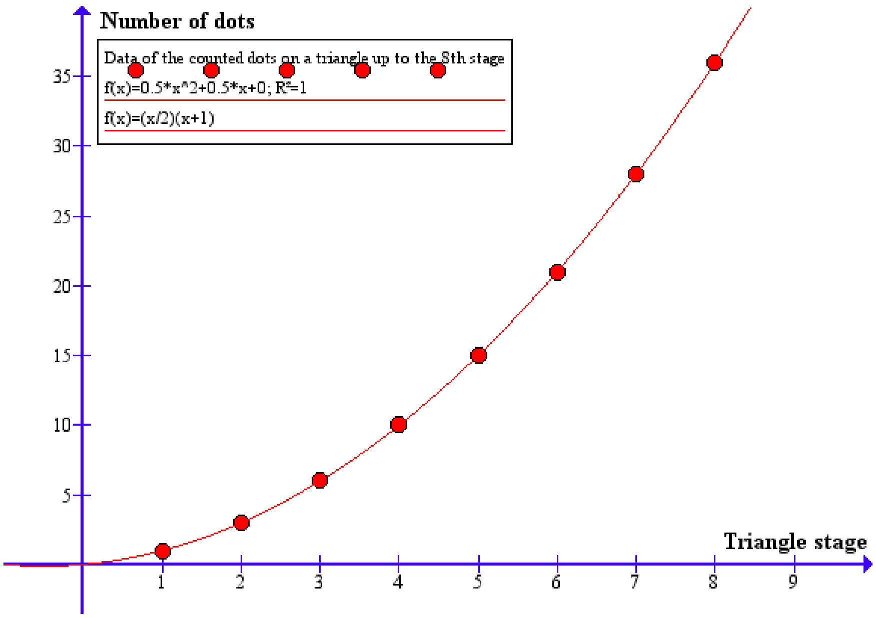
as there are "n" quantities of "n+1" the equation becomes "n(n+1)". Then the whole equation is divided by 2 to make the equation representative for the trian gle rather than the square.

The final general statement for the triangle will be expressed in terms of n:

$$\therefore F(n) = \frac{n(n+1)}{2}$$

if the general statement for the nth triangle were multiplied all together then the equation would become the quadratic function: $F(n) = 0.5n^2 + 0.5n + 0$. In order to test for validity the graphing program "Graph 4.3" was used to demonstrate the validity of the general statement. As demonstrated by the graph below, both equations satisfy the general statement for the triangular numbers.

<u>Figure 1: Graph of general statement of triangles with data of the 8 stages of the triangle</u>

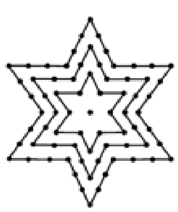


As shown in Figure 1, the R squared value is at 1 as shown on the graph which means that the formula created passes through all the given points perfectly. The R squared value was obtained through the "insert trend line function" of the "Graph 4.3" program. Thus, both the first and second function are valid and therefore both functions are verified. The second function could have been obtained through the use of technology, but instead it was obtained through another medium that of the multiplication of all terms of n in the obtained general statement, but both functions are essentially the same.

Stellar Numbers (Total number of dots in stellar number)

The second shapes that shall be considered are the stellar shapes with 6 -vertices.

The first stellar shapes to be considered have 6 -vertices. The counted values of the different stages of the stellar shapes are summed up in the table below:

S_n	Image of Stellar shape	n	Counted number of dots	$S_{n+1} - S_n = D_n$	$D_{n+1} - D_n = E_n$
S_1		1	1	12	12
S_2		2	13	24	12
S_3		3	37	36	
S_4		4	73		

The variables will be defined the same for tables to do with triangluar numbers:

-n will be defined as the stage number of the stellar shape

-S_n as the nth stage of the stellar shape

From the table it can be seen that there is no common difference (D_n) in the sequence of terms (S_n). On the other hand the difference between the two terms $D_{n+1} - D_n$ the value E_n is constant (12n+12),(12n+24),(12n+36)+...+12n. The pattern that S_n follows is represented in this chart:

Stage of stellar shape	n	S_n
S_1	1	1
S_2	2	1+12
S_3	3	1+12+24
S_4	4	1+12+24+36
S_n	n	1+12+24+...+(12n-24)+(12n-12)+12n

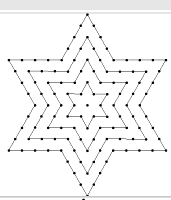
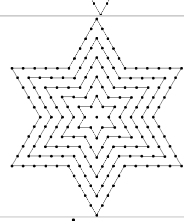
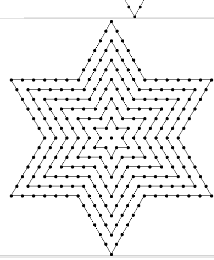
<u>Un-factorized form:</u>

$$F(n) = 1 + 12 + 24 + ... + (12n - 24) + (12n - 12) + 12n$$

Factorized form:

$$1 + 12 + 24 + ... + 12(n-2) + 12(n-1) + 12n$$

This pattern is clearly different as each subsequent value added to the next term is +12 greater than the value of the term before. From this finding the other stages of the stellar shape can be found.

Image of Stellar shape	n	Counted number of dots	$S_{n+1} - S_n = D_n$	$D_{n+1} - D_n = E_n$
	5	121	60	12
	6	181	72	12
	7	253	84	

Thus, from the above further data, a general statement may be formed from finding S_n, which will now be F(n) to the nth term.

In order to maintain consistency of the terms, and reduce complexity "1" will tem porarily be removed from the equation, then it will be returned into the general statement at the end. This means that all stellar terms will lose "1" from their total value hence,

$$F(n) = 1 + 12 + 24 + ... + (12n - 24) + (12n - 12) + 12n$$

$$to$$

$$F(n) = 12 + 24 + ... + (12n - 24) + (12n - 12) + 12n$$

n will be defined as the stage number of the s tellar shape
F(n) will be defined as the function of the nth stellar shape

The reason for this is that in the stellar shapes there is always one dot in the middle from S_1 to S_n hence taking out the 1 from the equation does not alter the equation, as the "1" is objective due to its constant presence in all stellar shapes observed.
The same method utilized in the triangle general statement will be utilized here:

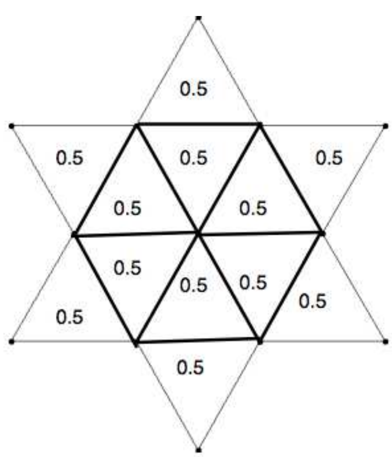
$$F(n) = 12n + (12n - 12) + ... + 24 + 12$$

$$+$$

$$F(n) = 12 + 24 + ... + (12n - 12) + 12n$$

$$\therefore 2F(n) = (12 + 12n) + (24 + 12n - 12) + ... + (12n - 12 + 24) + (12n + 12)$$

The reason for adding F(n)+F(n) is because of the geometric construction of the stellar shape as seen below there are 12 triangles which can be drawn in any stage of the stellar shape with 6 vertices. The 12 triangles all represent $F(n) = 0.5n^2 + 0.5n + 0$ and since there are 12 of them this means that it is 0.5 multiplied by 12 which is 6. Hence, in the equation the 2F(n) is to account for this 0.5 as to isolate F(n) it will be divided out.



Then,

$$2F(n) = n(12n + 12)$$

$$Factorized:$$

$$2F(n) = 12n(n + 1)$$

$$F(n) = \frac{12n}{2}(n + 1)$$

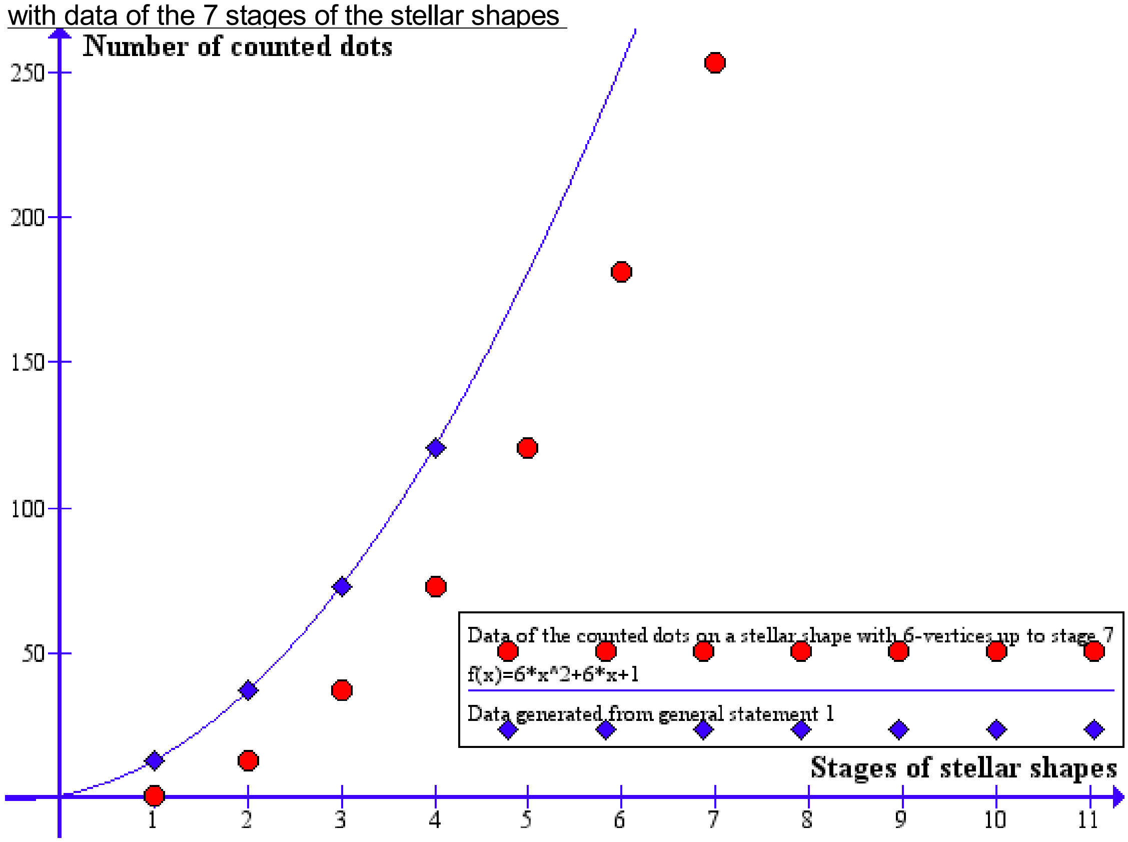
$$\therefore F(n) = 6n(n + 1) + 1$$

It is 12n(n+1) due to F(n) being n times of (12n+12). Then divided by 2 and one is added back in. The general statement here shall be tested by being graphed against the points of the known data as well as the terms generated by the general statement sh own below.

S_n	n	$F(n) = 6n(n+1)+1$	**Number of dots from formula**	$S_{n+1} - S_n = D_n$	$D_{n+1} - D_n = E_n$
S_1	1	$F(1) = 6(1)(1+1)+1$ $= 6(2)+1$ $= 13$	13	24	12
S_2	2	$F(2) = 6(2)(2+1)+1$ $= 12(3)+1$ $= 37$	37	36	12
S_3	3	$F(3) = 6(3)(3+1)+1$ $= 18(4)+1$ $= 73$	73	48	

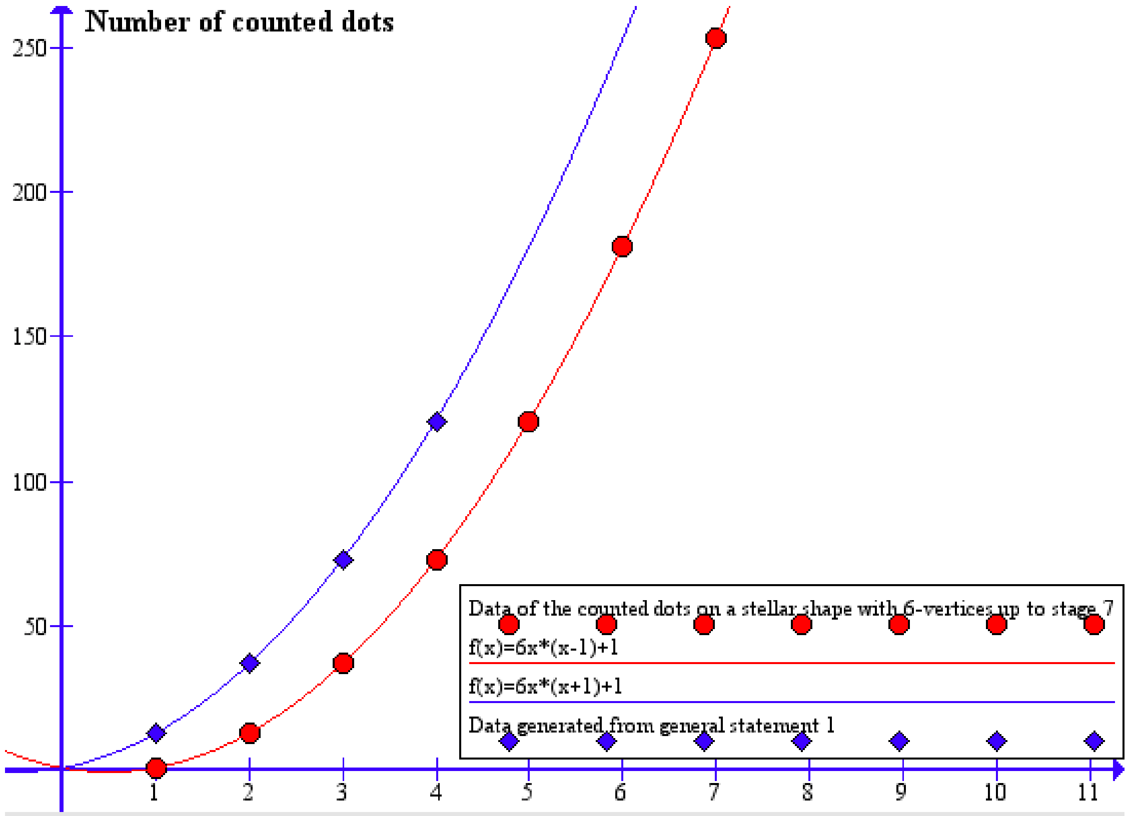
S_n	n	$F(n) = 6n(n+1)+1$	Number of dots from formula	$S_{n+1} - S_n = D_n$	$D_{n+1} - D_n = E_n$
S_4	4	$F(4) = 6(4)(4+1)+1$ $= 24(5)+1$ $= 121$	121		

Figure 2: Graph of general statement of stellar shapes (6 -vertices) and generated data, with data of the 7 stages of the stellar shapes



As it it is seen here the dat a generated from the general statement does not match the data which is known. The reason for this lies in the creation of the stellar numbers. With the general statement created, true stellar shapes are generated that is instead of starting with a dot, it starts with the first stellar shape, with the dot in the middle. With this in mind, in order to account for this, what must happen is that the “n+1” must change into a “n -1” sign. This is necessary as by moving the graph to the left, the parabola will fit the data as demonstrated below:

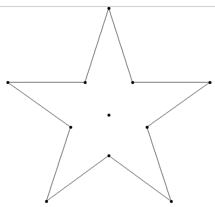
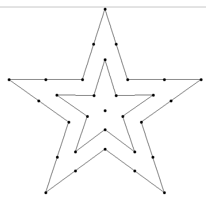
Figure 3: Graph of general statement of stellar shapes (6 -vertices) and generated data, with data of the 7 stages of the stellar shapes and general statement which fits known data



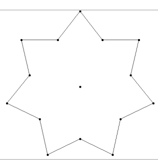
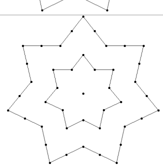
Based on the work above, the general statement for the 6-vertices star shapes are:
$\therefore 6n(n-1)+1$

Other Stellar Numbers (5-vertices, 7-vertices)
In order to obtain a general statement other stellar shapes must considered with different vertices therefore: stellar shapes with 5-vertices, 7-vertices shall be considered in order to formulate a general statement. Therefore, the subsequent tables will depict the 3 stages of a 5-vertices and 7-vertices stellar shape. Therefore, a stellar shape g enerating program shall be used to depict these new shapes.

5-vertices stellar shape

Image of Stellar shape	n	Counted number of dots	$S_{n+1} - S_n = D_n$	$D_{n+1} - D_n = E_n$
	1	1	10	10
	2	11	20	
	3	31		

7-vertices stellar shape

Image of Stellar shape	n	Counted number of dots	$S_{n+1} - S_n = D_n$	$D_{n+1} - D_n = E_n$
	1	1	14	14
	2	15	28	
	3	43		

5-vertices stellar shape and 7 -vertices stellar shape to the nth term:

5-vertices stellar shape to the nth term	7-vertices stellar shape to the nth term
$F(n)_1 = 1 + 10 + 20 + ... + 10(n-2) + 10(n-1) + 10n$ The "1" is removed at the start in order to maintain consistency, the similar constructions warrant this action. Due to similar constructed nature, it shall be also be multiplied by 2. As with the 6-vertices there are two layers, the pentagon then the extensions of the pentagon. Hence, $2F(n)_1 = 10 + 20 + ... + 10(n-1) + 10n$ $2F(n)_1 = (10n + 10) + (10(n-1) + 20)$ $\therefore 2F(n)_1 = 10n(n+1)$	$F(n)_2 = 1 + 14 + 28 + ... + 14(n-2) + 14(n-1) + 14n$ The "1" is removed at the start in order to maintain consistency, the similar constructions warrant this action. Due to similar constructed nature, it shall be also be multiplied by 2. As with the 6-vertices there are two layers, the heptagon then the extensions of the heptagon. Hence, $2F(n)_1 = 14 + 28 + ... + 14(n-1) + 14n$ $2F(n)_1 = (14n + 10) + (14(n-1) + 28)$ $\therefore 2F(n_1) = 14n(n+1)$
Due to their similar nature of construction in comparison to the 6 -vertices stellar shape, the "n+1" shall be changed to a "n-1." Plus one is added back in, to account for the dot. Ultimately giving:	
$\therefore F(n)_1 = 5n(n-1) + 1$	$\therefore F(n)_2 = 7n(n-1) + 1$
The similar increasing values follows a general pattern that can be expressed as 2p(n) times (n-1) divided by 2, for values of n=1,2,3,4...n. That is p is the number of vertices and n is the stages of the stellar shape. Hence the general statement can be produced: $S_n = S_{n-1} + \frac{2p(n-1)}{2}$	

$F(n)_1$ is the function of the 5-vertices stellar shape to the nth term

$F(n)_2$ is the function of the 7-vertices stellar shape to the nth term

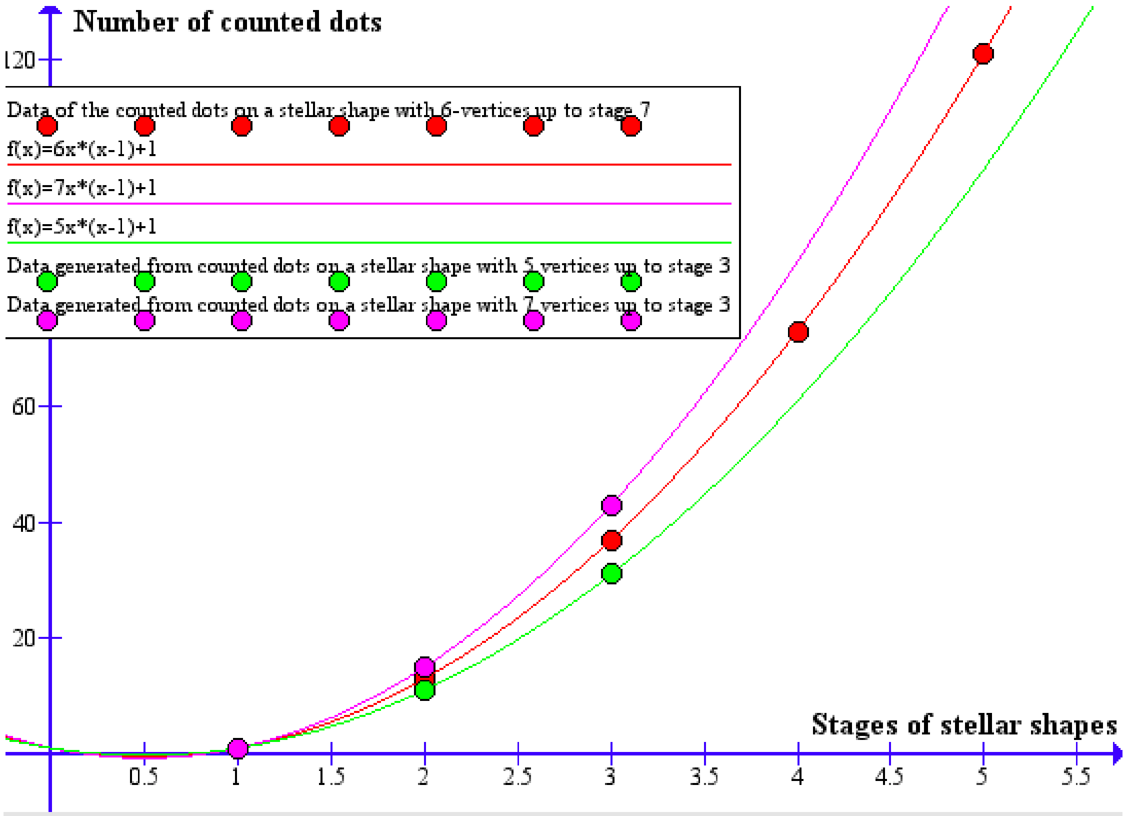
When comparing 3 functions it is clear that they are quite similar:

5-vertices function (p=5)	6-vertices function (p=6)	7-vertices function (p=7)
$\therefore F(n)_1 = 5n(n-1) + 1$	$\therefore F(n) = 6n(n-1) + 1$	$\therefore F(n)_2 = 7n(n-1) + 1$

The only difference seen is that the p - the number of vertices changes.

The graph below illustrates the functions given:

Title: Graph of general statements of 5,6,7 vertices and data on existing 5,6,7 vertices



As seen in the graph, the generated values match the general statement

Hence, a general statement can now be formed, from the patterns viewed above the general statement for all stars with p as vertices and n representing number of stages and S_n representing any stellar number stage with p -vertices:

$$\therefore S_n = pn(n-1)+1$$

Conclusion:

Scope and Limitations:

One of the limitations for this equation is for those shapes which do not have the minimum number of vertices required to make a stellar shape. Therefore there is a limit on the number of vertices this limit being $p \geq 3$, although it is possi ble to argue that a shape with 2-vertices is a star, the resulting shape does not look like the classical star. However, since the main concern for this investigation is counting the number of dots inside a "stellar" geometric shape, it must be inclusive t hat they form the shape of a classical star, then the limitation would be $p \neq 0, p \geq 3$ as when it is substituted into the equation the result it yields is always 1. The limitations for n on the other hand is that all integers of n must be positive as although negative integers when substituted into the general equation does yield results. To have negative stages of star development would be to discuss the imaginary numbers. Therefore, in the context of having practical applications for the general formula created, $n \geq 1, n \in \mathrm{Z}$,

Therefore, the general statement:

$$\therefore S_n = pn(n-1)+1$$

is true when

$n \geq 1, n \in \mathrm{Z}$
$p \geq 3, p \in \mathrm{Z}$