Shady Areas

I shall investigate different ways of finding a working rule to approximate the area under a curve using trapeziums. The area under the curve represents area between the f(x) and x values under the curve in the specified area. Therefore, through integration to find the area, integration of the integrated area will find the volume.

Consider the function $g(x) = x^2 + 3$

$g(x) = x^2 + 3$ (see Graph 1)

The area under this curve from x=0 to x=1 is approximated by the sum of the area of two trapeziums.

The approximation can be discovered by working out the area of the square in the trapezium and then by working out the triangle in the trapezium. The sum of this will give the area of the two trapeziums, which can be summed together.

The area of a trapezium can therefore be worked out using the formula

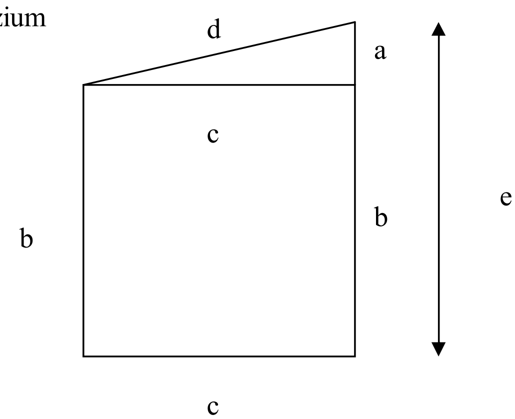
$(b \times c) + \frac{1}{2}(a \times d)$ = area of trapezium

$bc + \frac{1}{2}(e - b)c$

$c\,[\,b + \frac{1}{2}(e - b)\,]$

$c\,[\,2b/2 + (e-b)/2\,]$

$= c[\,(e + b)/2\,]$



x =	g(x) for $x^2 + 3 = g(x)$
0.2	3.04
0.4	3.16
0.6	3.36
0.8	3.64
1.0	4
0.0	3

Graph 1 area =

0.5 [(3.25 + 3)/2] = 1.5625 = area of first trapezium

0.5 [(4 + 3.25)/2] = 1.8125 = area of second trapezium

1.5625 + 1.8125 = 3.375

For Graph 1 the sum of the area of the two trapeziums gives a result of **3.375**

By increasing the number of trapeziums we can gain a more accurate estimate of the area under the curve. Using the same curve I shall try 5 separate trapeziums as apposed to the previous 2.

Area of first trapezium = 0.2 [(3 + 3.04)/2] = 0.604

Area of second trapezium = 0.2 [(3.04 + 3.16)/2] = 0.62

Area of third trapezium = 0.2 [(3.16 + 3.36)/2] = 0.652

Area of forth trapezium = 0.2 [(3.36 + 3.64)/2] = 0.7

Area of fifth trapezium = 0.2 [(3.64 + 4)/2] = 0.764

The sum of all the trapeziums and therefore the estimate for the area under the graphs is
0.604 + 0.62 + 0.652 + 0.7 + 0.764 = **3.34**

The difference between the two results is a decrease of 0.035 from 3.375 to 3.34 with an increase of 2 to 5 trapeziums

Using technology I can increase the number of trapeziums
For the first graph you can see that there are 10 trapeziums to give a greater accuracy of the estimate of the area under the graph.

0.3005(1st) + 0.3025(2nd) + 0.3065(3rd) + 0.3125(4th) + 0.3205(5th) + 0.3305(6th) + 0.3425(7th) + 0.3565(8th) + 0.3725(9th) + 0.3905(10th) = 3.335

For the second graph you can see that there is only one trapezium and therefore the estimate of area below will be inaccurate.

½(3 + 4) x 1 = 3.5

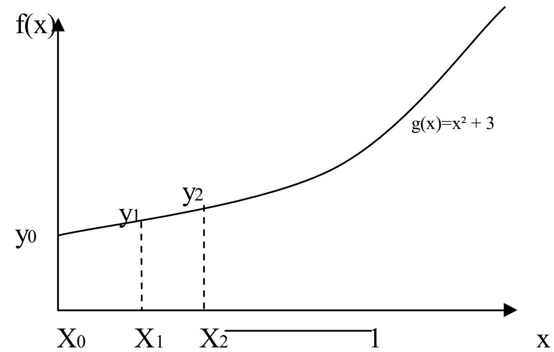
1 trapezium = 3.5
2 trapeziums = 3.375
5 trapeziums = 3.34
10 trapeziums = 3.335

The more trapeziums the lower the result as it increases in accuracy and therefore decreases the amount of wasted, inaccurate, area data.

Therefore to find a general expression for the area under the curve of g, from x=0 to x=1 using n trapeziums I can use my data and previous methods to work out and test this method.

Having already discovered how to discover the area of one trapezium:

(b x c) + ½(a x d) = area of trapezium



width of each trapezium = $\frac{1-0}{n} = \frac{1}{n}$

let $y_0 = g(X_0)$, $y_1 = g(X_1)$…..etc

$$\int_{x=0}^{x=1} g(x) = \text{width of each trapezium} \times \text{height of each trapezium}$$

$$= \frac{1}{n} \times \tfrac{1}{2}\left[(y_0 + y_1) + (y_1 + y_2) + \ldots(y_{n-1} + y_n)\right]$$

$$= \frac{1}{2n}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

Substituting for y_0, y_1, …etc

$$\int_{x=0}^{x=1} g(x) = \frac{1}{2n}\left[(3 + x\wedge n + 3) + 2\{(1^2 + 3) + (2^2 + 3) + (3^2 + 3) + \ldots + (x\wedge n-1 + 3)\}\right]$$

$$= \frac{1}{2n}\left[(0^2 + 3) + 2(1^2 + 3) + 2(2^2 + 3) + \ldots + (x\wedge n + 3)\right]$$

The general statement that will estimate the area under *any curve* can be found as follows for y = f(x) from x = a to x = b using n trapeziums for

As above with width of trapezium = $\frac{b-a}{n}$

$$\int_a^b f(x) = \text{Width of each trapezium} \times \text{height of each trapezium}$$

$$= \frac{b-a}{n} \times \tfrac{1}{2}\left[(y_0 + y_1) + (y_1 + y_2) + \ldots + (y_{n-1} + y_n)\right]$$

$$= \frac{b-a}{2n} \times \left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

$\int Y_1 = \int_1^3 f(x)\,dx \ (x/2)^{⅔}$ When n = 8 using the general formula

$$\int_1^3 f(x)\,dx \ (x/2)^{⅔} = \frac{(3 - 1)}{(8 \times 2)} \times [y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 2y_7 + y_8]$$

X	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
$(x/2)^{⅔}$	0.6299	0.7310	0.8254	0.9148	1.0000	1.0816	1.1603	1.2365	1.3103

Therefore $\int_2^3 f(x)\,dx$ = 1/8 [0.6299 + 1.4620 + 1.6508 + 1.8296 + 2.000 + 2.1632 + 2.3206 + 2.4730 + 1.3103]

= 1/8 x 15.8394
= 1.9799

X	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
$\frac{9x}{\sqrt{(x^3 + 9)}}$	2.8461	3.4616	4.0249	4.5348	4.9923	5.4000	5.7617	6.0815	6.3640

$\int Y_2 = \int_1^3 f(x)\,dx \ \frac{9x}{\sqrt{(x^3 + 9)}}$

$$\int_1^3 f(x)\,dx \ \frac{9x}{\sqrt{(x^3 + 9)}} = 1/8 \times [y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 2y_7 + y_8]$$

= 1/8 x [2.8461 + 6.9232 + 8.0498 + 9.0696 + 9.9846 + 10.8 + 11.5234 + 12.163 + 6.364]

= 10.9635

X	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
$(4x^3 - 23x^2 + 40x - 18)$	3.000	3.875	3.750	3.000	2.000	1.125	0.750	1.250	3.000

$\int Y_3 = \int_1^3 f(x)\,dx \ (4x^3 - 23x^2 + 40x - 18)$

$$\int_1^3 f(x)\,dx \ (4x^3 - 23x^2 + 40x - 18) = 1/8 \times [y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 2y_7 + y_8]$$

= 1/8 x [3.00 + 7.75 + 7.5 + 6 + 4 + 2.25 + 1.5 + 2.5 + 6)

= 5.0625

Integration of these functions to find the precise areas under the curve gives the following results

$$Y_1 = \int_1^3 f(x)\,dx \ (x/2)^{2/3} = \left[(3/10)\,x^{5/3}\right]_1^3$$

$$= \left[1.8721 - 0.3\right]$$

$$= 1.5721$$

$$Y_2 = \int_1^3 f(x)\,dx \ = (9x/\sqrt{(x^3 + 9)}) = \ ?$$

$$Y_3 = \int_1^3 f(x)\,dx \ (4x^3 - 23x^2 + 40x - 18)$$

$$= \left[x^4 - (23/3)x^3 + 20x^2 - 18x\right]_1^3$$

$$= (81 - 207 + 180 - 54) - (1 - (23/3) + 20 - 18)$$

$$= 0 - (-4.667)$$

$$= 4.67$$

	Using trapeziums	By integration	Absolute difference	% error
$\int_1^3 f(x)\,dx \ (x/2)^{2/3}$	1.98	1.57	0.41	15.6
$\int_1^3 f(x)\,dx \ (9x/\sqrt{(x^3 + 9)})$	10.96			
$\int_1^3 f(x)\,dx \ (4x^3 - 23x^2 + 40x - 18)$	5.06	4.67	0.39	7.7

The first approximation was 15.6% off from the integrated answer while the third approximation was 7.7% off from the integrated answer. The third would be closer as the curve has a minima and maxima and, as the trapeziums overestimate with maxima and underestimates with curves with a minima, and so reduces the percentage error as it cancels it out.