#1 Series and Induction

In this portfolio, I will be investigating a pattern and forming conjectures to explain what happens when k = 2, 3, or 4 in the $1^k + 2k^k + 3^k + 4^k + \ldots + n^k$ series. To do this, I will be using the knowledge that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

1. $\{a_n\}_{n=1}^{\infty}$ where $a_1 = 1 \times 2$

$a_1 = 1 \times 2$
$a_2 = 2 \times 3$
$a_3 = 3 \times 4$

$a_4 = 4 \times 5$

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Expression seems to show that $\mathbf{a_n = n(n+1)}$

2. Consider $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ where a_k is defined in #1 [$a_k = n(n+1)$]

a) Determine several values of S_k, including $S_1, S_2, S_3, \ldots, S_6$

$S_1 = a_1 = 1 \times 2 = 2$

$S_2 = a_1 + a_2 = S_1 + a_2 = 2 + 2 \times 3 = 8$

$S_3 = a_1 + a_2 + a_3 = S_2 + a_3 = 8 + 3 \times 4 = 20$

$S_4 = a_1 + a_2 + a_3 + a_4 = S_3 + a_4 = 20 + 4 \times 5 = 40$

$S_5 = = a_1 + a_2 + a_3 + a_4 + a_5 = S_4 + a_5 = 40 + 5 \times 6 = 70$

$S_6 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = S_5 + a_6 = 70 + 6 \times 7 = 112$

It seems that for every increasing value of k, the sum of the previous numbers plus the new number yields the new sum. All sums are multiples of 2.

b) Thus, the conjecture is that: $\mathbf{S_n = S_{n-1} + a_n}$

c) Prove conjecture by induction:

Step 1 Assume the conjecture to be true for n = 1

As shown above,

$S_1 = a_1 = 1 \times 2 = 2$

$S_2 = a_1 + a_2 = 2 + 2 \times 3 = 8$

$S_3 = a_1 + a_2 + a_3 = 8 + 3 \times 4 = 20$

Step 2 Assume the conjecture to be true for n = k (done in part a of 2)

So $S_k = a_1 + a_2 + a_3 + \ldots + a_k = S_{k-1} + a_k$

Step 3 Observe for if n = k + 1

Should be: $S_{k+1} = \mathbf{S_k + a_{k+1}}$

So substitute $S_{k-1} + a_k$ for S_k (in step 2) and $k(k+1)$ for a_k where $n = k$ (refer to #1):

$S_{k+1} = a_1 + a_2 + a_3 + \ldots + a_k + a_{k+1}$

$= S_{k-1} + a_k + a_{k+1} = \mathbf{S_k + a_{k+1}}$

We got the answer that we wanted, so the conjecture is true!

d) Using the above result, calculate $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2$

It can be seen that the differences of consecutive terms (squares) turns out to be a common number of 2:

$1^2 \quad 2^2 \quad 3^2 \quad 4^2 \quad 5^2 \quad 6^2 \quad 7^2$

3 5 7 9 11 13

2 2 2 2 2

Also, taking the terms $a_1 = 1 \times 2$, $a_2 = 2 \times 3$, $a_3 = 3 \times 4$, $a_4 = 4 \times 5\ldots$ from #1, the differences of consecutive terms also turns out to be a common number of 2:

$$\begin{array}{cccccc} 2 & 6 & 12 & 20 & 30 & 42 \\ & 4 & 6 & 8 & 10 & 12 \\ & & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} \end{array}$$

Taking the sums of these terms that were investigated in part a, it can be seen that:

$$\begin{array}{cccccc} 2 & 8 & 20 & 40 & 70 & 112 \\ & 6 & 12 & 20 & 30 & 42 \\ & & 6 & 8 & 10 & 12 \\ & & & \mathbf{2} & \mathbf{2} & \mathbf{2} \end{array}$$

In the last case, the numbers in my first two rows of differences are not all the same, but in the third row of differences, the differences are all the same. Because the difference is 2 after three rounds of subtracting in this fashion, the equation that I will set as my conjecture will be of degree 3, in the form $S_n = an^3 + bn^2 + cn + d$. There are 4 unknowns, so I set up 4 equations:

$S_1 = a + b + c + d$ which we know equals 2 (from part a)
$S_2 = a2^3 + b2^2 + 2c + d$
$= 8a + 4b + 2c + d = 8$
$S_3 = a3^3 + b3^2 + 3c + d$
$= 27a + 9b + 3c + d = 20$
$S_4 = a4^3 + b4^2 + 4c + d$
$= 64a + 16b + 4c + d = 40$

These values were entered into the GDC as matrix A $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix}$ and B $\begin{bmatrix} 2 \\ 8 \\ 20 \\ 40 \end{bmatrix}$ and it was determined by reduced row echelon form that $a = \frac{1}{3}$, $b = 1$, $c = \frac{2}{3}$, and $d = 0$.

Therefore, $\mathbf{S_n} = \frac{1}{3}n^3 + n^2 + \frac{2}{3}n$

$S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n = 1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n+1) = \sum_{i=1}^{n} n(n+1) = \sum_{i=1}^{n} n^2 + n$

$= \sum_{i=1}^{n} n^2 + \sum_{i=1}^{n} n$

Thus, $\sum_{i=1}^{n} n^2 = S_n - \sum_{i=1}^{n} n$

We know that $\sum_{i=1}^{n} n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ and that $S_n = \frac{1}{3}n^3 + n^2 + \frac{2}{3}n$, so plug these values in and:

$$\sum_{i=1}^{n} n^2 = S_n - \sum_{i=1}^{n} n = \frac{1}{3}n^3 + n^2 + \frac{2}{3}n - \frac{n(n+1)}{2} = \frac{2n^3 + 6n^2 + 4n - 3n^2 - 3n}{6} = \frac{n(n+1)(2n+1)}{6}$$

Thus, my conjecture is that: $\sum_{i=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$

Check:

n = 1; $\sum_{i=1}^{n} i^2 = 1^2 = \mathbf{1}$

now try this with the above equation… $\frac{1(1+1)(2\times1+1)}{6} = \mathbf{1}$ [it fits!]

n = 2; $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 = \mathbf{5}$

now with the above equation… $\frac{2(2+1)(2\times2+1)}{6} = \mathbf{5}$ [it fits!]

n = 3; $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 = \mathbf{14}$

now with the above equation… $\frac{3(3+1)(2\times3+1)}{6} = \mathbf{14}$ [it fits!]

Let's check if this conjecture works all other values of n using an induction proof***:***

Step 1 Assume the conjecture to be true for n = 1

As shown above,

$$\sum_{i=1}^{n} n^2 = 1^2 = \frac{1(1+1)(2\times1+1)}{6} = 1$$

Step 2 Assume the conjecture to be true for n = k

So $\sum_{i=1}^{k} k^2 = \frac{k(k+1)(2k+1)}{6}$

Step 3 Observe for if n = k + 1

$\sum_{i=1}^{k+1} (k+1)^2 = 1 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2$ which, according to the formula, should equal

$$\frac{(k+1)(k+1+1)(2k+2+1)}{6} = \boxed{\frac{(k+1)(k+2)(2k+3)}{6}}$$

$$\sum_{i=1}^{k+1} (k+1)^2 = \sum_{i=1}^{k+1} k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} = \frac{(k+1)(2k^2 + 7k + 6)}{6} = \boxed{\frac{(k+1)(k+2)(2k+3)}{6}}$$

$\frac{(k+1)(k+2)(2k+3)}{6}$ is equal to the equation above, so $\sum_{i=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$ is true.

3. Consider T_n = 1x2x3 + 2x3x4 + 3x4x5 + … + n(n+1)(n+2)
   a) Determine several values of T_k including $T_1, T_2, T_3, …, T_6$

      $T_1 = a_1 = 1 \text{ x } 2 \text{ x } 3 = 6$
      $T_2 = a_1 + a_2 = T_1 + a_2 = 6 + 2 \text{ x } 3 \text{ x } 4 = 30$
      $T_3 = a_1 + a_2 + a_3 = T_2 + a_3 = 30 + 3 \text{ x } 4 \text{ x } 5 = 90$
      $T_4 = a_1 + a_2 + a_3 + a_4 = T_3 + a_4 = 90 + 4 \text{ x } 5 \text{ x } 6 = 210$
      $T_5 = = a_1 + a_2 + a_3 + a_4 + a_5 = T_4 + a_5 = 210 + 5 \text{ x } 6 \text{ x } 7 = 420$
      $T_6 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = T_5 + a_6 = 420 + 6 \text{ x } 7 \text{ x } 8 = 756$
      $T_7 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = T_6 + a_7 = 756 + 7 \text{ x } 8 \text{ x } 9 = 1260$

   It seems that for every increasing value of k, the sum of the previous numbers plus the new number yields the new sum.
   b) Thus, the conjecture is that: $\mathbf{T_k = T_{k-1} + a_k}$
   c) Prove conjecture by induction:

      <u>Step 1</u> Assume the conjecture to be true for n = 1
      As shown above,

      $T_1 = a_1 = 1 \text{ x } 2 \text{ x } 3 = 6$
      $T_2 = a_1 + a_2 = 6 + 2 \text{ x } 3 \text{ x } 4 = 30$
      $T_3 = a_1 + a_2 + a_3 = 30 + 3 \text{ x } 4 \text{ x } 5 = 90$

      So $\mathbf{T_n = T_{n-1} + a_n}$

      <u>Step 2</u> Assume the conjecture to be true for n = k (done in part a of 2)
      So $T_k = T_1 + T_2 + T_3 + … + T_k = T_{k-1} + a_k$
      $T_k = T_{k-1} + a_k$

      <u>Step 3</u> Observe for if n = k + 1
      According to conjecture it should be: $T_{k+1} = T_k + a_{k+1}$
      So substitute $T_{k-1} + a_k$ for T_k (in step 2):
      $T_{k+1} = (T_{k-1} + a_k) + a_{k+1} = T_{k-1} + a_k + a_k(a_1)$

   d) Using the above result, calculate $1^3 + 2^3 + 3^3 + 4^3 + … + n^3$

   Taking the sums of terms a_1= 1x2x3, a_2 = 2x3x4, a_3 = 3x4x5, a_4 = 4x5x6… from part a, it can be seen that, as it was with the differences of squares, the differences of consecutive terms (cubes) turns out to be a common number. Except this time, this number was 6 after 4 rounds of subtracting in this fashion:

   6 30 90 210 420 756 1260
   24 60 120 210 336 504
   36 60 90 126 168
   24 30 36 42
   6 6 6

   The equation that I will set as my conjecture will be of degree 4, in the form $S_n = an^4 + bn^3 + cn^2 + dn + e$. There are 5 unknowns, so I set up 5 equations:

$T_1 = a + b + c + d + e$ which we know equals 6 (from part a)

$T_2 = a2^4 + b2^3 + c2^2 + 2d + e$

$= 16a + 8b + 4c + 2d + e = 30$

$T_3 = a3^4 + b3^3 + c3^2 + 3d + e$

$= 81a + 27b + 9c + 3d + e = 90$

$T_4 = a4^4 + b4^3 + c4^2 + 4d + e$

$= 256a + 64b + 16c + 4d + e = 210$

$T_5 = a5^4 + b5^3 + c5^2 + 5d + e$

$= 625a + 125b + 25c + 5d + e = 420$

These values were entered into the GDC as matrix A $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 256 & 64 & 16 & 4 & 1 \\ 625 & 125 & 25 & 5 & 1 \end{bmatrix}$ and B $\begin{bmatrix} 6 \\ 30 \\ 90 \\ 210 \\ 420 \end{bmatrix}$ and it was determined by reduced row echelon form that a $= \frac{1}{4}$, b = $\frac{3}{2}$, c = $\frac{11}{4}$, d = $\frac{3}{2}$, and e = 0.

Therefore, $T_n = \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + \frac{3}{2}n$

$T_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n = T_n = 1x2x3 + 2x3x4 + 3x4x5 + \ldots + n(n+1)(n+2)$

$= \sum_{i=1}^{n} n(n+1)(n+2) = \sum_{i=1}^{n} (n^3 + 3n^2 + 2n)$

$= \sum_{i=1}^{n} n^3 + \sum_{i=1}^{n} 3n^2 + \sum_{i=1}^{n} 2n$

Thus, $\sum_{i=1}^{n} n^3 = T_n - \sum_{i=1}^{n} 3n^2 - \sum_{i=1}^{n} 2n$

We know that $\sum_{i=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$ so $\sum_{i=1}^{n} 3n^2 = \frac{3n(n+1)(2n+1)}{6} = \frac{n(n+1)(2n+1)}{2}$ and that $\sum_{i=1}^{n} 2n$ =2+4+6+8+…+2n can be rewritten by factoring out the 2 to make it 2(1+2+3+…+n). We already know 1+2+3+…+n=$\frac{n(n+1)}{2}$ so, 2 times $\frac{n(n+1)}{2}$ becomes simply n(n+1). Also, we know that $T_n = -\frac{1}{24}n^4 + \frac{5}{12}n^3 - \frac{35}{24}n^2 + \frac{25}{12}n$ so plug these values in and:

$\sum_{i=1}^{n} n^3 = T_n - \sum_{i=1}^{n} 3n^2 - \sum_{i=1}^{n} 2n = \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + \frac{3}{2}n - \frac{n(n+1)(2n+1)}{2} - n(n+1)$

$= \frac{1n^4 + 6n^3 + 11n^2 + 6n - 2(2n^3 + 3n^2 + n) - 4n^2 - 4n}{4} =$

$\frac{1n^4 + 6n^3 + 11n^2 + 6n - 2(2n^3 + 3n^2 + n) - 4n^2 - 4n}{4} = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n^2 + 2n + 1)}{4}$

$$=\frac{n^2(n+1)^2}{4}$$

Thus, my conjecture is that: $\sum_{i=1}^{n} n^3 = \frac{n^2(n+1)^2}{4}$

***Check*:**

n = 1; $\sum_{i=1}^{n} n^3 = 1^3 = \mathbf{1}$

now try this with the above equation… $\frac{1^2(1+1)^2}{4} = \mathbf{1}$ [it fits!]

n = 2; $\sum_{i=1}^{n} n^3 = 1^3 + 2^3 = \mathbf{9}$

now with the above equation… $\frac{2^2(2+1)^2}{4} = \mathbf{9}$ [it fits!]

n = 3; $\sum_{i=1}^{n} n^3 = 1^3 + 2^3 + 3^3 = \mathbf{36}$

now with the above equation… $\frac{3^2(3+1)^2}{4} = \mathbf{36}$ [it fits!]

Let's check if this conjecture works all other values of n using an induction proof:

<u>Step 1</u> Assume the conjecture to be true for n = 1

As shown above,

$$\sum_{i=1}^{n} n^3 = 1^3 = \frac{1^2(1+1)^2}{4} = 1$$

<u>Step 2</u> Assume the conjecture to be true for n = k

So $\sum_{i=1}^{k} k^3 = \frac{k^2(k+1)^2}{4}$

<u>Step 3</u> Observe for if n = k + 1

$\sum_{i=1}^{k+1} (k+1)^3 = 1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3$ which, according to the formula, should equal

$$\boxed{\frac{(k+1)^2(k+2)^2}{4}}$$

$$\sum_{i=1}^{k+1} (k+1)^3 = \sum_{i=1}^{k+1} k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4k^3 + 12k^2 + 12k + 4}{4}$$

$$= \frac{k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4}{4} = \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} = \boxed{\frac{(k+1)^2(k+2)^2}{4}}$$

$\frac{(k+1)^2(k+2)^2}{4}$ is equal to the equation above, so $\sum_{i=1}^{n} n^3 = \frac{n^2(n+1)^2}{4}$ is true.

4. Consider U_n = 1x2x3x4 + 2x3x4x5 + 3x4x5x6 + … + n(n+1)(n+2)(n+3)
   a) Determine several values of U_k including $U_1, U_2, U_3, \ldots, U_6$

      $U_1 = a_1$ = 1 x 2 x 3 x 4 = 24
      $U_2 = a_1 + a_2 = U_1 + a_2$ = 24 + 2 x 3 x 4 x 5 = 144
      $U_3 = a_1 + a_2 + a_3 = U_2 + a_3$ = 144 + 3 x 4 x 5 x 6 = 504
      $U_4 = a_1 + a_2 + a_3 + a_4 = U_3 + a_4$ = 504 + 4 x 5 x 6 x 7 = 1344
      $U_5 = = a_1 + a_2 + a_3 + a_4 + a_5 = U_4 + a_5$ = 1344 + 5 x 6 x 7 x 8 = 3024
      $U_6 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = U_5 + a_6$ = 3024 + 6 x 7 x 8 x 9 = 6048
      $U_7 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = U_6 + a_7$ = 6048 + 7 x 8 x 9 x 10 = 11088
      $U_8 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 = U_7 + a_8$ = 11088 + 8 x 9 x 10 x 11 = 19008

   It seems that for every increasing value of k, the sum of the previous numbers plus the new number yields the new sum.

   b) Thus, the conjecture is that: $\mathbf{U_k = U_{k-1} + a_k}$
   c) Prove conjecture by induction:

      Step 1 Assume the conjecture to be true for n = 1
      As shown above,

      $U_1 = a_1$ = 1 x 2 x 3 x 4 = 24
      $U_2 = a_1 + a_2 = T_1 + a_2$ = 24 + 2 x 3 x 4 x 5 = 144
      $U_3 = a_1 + a_2 + a_3 = T_2 + a_3$ = 144 + 3 x 4 x 5 x 6 = 504

      So $\mathbf{U_n = U_{n-1} + a_n}$

      Step 2 Assume the conjecture to be true for n = k (done in part a of 2)
      So $U_k = U_1 + U_2 + U_3 + \ldots + U_k = U_{k-1} + a_k$
      $U_k = U_{k-1} + a_k$

      Step 3 Observe for if n = k + 1
      According to conjecture it should be: $U_{k+1} = U_k + a_{k+1}$
      So substitute $U_{k-1} + a_k$ for U_k (in step 2):
      $U_{k+1} = (U_{k-1} + a_k) + a_{k+1} = U_{k-1} + a_k + a_k(a_1)$

   d) Using the above result, calculate $1^4 + 2^4 + 3^4 + 4^4 + \ldots + n^4$

   Taking the sums of terms a_1= 1x2x3x4, a_2 = 2x3x4x5, a_3 = 3x4x5x6, a_4 = 4x5x6x7… from #5, it can be seen that, as it was with the differences of cubes and squares, the differences of consecutive terms to the fourth power turns out to be a common number. Except this time, this number was 24 after 5 rounds of subtracting in this fashion:

```
24   144   504   1344   3024   6048   11088   19008
   120   360   840   1680   3024   5040   7920
      240   480   840   1344   2016   2880
         240   360   504   672   864
            120   144   168   192
               24    24    24
```

The equation that I will set as my conjecture will be of degree 5, in the form $S_n = an^5 + bn^4 + cn^3 + dn^2 + en + f$. There are 6 unknowns, so I set up 6 equations:

$U_1 = a + b + c + d + e + f$ which equals 24 (from part a)
$U_2 = a2^5 + b2^4 + c2^3 + d2^2 + 2e + f$
$= 32^a + 16b + 8c + 4d + 2e + f = 144$
$U_3 = a3^5 + b3^4 + c3^3 + d3^2 + 3e + f$
$= 243a + 81b + 27c + 9d + 3e + f = 504$
$U_4 = a4^5 + b4^4 + c4^3 + d4^2 + 4e + f$
$= 1024a + 256b + 64c + 16d + 4e + f = 1344$
$U_5 = a5^5 + b5^4 + c5^3 + d5^2 + 5e + f$
$= 3125a + 625b + 125c + 25d + 5e + f = 3024$
$U_6 = a6^5 + b6^4 + c6^3 + d6^2 + 6e + f$
$= 7776a + 1296b + 216c + 36d + 6e + f = 6048$

These values were entered into the GDC as A $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 32 & 16 & 8 & 4 & 2 & 1 \\ 243 & 81 & 27 & 9 & 3 & 1 \\ 1024 & 256 & 64 & 16 & 4 & 1 \\ 3125 & 625 & 125 & 25 & 5 & 1 \\ 7776 & 1296 & 216 & 36 & 6 & 1 \end{bmatrix}$ and B $\begin{bmatrix} 24 \\ 144 \\ 504 \\ 1344 \\ 3024 \\ 6048 \end{bmatrix}$ and it was seen by reduced row echelon form that $a = \frac{1}{5}$, b = 2, c = 7, d = 10, $e = \frac{24}{5}$, f = 0.

Therefore, $U_n = \frac{1}{5}n^5 + 2n^4 + 7n^3 + 10n^2 + \frac{24}{5}n$

$U_n = 1x2x3x4 + 2x3x4x5 + 3x4x5x6 + \ldots + n(n+1)(n+2)(n+3)$

$$= \sum_{i=1}^{n} n(n+1)(n+2)(n+3) = \sum_{i=1}^{n} (n^4 + 6n^3 + 11n^2 + 6n)$$

$$= \sum_{i=1}^{n} n^4 + \sum_{i=1}^{n} 6n^3 + \sum_{i=1}^{n} 11n^2 + \sum_{i=1}^{n} 6n$$

Thus, $\sum_{i=1}^{n} n^4 = U_n - \sum_{i=1}^{n} 6n^3 - \sum_{i=1}^{n} 11n^2 - \sum_{i=1}^{n} 6n$

We know that $\sum_{i=1}^{n} n^3 = \frac{n^2(n+1)^2}{4}$ so $\sum_{i=1}^{n} 6n^3 = \frac{3n^2(n+1)^2}{2}$

Also, we know $\sum_{i=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$ so $\sum_{i=1}^{n} 11n^2 = \frac{11n(n+1)(2n+1)}{6}$ and that $\sum_{i=1}^{n} 6n$ =6+12+18+…+6n can be rewritten by factoring out the 6 to make it 6(1+2+3+…+n). We already know 1+2+3+…+n=$\frac{n(n+1)}{2}$ so, 6 times $\frac{n(n+1)}{2}$ becomes simply 3n(n+1). Finally, we know that $U_n = \frac{1}{5}n^5 + 2n^4 + 7n^3 + 10n^2 + \frac{24}{5}n$ so plug these values in and:

$$\sum_{i=1}^{n} n^4 = U_n - \sum_{i=1}^{n} 6n^3 - \sum_{i=1}^{n} 11n^2 - \sum_{i=1}^{n} 6n = \frac{1}{5}n^5 + 2n^4 + 7n^3 + 10n^2 + \frac{24}{5}n - \frac{3n^2(n+1)^2}{2} - \frac{11n(n+1)(2n+1)}{6} - 3n(n+1)$$

$$= \frac{6n^5 + 60n^4 + 210n^3 + 300n^2 + 144n - 45n^2(n^2 + 2n^2 + 1) - 55(2n^3 + 4n^2 + n) - 90n^2 - 90n}{30}$$

$$= \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$$

Thus, my conjecture is that: $\sum_{i=1}^{n} n^4 = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}$

***Check*:**

n = 1; $\sum_{i=1}^{n} n^4 = 1^4 =$ **1**

now try this with the above equation… $\frac{1(1+1)(2\times1+1)(3\times1^2 + 3\times1 - 1)}{30} =$ **1** [it fits!]

n = 2; $\sum_{i=1}^{n} n^4 = 1^4 + 2^4 =$ **17**

now with the above equation… $\frac{2(2+1)(2\times2+1)(3\times2^2 + 3\times2 - 1)}{30} =$ **17** [it fits!]

n = 3; $\sum_{i=1}^{n} n^4 = 1^4 + 2^4 + 3^4 =$ **98**

now with the above equation… $\frac{3(3+1)(2\times3+1)(3\times3^2 + 3\times3 - 1)}{30} =$ **98** [it fits!]

Let's check if this conjecture works all other values of n using an induction proof*:*

Step 1 Assume the conjecture to be true for n = 1

As shown above,

$$\sum_{i=1}^{n} n^4 = 1^4 = \frac{1(1+1)(2\times1+1)(3\times1^2 + 3\times1 - 1)}{30} = 1$$

Step 2 Assume the conjecture to be true for n = k

So $\sum_{i=1}^{k} k^4 = \frac{k(k+1)(2k+1)(3k^2 + 3k - 1)}{30}$

Step 3 Observe for if n = k + 1

$\sum_{i=1}^{k+1} (k+1)^4 = 1^4 + 2^4 + 3^4 + ... + k^4 + (k+1)^4$ which, according to the formula, should equal

$$\frac{(k+1)(k+2)(2k+3)[3(k+1)^2+3k+2]}{30} = \boxed{\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}}$$

$$\sum_{i=1}^{k+1}(k+1)^4 = \sum_{i=1}^{k}k^4 + (k+1)^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4$$

$$= \frac{k(k+1)(2k+1)(3k^2+3k-1)+30(k+1)^2(k+1)^2}{30}$$

$$= \boxed{\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}}$$

$\frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}$ is equal to the equation above, so

$\sum_{i=1}^{n} n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ is true.