Ecolint – DP 2009/10

Properties of Quartics

Math HL – Portfolio Assignment

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Introduction

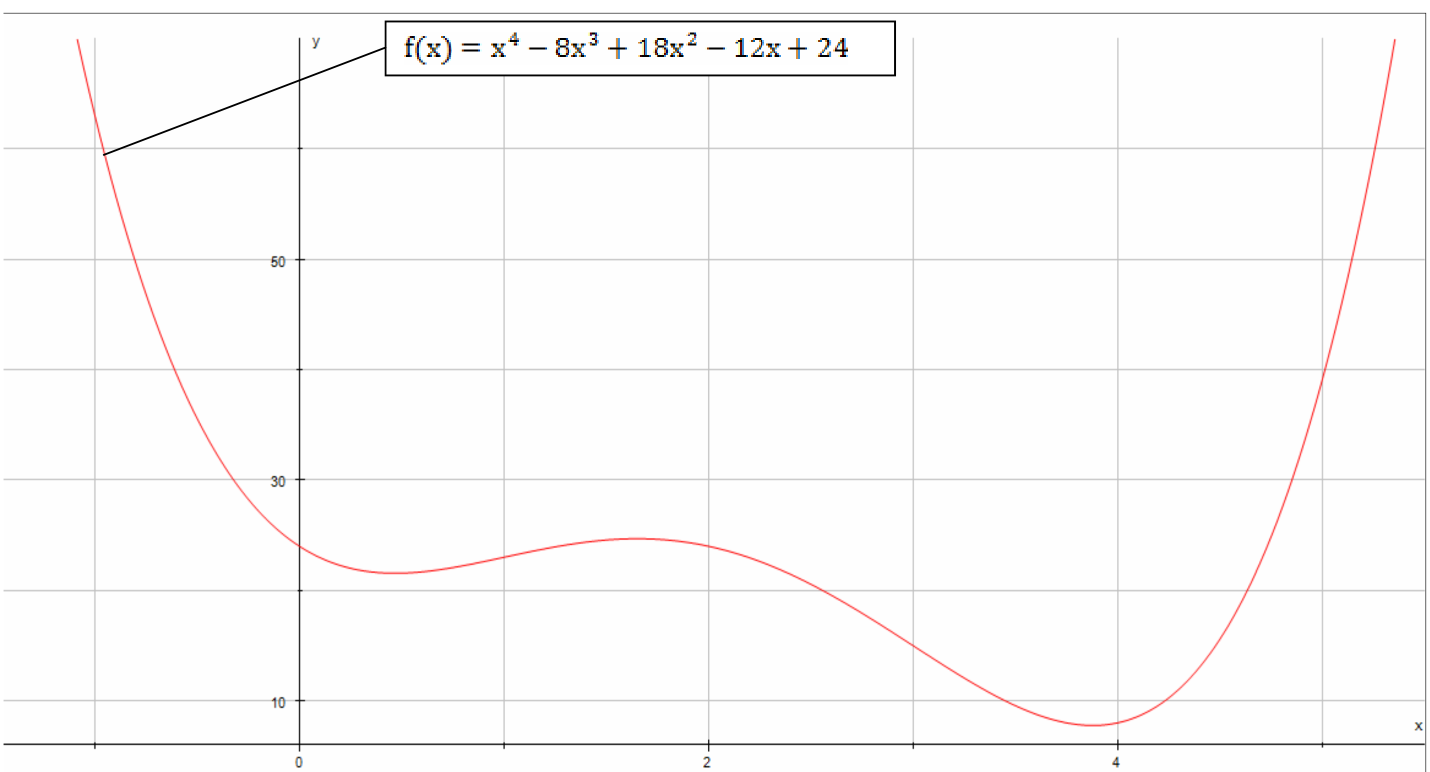
Quartic functions are functions that the highest exponent is 4. These types of functions are of the form

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

The graphs of a Quartic functions usually exert two shapes; "W" shape or "M" shape. For this investigation, an analysis of a "W" shaped function is to be carried out to explore the properties of the function. The points of inflection of the Quartic function, will be looked at very closely so that the ratio between the distances of the points if intersection when the Quartic graph is cut by a straight line is found.

Analysis

Let's take $f(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$. The second derivative of this function $f(x)$ will gives the points inflection at $x = 0$; provided $f''(x) = 0$.



<u>FIND 2ND DERIVATIVE OF f''(x)</u>

$$f(\mathrm{x}) = \mathrm{x}^4 - 8\mathrm{x}^3 + 18\mathrm{x}^2 - 12\mathrm{x} + 24$$
$$\therefore f'(\mathrm{x}) = 4\mathrm{x}^3 - 24\mathrm{x}^2 + 36\mathrm{x} - 12$$
$$\therefore f''(\mathrm{x}) = 12\mathrm{x}^2 - 48\mathrm{x} + 36$$
$$= 12(\mathrm{x}^2 - 4x + 3)$$
$$= 12(x-3)(x-1)$$

$\therefore f''(x) = 0$ when $\mathrm{x} = 3$ and $\mathrm{x} = 1$

At $x = 3$ and $x = 1$ is where the points of inflection are located at the original function i.e. $f(x)$

$\therefore$ Substitute the x values in the $f(x)$

$(i)\ f(3) = (3)^4 - 8(3)^3 + 18(3)^2 - 12(3) + 24$

$=$ R (3, 15)

$(ii)\ f(1) = (1)^4 - 8(1)^3 + 18(1)^2 - 12(1) + 24$

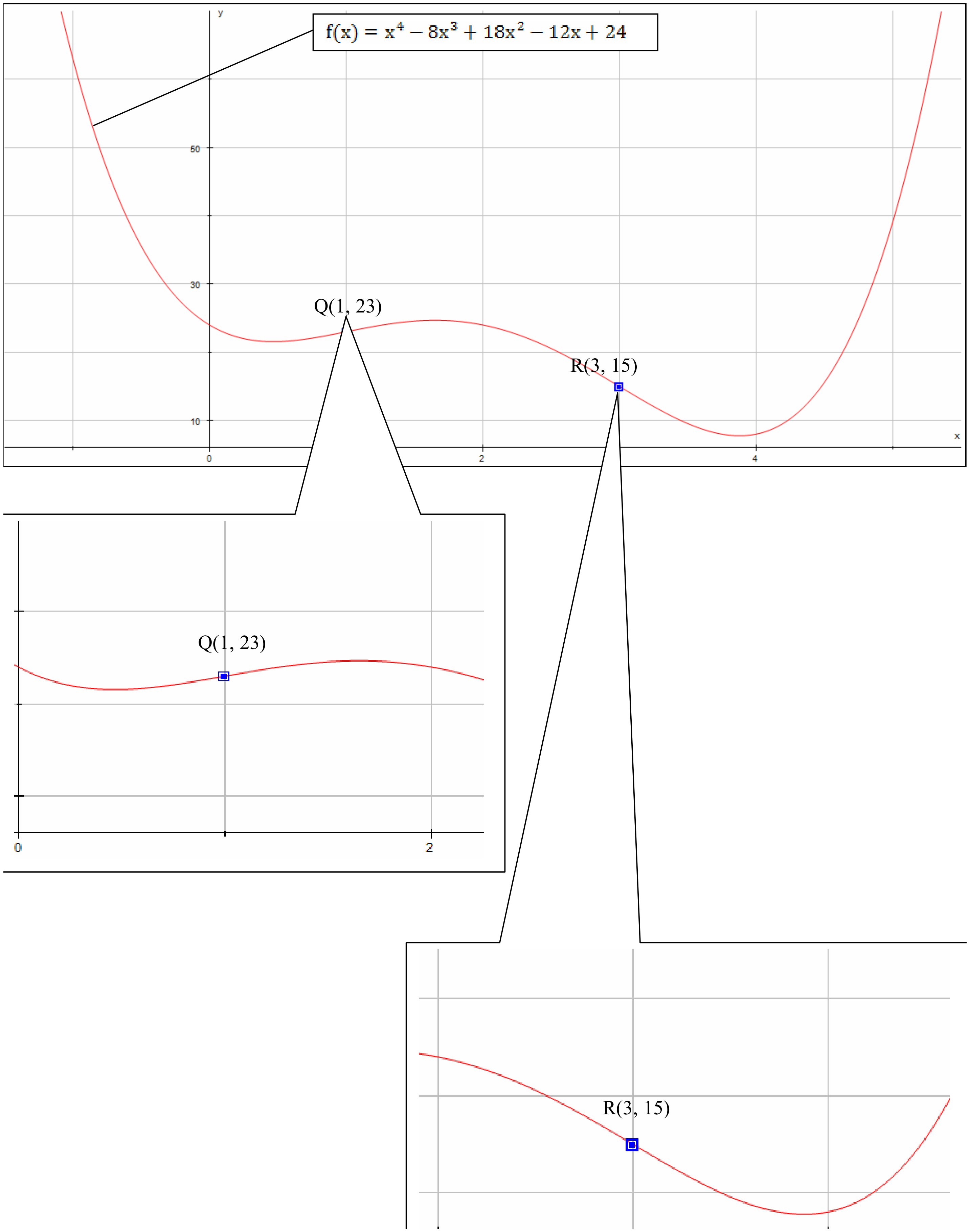
$= \mathrm{Q}\ (1, 23)$

Since $f'(x) \neq 0$ for both x points $\therefore$ both points of inflection are non-horizontal points of inflection.

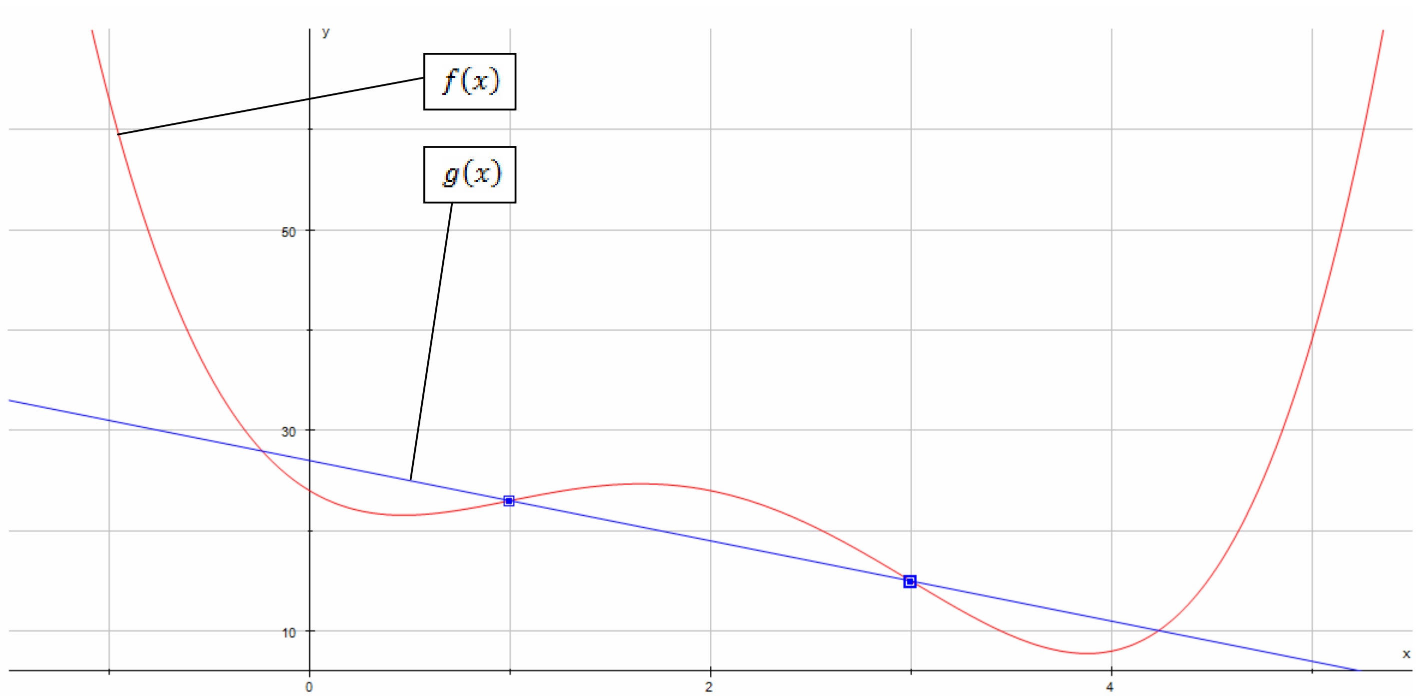
Once the two points of inflection are found, a straight line is drawn so that the two points of inflection (Q and R) meet the quartic function i.e. $f(x)$. When this straight line is drawn, the line meets the Quartic again at another two points P and S creating three identical segments.

It is important the coordinates of these two points, so that the ratio PQ:QR:RS is determined. The next step for completing this investigation is to calculate the equation of the straight line. From the equation of the straight line, the coordinates of point P and S will be obtained, hence, the distance between PQ→QR→RS, thus, leading to the ratio segments.

So, the graph of $f(x)$ with the points of inflection would look like this.


$f(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$
Q(1, 23)
R(3, 15)
Q(1, 23)
R(3, 15)
y
x
50
30
10
0
2
4
0
2

As mentioned before a line is to be drawn, passing through the points of inflection (Q and R). After this is done the equation of the line can be calculated. The new graph would look like



this.

FINDING THE EQUATION OF THE STRAIGHT LINE

Find the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\therefore m = \frac{15 - 23}{3 - 1}$$

$$\therefore m = -4$$

Find the equation of the line

$$y - y_1 = m(x - x_1)$$

$$\therefore y - 23 = -4(x - 1)$$

$$\therefore y - 23 = -4x + 4$$

$$\therefore y = -4x + 27$$

Now with the equation of line, the coordinates of the remaining points of intersection – between Quartic function $f(x)$ and the straight line $g(x)$ – may be calculated. This calculation can be done from combining both equations i.e. $f(x)$ and $g(x)$ obtained from the graph. After equating the result will be a Quartic function.

[1] $y = -4x + 27$

[2] $f(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$

$$\therefore \; -4x + 27 = x^4 - 8x^3 + 18x^2 - 12x + 24$$

$$\therefore = x^4 - 8x^3 + 18x^2 - 12x + 24 + 4x - 27$$

$$\therefore = x^4 - 8x^3 + 18x^2 - 8x - 3$$

Since the final result of the Quartic equation, it is likely that 4 roots are going to be available. However, there are two roots known from previously calculations – 1 and 3 – in this case Q and R. Thus, to find the other two remaining roots we would need then to reduce the Quartic equation to a square equation so that only two roots are found. Yet, It is vital to take into account the roots already found thus eliminating the possibility of second derivative of the quartic function. The only way to perform this process would be the synthetic division. Synthetic division is the division of polynomials by a factor of form $(x - \alpha)$ in this case $(x - 1)$ and $(x - 3)$, resulting on a quotient and a constant (*the constant of this calculation should be 0*).

SYNTHETIC DIVISION

	1	-8	18	-8	-3
1		1	-7	11	3
	1	-7	11	3	0
3		3	-12	-3	
	1	-4	-1	0	

– these are the coefficient of

$$\therefore\ x^2 - 4x - 1$$

The quadratic equation will be used to obtain the roots which are equivalent to the missing points P and S.

$$0 = \ x^2 - 4x - 1$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$$

$$\therefore x = 2 - \sqrt{5} \text{ or } 2 + \sqrt{5}$$

The two roots found (above) are the x coordinates of the missing points and they happen to be irrational numbers.

P since $x = 2 + \sqrt{5}$ $\therefore P((2 + \sqrt{5}), f(2 + \sqrt{5}))$

$$\therefore f(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$$

$$\therefore f(2 - \sqrt{5}) = (2 - \sqrt{5})^4 - 8(2 - \sqrt{5})^3 + 18(2 - \sqrt{5})^2 - 12(2 - \sqrt{5}) + 24$$

$$\therefore y = 27.94$$

$$\therefore P\ ((2 - \sqrt{5}), 27.94)$$

S since $x = (2 + \sqrt{5})$ $\therefore$ S $((2 + \sqrt{5}), f(2 + \sqrt{5}))$

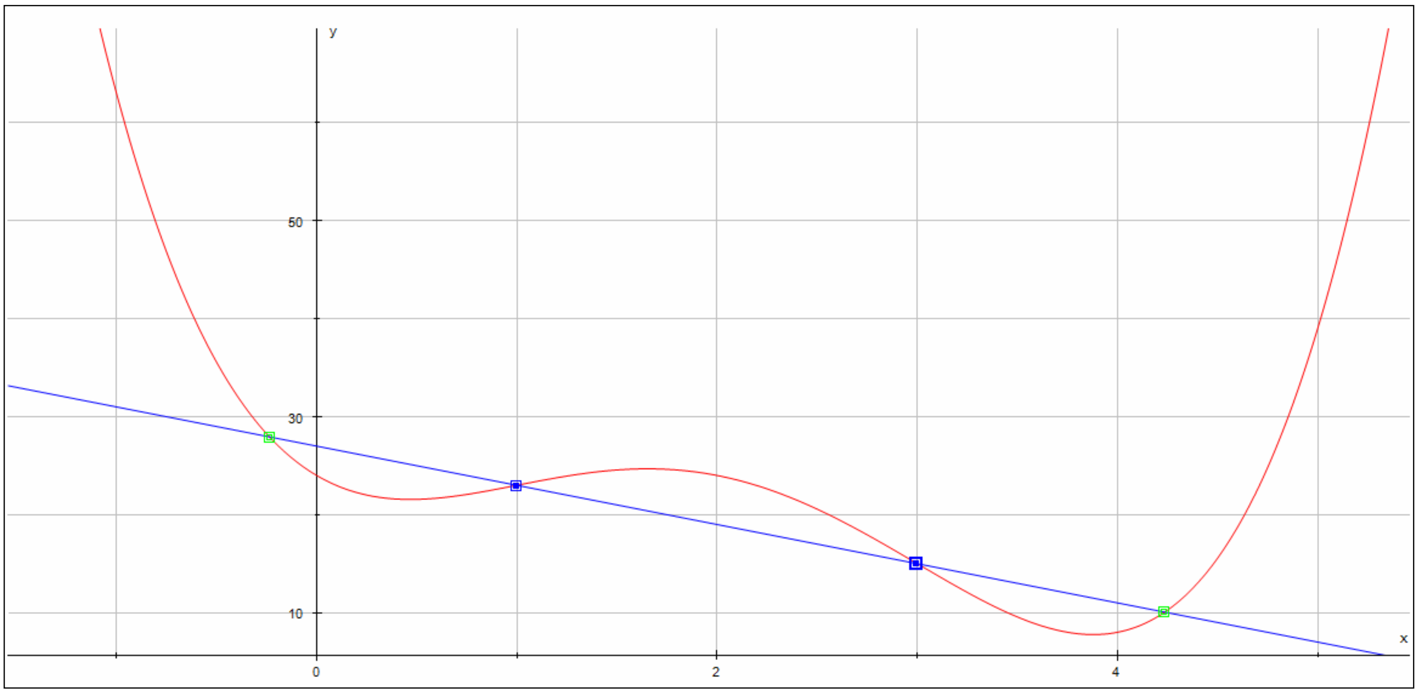
$$\therefore\ f(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$$

$$\therefore f\left(2-\sqrt{5}\right) = \left(2-\sqrt{5}\right)^4 - 8\left(2-\sqrt{5}\right)^3 + 18\left(2-\sqrt{5}\right)^2 - 12\left(2-\sqrt{5}\right) + 24$$

$$\therefore y = 10.06$$

$$\therefore S\left(\left(2-\sqrt{5}\right), 10.06\right)$$

Once found the points P and S, the new graph would look like the one below.



The goal of this investigation is with the points of inflections and the extension of the line passing through them – $g(x) = x^2 - 4x - 1$ – the ratio between the points would be found. The distance formula has its roots from the Pythagoras' theorem. Hence, if a triangle is drawn between the points linking them, a triangle should be visible (see the graph below). Again, the y coordinates can be ignored as to they are not exact. Thus, when calculating the ratio the x coordinates are used.