In this investigation you will attempt to find a rule to approximate the area under a curve (i.e. between the curve and the x-axis) using trapeziums (trapezoids).

Consider the function $g(x) = x^2 + 3$

The diagram below shows the graph of g. The area under this curve from x = 0 to x = 1 is approximated by the sum of the area of two trapeziums. Find this approximation.

In this investigation I will attempt to find the approximate area under a curve (i.e. between the curve and the x-axis) using trapeziums (trapezoids).

The area of a trapezoid can be expressed as:

$$A = (1/2)(b)(h_1+h_2)$$

Where:

b is the length of the base of the trapezium along the x-axis

h_1 is the vertical length (height) of one side on the trapzium, the equivalent y- value for one of the endpoints on the base

h_2 is the other height of the trapezium for which it is the corresponding y-value for the second endpoint of the base.

A is the area of a specific trapezium in units squared.

The total area underneath the curve can be expressed as:

$$T_A = A_A + A_B$$

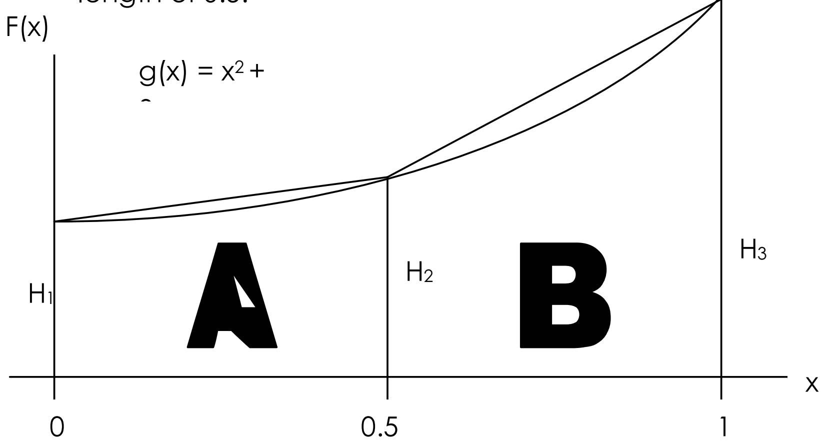
Where:

T_A is the total area (or sum) of the two trapeziums, which is equivalent to the approximated area underneath the curve, in units squared.

A_A is the estimated, calculated area of the trapezium labeled as A, in units squared.

A_B is the estimated, calculated area of the trapezium labeled as B, in units squared.

In order to determine the individual area of each trapezium, the length of each base has to be estimated, since only the overall domain [0, 1] is given. Here, an assumption is being made that the domain was cut in half when dividing the area underneath the curve, which would result in both trapeziums sharing an equal base length of 0.5.



TRAPEZIUM A [0, 0.5]:

1. $\underline{h_1}$: $g(x) = x^2 + 3$
   $g(0) = (0)^2 + 3$
   $= 3$

2. $\underline{h_2}$: $g(x) = x^2 + 3$
   $g(0.5) = (0.5)^2 + 3$
   $= 3.25$

3. $\underline{b} = 0.5$

4. $\underline{A_A} = (1/2)\ (b)\ (h_1 + h_2)$
   $=(1/2)\ (0.5)\ (3 + 3.25)$
   $= 1.5625$

TRAPEZIUM B [0.5, 1]:

1. $\underline{h_2}$: $g(x) = x^2 + 3$
   $g(0.5) = (0.5)^2 + 3$
   $= 3.25$

2. $\underline{h_3}$: $g(x) = x^2 + 3$
   $g(1) = (1)^2 + 3$
   $= 4$

3. $\underline{b} = 0.5$

4. $\underline{A_B} = (1/2)\ (b)\ (h_1 + h_2)$
   $=(1/2)\ (0.5)\ (3.25 + 4)$
   $= 1.8125$

$$T_A = A_A + A_B$$

$$T_A = 1.5625 + 1.8125$$

$$\boxed{T_A = 3.375}$$

Increase the number of trapeziums to five and find a second approximation for the area.

In the next part, the number of trapeziums has increased to five and the goal is to find a second approximation for the area. The same techniques and equations will be used in the calculations. It is predicted that since there are more trapeziums, the predicted area will be closer to the actual area (more precise).

However, since there are more trapeziums, the equation for the total area is now:

$$T_A = A_A + A_B + A_C + A_D + A_E$$

Where:

T_A is the total area (or sum) of the five trapeziums, which is equivalent to the approximated area underneath the curve, in units squared.

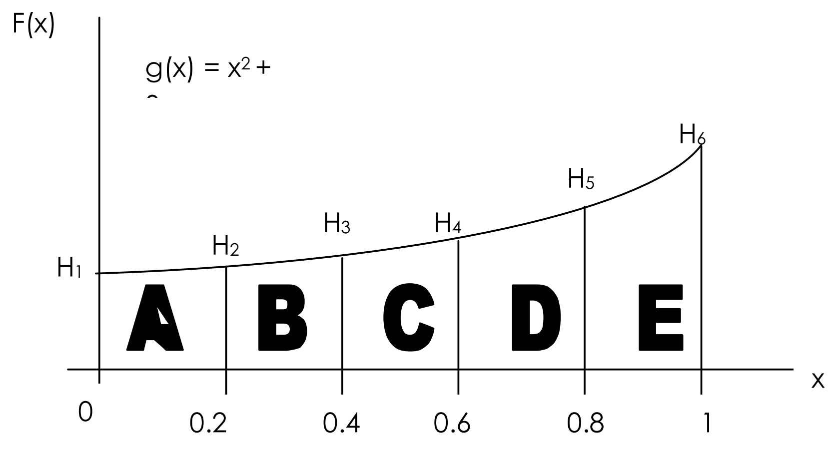
A_A is the approximated area for trapezium A, in units squared

A_B is the approximated area for trapezium B, in units squared

A_C is the approximated area for trapezium C, in units squared

A_D is the approximated area for trapezium D, in units squared

A_E is the approximated area for trapezium E, in units squared



Again an assumption must be made in regards to the length of the bases for each of the individual trapeziums. It is being postulated that, like before, the domain [0, 1] is divided equally between the five bases resulting in a length of 0.2 for each base.

TRAPEZIUM A [0, 0.2]:

1. $\underline{h_1}$: $g(x) = x^2 + 3$
   $g(0) = (0)^2 + 3$
   $= 3$

2. $\underline{h_2}$: $g(x) = x^2 + 3$
   $g(0.2) = (0.2)^2 + 3$
   $= 3.04$

3. $\underline{b} = 0.2$

4. $\underline{A_A} = (1/2)\ (b)\ (h_1 + h_2)$
   $=(1/2)\ (0.2)\ (3 + 3.04)$
   $= 0.604$

TRAPEZIUM B [0.2, 0.4]:

1. $\underline{h_2}$: $g(x) = x^2 + 3$
   $g(0.2) = (0.2)^2 + 3$
   $= 3.04$

2. $\underline{h_3}$: $g(x) = x^2 + 3$
   $g(0.4) = (0.4)^2 + 3$
   $= 3.16$

3. $\underline{b} = 0.2$

4. $\underline{A_B} = (1/2)\ (b)\ (h_1 + h_2)$
   $=(1/2)\ (0.2)\ (3.04 + 3.16)$
   $= 0.62$

TRAPEZIUM C [0.4, 0.6]:

1. $\underline{h_3}$: $g(x) = x^2 + 3$
   $g(0.4) = (0.4)^2 + 3$
   $= 3.16$

2. $\underline{h_4}$: $g(x) = x^2 + 3$
   $g(0.6) = (0.6)^2 + 3$
   $= 3.36$

3. $\underline{b} = 0.2$

4. $\underline{A_C} = (1/2)\ (b)\ (h_1 + h_2)$
   $=(1/2)\ (0.2)\ (3.16 + 3.36)$
   $= 0.652$

TRAPEZIUM D [0.6, 0.8]:

1. $\underline{h_4}$: $g(x) = x^2 + 3$
   $g(0.6) = (0.6)^2 + 3$
   $= 3.36$

2. $\underline{h_5}$: $g(x) = x^2 + 3$
   $g(0.8) = (0.8)^2 + 3$
   $= 3.64$

3. $\underline{b} = 0.2$

4. $\underline{A_D} = (1/2)\ (b)\ (h_1 + h_2)$
   $=(1/2)\ (0.2)\ (3.36 + 3.64)$
   $= 0.7$

TRAPEZIUM E [0.8, 1.0]:

1. $\underline{h_5}$: $g(x) = x^2 + 3$
   $g(0.8) = (0.8)^2 + 3$
   $= 3.64$

2. $\underline{h_6}$: $g(x) = x^2 + 3$
   $g(1.0) = (1.0)^2 + 3$
   $= 4.0$

3. $\underline{b} = 0.2$

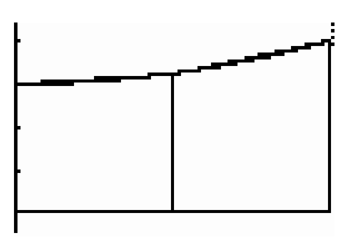
4. $\underline{A_C} = (1/2)\ (b)\ (h_1 + h_2)$
   $=(1/2)\ (0.2)\ (3.64 + 4.0)$
   $= 0.764$

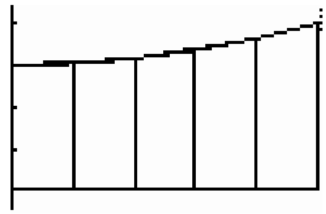
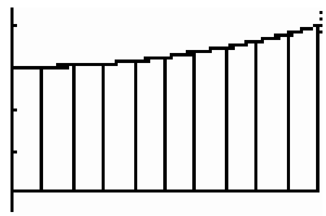
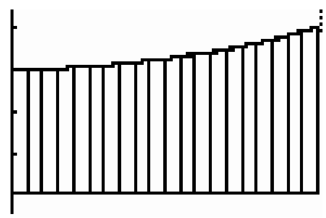
$$T_A = A_A + A_B + A_C + A_D + A_E$$

$$T_A = 0.604 + 0.62 + 0.652 + 0.7 + 0.764$$
$$\boxed{T_A = 3.34}$$

With the help of technology, create diagrams showing and increasing number of trapeziums. For each diagram, find the approximation for the area. What do you notice?

Various diagrams will be created to depict the increasing number of trapeziums, which result in a greater precision for the approximation for the area.

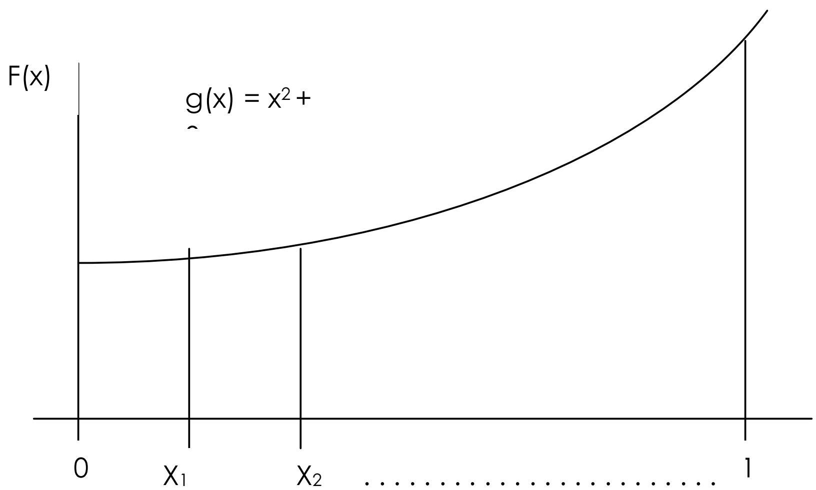
Number of Trapeziums	Manual Calculations	Riemann Sum Application	
		Graphs ($Y = x^2 + 3$) [0,1]	Sum
2	3.375		3.375

5	3.34		3.34
10	3.33		3.336
20	3.33		3.33375

First I calculated the area manually by hand. Then I tested these results with the use of technology, which in this case was the Riemann Sum Application on a TI-84+, which I used to create the table above.

The smaller the trapeziums, the more precise the area is because a smaller unit of measurement was utilized. It is possible to go on to infinity trapeziums, in which case the uncertainty would decrease more and more resulting in the relative error heading towards zero. This is because with more trapeziums it is possible to get a better approximation of the area since the line is going to a microscopic level, nearer to the original line. It gets nearer to the real value.

Use the diagram below to find a general expression for the area under the curve of g, from x = 0 to x = 1, using n trapeziums.



Based on the pattern observed from the diagrams above, a general equation can be written for the area under the curve of g, from x=0 to x=1, using n trapeziums. [$g(x) = x^2 + 3$]

$$\int_a^c f(x)\,dx \rightarrow \int_0^1 x^2 + 3\,dx$$

Where:

c is equal to the end x value of the domain

a is equal to the beginning x value for the domain

Y= f(x), the equation of the curve

dx indicates that everything is being taken in respects to x

The above equation in its expanded form is:

$$\int_0^1 x^2 + 3\,dx = (1/2)\,b\,[g(0) + g(0.5)] + (1/2)\,b\,[g(0.5) + g(1)]$$

Where:

b is base which can be calculated by ([c-a]/n)

This can be simplified to:

$$\int_0^1 x^2 + 3\,dx = (1/2)\, b\, [g(0) + 2g(0.5) + g(1)]$$

For example, if the question with two trapeziums is solved using this formula:
$g(x) = x^2 + 3$ from $x = 0$ to $x = 1$

$\int_0^1 = ½\, b\, [g(x_0) + g(x_{0.5})] + ½\, h[g(x_{0.5}) + g(x_1)]$

$= ½\,(0.5)\,(3 + 3.25) + ½\,(0.5)\,(3.25 + 4)$

$= ½\,(0.5)\,[3 + 4 + (2)(3.25)]$ = *3.375*

Use your results to develop the general statement that will estimate the area under any curve y = f(x) from x = a to x = b using n trapeziums. Show clearly how you developed your statement.

$\Sigma = ½\, b\, [g(x_0) + g(x_1)] + ½\, b\, [g(x_1) + g(x_2)] + ½\, b\, [g(x_2) + g(x_3)] + ... + ½\, b\, [g(x_{n-1}) + (g(x_n)]$

$$b = ½\,(c – a)/(n) = (c – a)/(2n) = (x_n – x_0)/(2n)$$

However, since b is a common factor it can be brought outside in a bracket. Also all the g(x) co-ordinates occur twice (in two consecutive trapeziums) apart from $g(x_0)$ and the last $g(x_n)$ co-ordinate. All the terms are halved which means the first and last g(x) values are halved and every other g(x) value should be counted once.

This can be re-written to, if using n trapeziums:

$$\int_a^c f(x)\,dx =(1/2)\, b\, [g(x_0) + g(x_n)+ 2(g(x_1)+ g(x_2)+ g(x_3)+ . . . + g(x_{n-1})]$$

$$\int_a^c f(x)\,dx = [(x_n – x_0)/(2n)]\, (g(x_0) + g(x_n) + 2[\Sigma\, g(x_i)])$$

Where:

n is the number of trapeziums the area of the curve is divided into

b is base which can be calculated by ([c-a]/n)

c is equal to 1 in this problem or (x_n)

a is equal to 0 in this question or (x_0)

$g(x_0)$… $g(x_n)$ are the y values for each respective x value.

Use your general statement, with eight trapeziums, to find approximations for these areas.

$$y_1 = \left(\frac{x}{2}\right)^{\frac{2}{3}}$$

$\int^{c}_{a}$ (x/2)^(2/3) dx = $(x_n - x_0)/(2n)$ $(g(x_0) + g(x_n) + 2\left[\sum g(x_i)\right]$

$\int^{3}_{1}$(x/2)^(2/3) dx =(3-1)/(2x8) (g(3) + g(1) + 2[g(1.25) + g(1.5) + g(1.75) + g(2) + g(2.25) + g(2.5) + g(2.75)]

= (2)/(16)[0.63 + 1.31 + 2 (0.73 + 0.83 + 0.91 + 1 + 1.08 + 1.16 + 1.24)]

=1.98

$$y_2 = \frac{9x}{\sqrt{x^3 + 9}}$$

$\int^{c}_{a}$ (9x)/ ($\sqrt{}x^3$ + 9) dx = $(x_n - x_0)/(2n)$ $(g(x_0) + g(x_n) + 2\left[\sum g(x_i)\right]$

$\int^{3}_{1}$(9x)/ ($\sqrt{}x^3$ + 9) dx =(2)/(16) (g(3) + g(1) + 2[g(1.25) + g(1.5) + g(1.75) + g(2) + g(2.25) + g(2.5) + g(2.75)]

= (2)/(16)[2.85 + 4.5 + 2 (3.40 + 3.84 + 4.16 + 4.37 + 4.48 + 4.53 + 4.54)]

= 8.24625

$$y_3 = 4x^3 - 23x^2 + 40x - 18$$

$\int^{c}_{a}$ (9x)/ ($\sqrt{}x^3$ + 9) dx = $(x_n - x_0)/(2n)$ $(g(x_0) + g(x_n) + 2\left[\sum g(x_i)\right]$

$\int^{3}_{1}$($4x^3 - 23x^2$ + 40x - 18)dx =(2)/(16) (g(3) + g(1) + 2[g(1.25)+g(1.5)+g(1.75)+g(2)+g(2.25)+g(2.5)+g(2.75)]

= (2)/(16)([3 + 3 + 2(3.875 + 3.75 + 3 + 2 + 1.125 + 0.75 + 1.25)]

= 4.6875

Find $\int_1^3 \left(\frac{x}{2}\right)^{\frac{2}{3}} \mathrm{d}x$, $\int_1^3 \left(\frac{9x}{\sqrt{x^3+9}}\right) \mathrm{d}x$, $\int_1^3 (4x^3 - 23x^2 + 40x - 18)\,\mathrm{d}x$, **and compare these answers with your approximations. Comment on the accuracy of your approximations.**

KEY STROKES: math-> 9. fnInt (enter equation ,X ,1 ,3) ENTER

$$y_1 = \left(\frac{x}{2}\right)^{\frac{2}{3}}$$

∫f(x) dx = 1.9806909

$$y_2 = \frac{9x}{\sqrt{x^3+9}}$$

∫f(x) dx = 8.2597312

$$y_3 = 4x^3 - 23x^2 + 40x - 18$$

∫f(x) dx = 4.6666667

The approximated results that were achieved by manual calculation were very precise from the actual area underneath the curve. Based on the equation, the percent of variation changes; however, the data is precise at least up to the tenths place.

Use other functions to explore the scope and limitations of your general statement. Does it always work? Discuss how the shape of a graph influences your approximation.

For each function n=4 trapeziums and the interval is [0,1], and $[(x_n - x_0)/(2n)] = (1/8)$

1. $y = x^4 + 12x + 4$

$$\int_a^c f(x)\,dx = [(x_n - x_0)/(2n)]\,(g(x_0) + g(x_n) + 2[\sum g(x_i)])$$

x_0 : $g(x) = x^4 + 12x + 4$

$g(0) = (0)^4 + 12(0) + 4$

$= 4$

x_n : $g(x) = x^4 + 12x + 4$

$g(1) = (1)^4 + 12(1) + 4$

$= 17$

x_1 : $g(x) = x^4 + 12x + 4$

$g(0.25) = (0.25)^4 + 12(0.25) + 4$

$= 7.00390625$

x_2: $g(x) = x^4 + 12x + 4$

$g(0.5) = (0.5)^4 + 12(0.5) + 4$

x_3: $g(x) = x^4 + 12x + 4$

$g(0.75) = (0.75)^4 + 12(0.75) + 4$

$= 13.31640625$

$$\int_0^1 (x^4 + 12x + 4)\,dx = (1/8)\ (4 + 17 + 2[7.00390625 + 10.0625 + 13.31640625]$$

= 10.22070313

Calculator:

[math-> 9. fnInt (] enter [x⁴ + 12x +4] [,X] [,1] [,3] [)] [ENTER] OR [Y=] enter y = x⁴ + 12x + 4 [2nd] [TRACE] [7] [ENTER] graph displayed [0] [ENTER 1] [ENTER]

$\int f(x)dx = 10.2$

2. $y = x^3 + 10$

$$\int_a^c f(x)\,dx = [(x_n - x_0)/(2n)]\,(g(x_0) + g(x_n) + 2[\sum g(x_i)])$$

x_0 : $g(x) = x^3 + 10$

$g(0) = (0)^3 + 10$

$= 10$

x_n : $g(x) = x^3 + 10$

$g(1) = (1)^3 + 10$

$= 11$

x_1 : $g(x) = x^3 + 10$

$g(0.25) = (0.25)^3 + 10$

$= 10.015625$

x_2: $g(x) = x^3 + 10$

$g(0.5) = (0.5)^3 + 10$

$= 10.000125$

x_3: $g(x) = x^3 + 10$

$g(0.75) = (0.75)^3 + 10$

$= 10.421875$

$$\int_0^1 (x^3 + 10)\, dx = (1/8)\ (10 + 11 + 2[10.015625 + 10.000125 + 10.421875]$$

= 10.23440625

Calculator:

math-> 9. fnInt (enter x³ + 10 ,X ,1 ,3) ENTER OR Y= enter y = x³ + 10 2nd

TRACE 7 ENTER graph displayed 0 ENTER 1 ENTER

∫f(x)dx = 10.25

3. *y = 2/(x² + 4)*

$$\int_a^c f(x)\, dx = [(x_n - x_0)/(2n)]\ (g(x_0) + g(x_n) + 2[\sum g(x_i)])$$

x_0 : $g(x) = 2/(x^2 + 4)$

$g(0) = 2/(0^2 + 4)$

$= 0.5$

x_1 : $g(x) = 2/(x^2 + 4)$

$g(0.25) = 2/(0.25^2 + 4)$

$= 0.4923076923$

x_3: $g(x) = 2/(x^2 + 4)$

$g(0.75) = 2/(0.75^2 + 4)$

$= 0.4383561644$

x_n : $g(x) = 2/(x^2 + 4)$

$g(1) = 2/(1^2 + 4)$

$= 0.4$

x_2: $g(x) = 2/(x^2 + 4)$

$g(0.5) = 2/(0.5^2 + 4)$

$= 0.4705882353$

$$\int_0^1 (2/(x^2 + 4))\, dx = (1/8)\ (0.5 + 0.4 + 2[0.4923076923 + 0.4705882353 + 0.4383561644]$$

= 0.462813023

Calculator:

math-> 9. fnInt (enter 2/(x² + 4) ,X ,1 ,3) ENTER OR Y= enter y = 2/(x² + 4)

2nd TRACE 7 ENTER graph displayed 0 ENTER 1 ENTER

∫f(x)dx = 0.463647609

4. $y = \sqrt{(2x+1)}$

$$\int_a^c f(x)\,dx = [(x_n - x_0)/(2n)]\ (g(x_0) + g(x_n) + 2\left[\sum g(x_i)\right])$$

x_0 : $g(x) = \sqrt{(2x+1)}$

$g(0) = \sqrt{(2(0)+1)}$

$= 1$

x_1 : $g(x) = \sqrt{(2x+1)}$

$g(0.25) = \sqrt{(2(0.25)+1)}$

$= 1.224744871$

x_3: $g(x) = \sqrt{(2x+1)}$

$g(0.75) = \sqrt{(2(0.75)+1)}$

$= 1.58113883$

x_n : $g(x) = \sqrt{(2x+1)}$

$g(1) = \sqrt{(2(1)+1)}$

$= 1.732050808$

x_2: $g(x) = \sqrt{(2x+1)}$

$g(0.5) = \sqrt{(2(0.5)+1)}$

$= 1.414213562$

$$\int_0^1 (\sqrt{(2x+1)})\,dx = (1/8)\ (1 + 1.73 + 2\left[1.224744871 + 1.414213562 + 1.58113883\right]$$

= 1.396530667

Calculator:

$\boxed{\text{math-> 9. fnInt (}}$ enter $\boxed{2/(x^2 + 4)}$ $\boxed{,X}$ $\boxed{,1}$ $\boxed{,3}$ $\boxed{)}$ $\boxed{\text{ENTER}}$ OR $\boxed{\text{Y=}}$ enter $y = 2/(x^2 + 4)$

$\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ $\boxed{7}$ $\boxed{\text{ENTER}}$ graph displayed $\boxed{0}$ $\boxed{\text{ENTER}\ \ 1}$ $\boxed{\text{ENTER}}$

$\int f(x)dx$ = 1.398717474

CONCLUSION

The manual calculations derived from the general statement appeared to be very close to the exact answers as computed by the calculator. Based on the results, it appears that no matter the shape, using the trapezium rule provides with a fairly accurate and precise approximation.

However, it can be determined in two ways whether the approximation is an overestimate or an underestimate. The first is to sketch a graph and draw the trapeziums. If the tops of the trapeziums are above the curve, there is an overestimate, and if the trapeziums are below the curve, it is an underestimate. The second method is to examine the second derivative of the graph. If it is negative, indicating concave down, then the curve will have a lesser gradient at any given interval in the positive x-direction, and therefore the trapeziums will be underneath the curve. If the second derivative is positive, indicating upwards concavity, trapeziums will extrude, and thus give an overestimate.

Overall, to achieve the most precise and accurate approximation for an area under the curve, n, the number of trapeziums, needs to be divided into smaller subintervals.