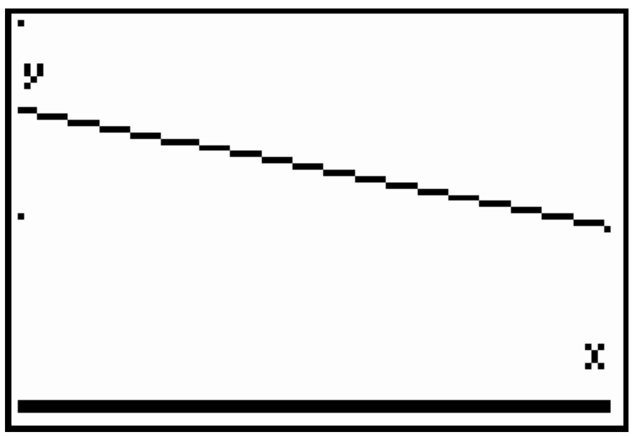
We were first given the equation $u_{n+1} = r \cdot u_n$ where r is the growth factor and u_n is the population at year n. A logistic model is one that grows and then stabilizes over time. This will be shown in the following set of problems.

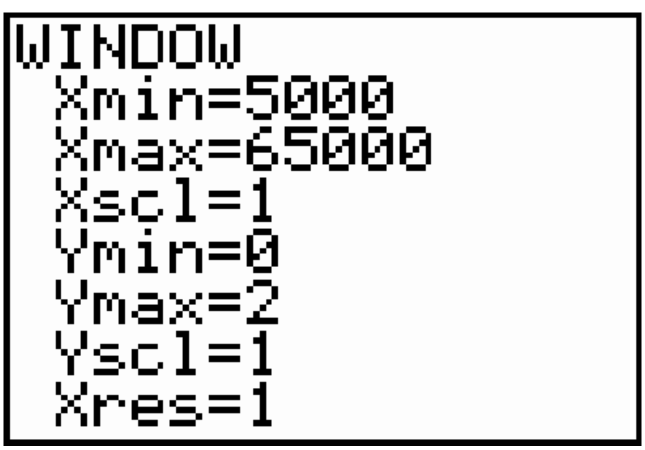
1. The biologist estimates that if 10,000 fish were introduced into a lake, then the population of the fishes would increase by 50% the first year but then level off and never exceed 60,000 fishes. Therefore we are given a set of initial and eventual numbers that we can assign variables to:

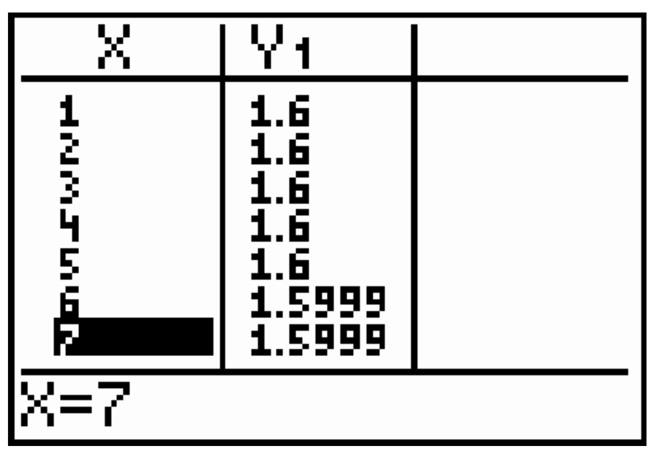
$$u_0 = 10000$$
$$u_n = 60000$$
$$r_0 = \frac{u_{n+1}}{u_n} = \frac{15000}{10000} = 1.5$$
$$r_n = 1$$

Thus we can fill the ordered pairs (u_0, r_0), (u_n, r_n) as $(10000, 1.5)$ and $(60000, 1)$ and these points are indicated on the graph below:



The window and table values for the graph above are:





The window of the graph shows the scale of the graph and the values give evidence for the points on the graph.

The slope of the line with coordinates $(10000, 1.5)$ and $(60000, 1)$ is:

$$m = \frac{1-1.5}{60000-10000} = \frac{-0.5}{50000} = -0.00001 = -1\times 10^{-5}$$

$$r - 1.5 = (-1\times 10^{-5})(u_n - 10000)$$

Therefore $r - 1.5 = (-1\times 10^{-5})u_n + 0.1$

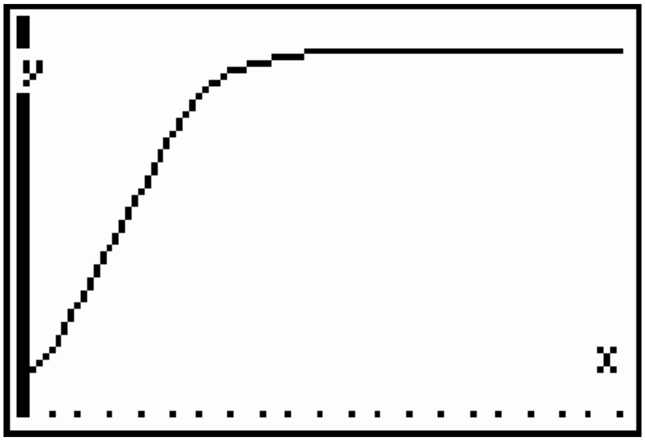
$$r = (-1\times 10^{-5})u_n + 1.6.$$

$\therefore r = (-1\times 10^{-5})u_n + 1.6$ is the linear growth factor in terms of u_n with a slope, $m = (-1\times 10^{-5})$.

2. Since $u_{n+1} = r \cdot u_n$ and we figured $r = (-1\times 10^{-5})u + 1.6$, we can derive:

$$u_{n+1} = [(-1\times 10^{-5})u_n + 1.6]u_n .$$

3. In order to determine the fish population over the next 20 years, we can use the logistic function model found in 2 and graph this. The graph of the function and fish population over the next 20 years is shown below:



The window set for the above graph is shown below as well:

WINDOW
 nMin=0
 nMax=20
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=20
↓Xscl=1

WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=20
 Xscl=1
 Ymin=5000
 Ymax=65000
 Yscl=1

The table values for the above graph are shown below:

n	u(n)
0	10000
1	15000
2	21750
3	30069
4	39069
5	47247
6	53272

n=0

n	u(n)
7	56856
8	58644
9	59439
10	59772
11	59908
12	59963
13	59985

n=7

n	u(n)
14	59994
15	59998
16	59999
17	60000
18	60000
19	60000
20	60000

n=20

As we can conclude from the graph is that it is the shape of a typical logistic graph since it greatly increase initially and then over time levels off which is what will happen to the fish population as well. The table values further prove this because the number are rapidly increasing but than after the 17th year begin to become stable at the projected upper bound, 60000.

4. In order to compare the graphs with different initial growth rates of $r = 2$, 2.3, and 2.5 we will have different coordinate points leading to different linear growth equations and different logistic function models.

When $r = 2$:

Coordinates: $(10000, 2)$ and $(60000, 1)$

$$m = \frac{1-2}{60000-10000} = \frac{-1}{50000} = -0.00002 = -2\times 10^{-5}$$

$$r = (-2\times 10^{-5})u + 2.2$$

$$u_{n+1} = [(-2\times 10^{-5})u_n + 2.2]u_n$$

When $r = 2.3$:

Coordinates: $(10000, 2.3)$ and $(60000, 1)$

$$m = \frac{1-2.3}{60000-10000} = \frac{-1.3}{50000} = -0.000026 = -2.6\times 10^{-5}$$

$$r = (-2.6\times 10^{-5})u + 2.56$$

$$u_{n+1} = [(-2.6\times 10^{-5})u_n + 2.56]u_n$$

When $r = 2.5$:

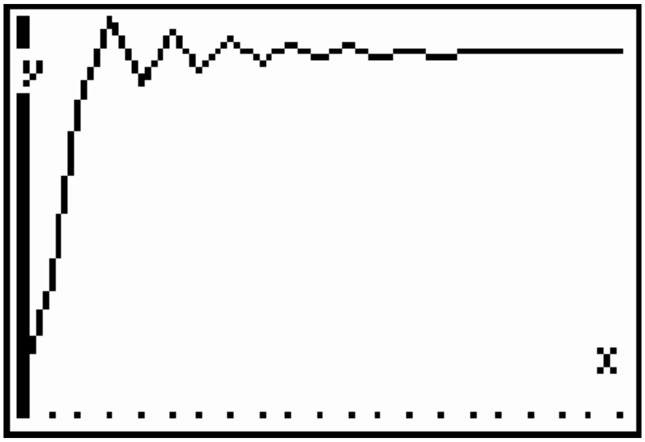
Coordinates: $(10000, 2.5)$ and $(60000, 1)$

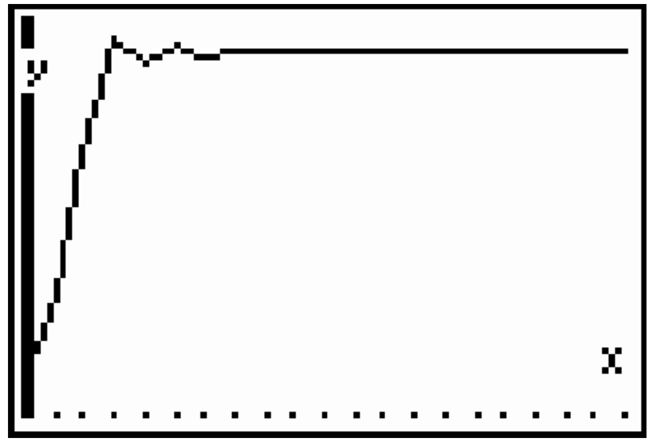
$$m = \frac{1-2.5}{60000-10000} = \frac{-1.5}{50000} = -0.00003 = -3\times 10^{-5}$$

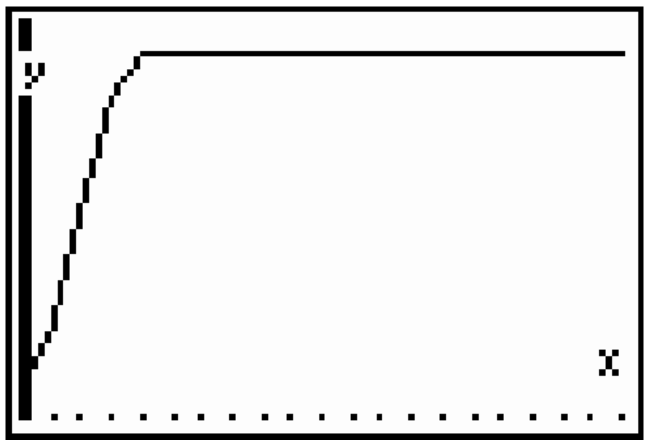
$$r = (-3\times 10^{-5})u + 2.8$$

$$u_{n+1} = [(-3\times 10^{-5})u_n + 2.8]u_n$$

Their respective graphs in order when $r = 2$, 2.3, and 2.5 are shown below:







All three graphs vary initially but over time level off and become stable. Like the graph with an initial growth rate $r = 2$, all three graphs are logistic graphs.

5. When $r = 2.9$:

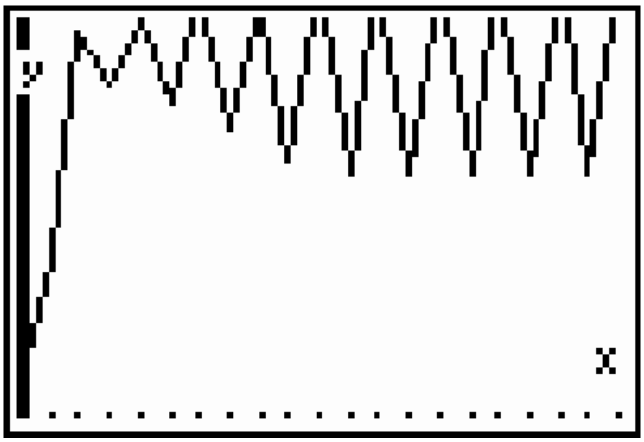
Coordinates: $(10000, 2.9)$ and $(60000, 1)$

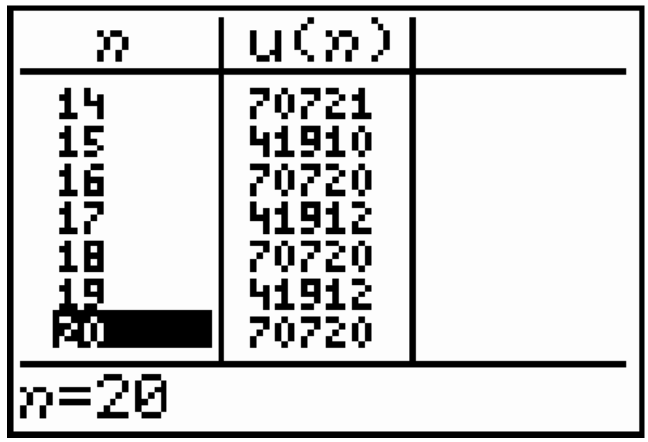
$$m = \frac{1-2.9}{60000-10000} = \frac{-1.9}{50000} = -0.000038 = -3.8\times10^{-5}$$

$$r = (-3.8\times10^{-5})u + 3.28$$

$$u_{n+1} = [(-3.8\times10^{-5})u_n + 3.28]u_n$$

Also, its graph and table values are shown below:





n	u(n)
14	70721
15	41910
16	70720
17	41912
18	70720
19	41912
20	70720

n=20

The peculiar outcome of the graph is noted and can be accredited to higher initial growth rate which makes it harder for the fish population to stabilize. From the table values one can observe that after the 14th year, the fish population fluctuates from 41910 and 70720 because of the higher growth rate. Looking past 20 years the population continues to fluctuate never seeming to stabilize.

6. Realizing the fish population has stabilized after 20 years, if the biologist and regional managers would like to see the commercial possibility of an annual 5000 harvest value the difference equation would then be $u_{n+1} = ([(-1\times 10^{-5})u_n + 1.6]u_n) - 5000$. The graph, and table values would be the graph shown below:

n	u(n)
20	60000
21	55000
22	52750
23	51574
24	50920
25	50543
26	50323

n=20

Observing the graph and its values, one can see that the fish population will decrease and then slowly stabilize at a fish population of 50000 which is proved in the table value of the above graph below:

n	u(n)
35	50003
36	50002
37	50001
38	50001
39	50000
40	50000
41	50000

n=35

After the 39th year, we can see that the fish population stabilizes at 50000.

7. I initially planned to investigate the fish population with harvest sizes of 8000, 9000, and 10000 but was not able to because the population decreased too fast. I figured that the fish population died out between an annual harvest size of 9000 and 10000. Then, I changed the values to be more specific to 8500, 9000, and 9500 but still could not prove the death of the population and figured that the population died out between harvest sizes of 9000 and 9500. I found an annual harvest size of 9400 to be the last value in which I could prove that the fish population would eventually die out, therefore I investigated annual harvest sizes of 7000, 8000, and 9000 between year 20 and 70. The graphs or harvest sizes 7000, 8000, and 9000 are shown below in that order from year 20 to 70.

Noticing that the graphs were decreasing in later years, I tested different values of annual harvest after 9000 and found that an annual harvest of 9300 rapidly decreased after year 65 and again a large decrease after year 69 in the table value shown below:

n	u(n)	
64	19673	
65	18307	
66	16639	
67	14554	
68	11868	
69	8281	
70	3263.8	

n=70

The fish population would die out after 70 years with a growth rate $r = 1.5$ and an annual harvest of 9300.

8. Similar to the difference equation in 6, since the harvest size in unknown we replace it with a variable for the harvest size, h , and in order to achieve stability u_n and u_{n+1} will be equal therefore can be replaced by x in the below equation:

$$x = [(-1 \times 10^{-5})x + 1.6]x - h$$
$$x = -0.00001x^2 + 1.6x - h$$
$$x^2 - 60000x + 100000h = 0$$
$$x = \frac{60000 \pm \sqrt{(60000)^2 - 4(1)(100000h)}}{2}$$

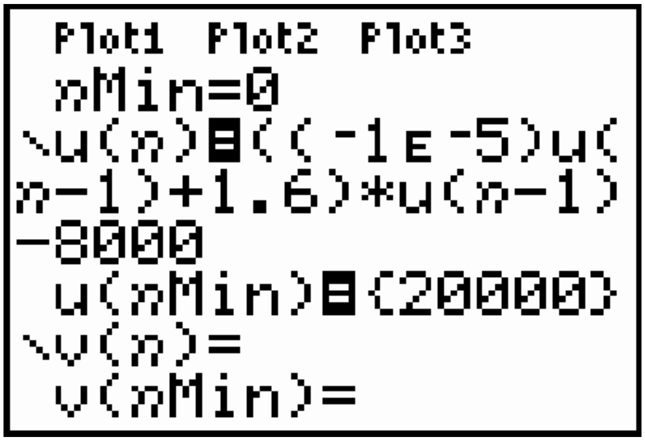
Since the equation inside the square root must be greater than zero to satisfy the equation, we can solve for the highest value of h by setting the value inside the square root having to be greater than zero.

$$(60000)^2 - 400000h > 0$$
$$400000h < (60000)^2$$
$$h < 9000.$$

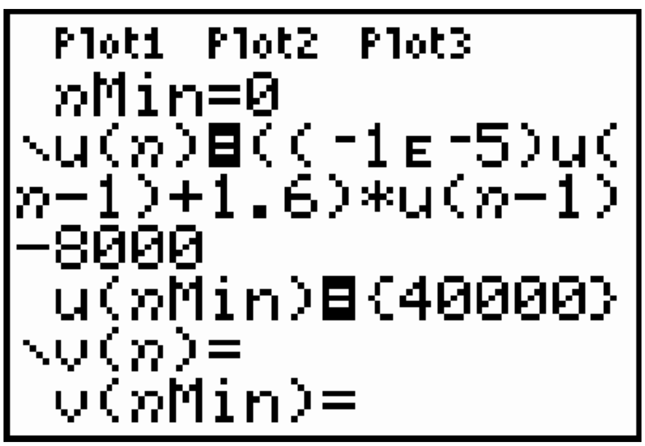
Therefore we can conclude that the maximum annual sustainable harvest is when $h = 9000$.

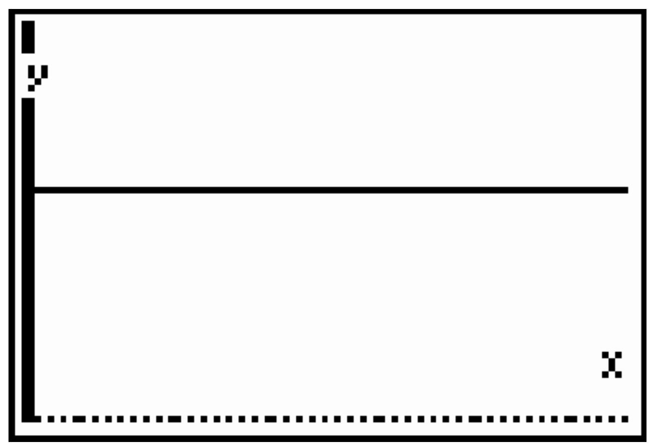
9. Knowing that $u_n = [(-1\times 10^{-5})u_n + 1.6]u_n - 8000$, we are able to use different values of u_n other than 60000. I chose initial populations of 20000 and 40000 because both show that with a harvest size of 8000 they become steady well before reaching 60000 fish.

When $u_0 = 20000$:

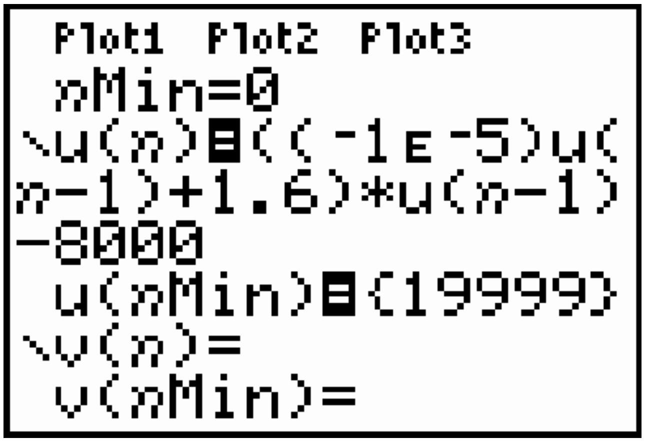


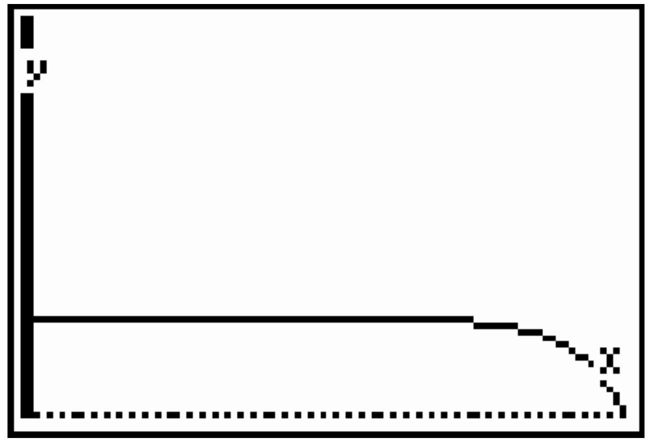
When $u_0 = 40000$:





But, since these values both lead us to stable populations we must find the initial population which does not become stable. This is shown when the initial population is 19999 in the graph below:





Thus, we can conclude that the politicians will have to wait until the population reaches 20000 before an annual harvest of 8000. They will only have to wait 2 years.