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Math Portfolio SL

MATRIX BINOMIALS

NONAME STUDENT

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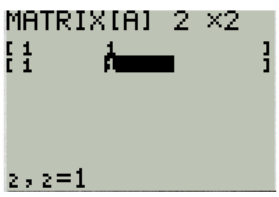
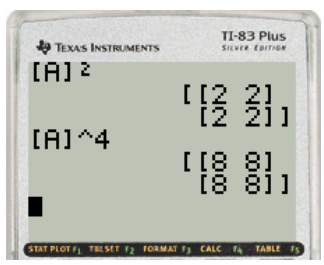
MATRIX BINOMIALS

In this portfolio I will try to find a general statement and patterns for given matrix binomials exercises. For data processing I will use *TI-83 Plus (Sliver Edition)* graphing calculator. I will use my knowledge from patterns and matrix in order to find suitable formulas.

X=$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, **Y**=$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

<u>Find **X**^{2}, **X**^{3}, **X**^{4}; **Y**^{2}, **Y**^{3}, **Y**^{4}.</u>





X^2=$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$= $\begin{pmatrix} 1+1 & 1+1 \\ 1+1 & 1+1 \end{pmatrix}$ **X**^{3}=$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ **X**^{4}=$\begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$

Y^2=$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ **Y**^{3}=$\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$ **Y**^{4}=$\begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}$

<u>Find expressions for **X**^{n}, **Y**^{n}, **(X+Y)**^{n}</u>

- The entries double for every higher power of **X**, i.e.:

X^2= 2**X**= 2^1**X**
X^3= 4**X**= 2^2**X**
X^4= 8**X** = 2^3**X** follows
X^n=2^{n-1}**X**

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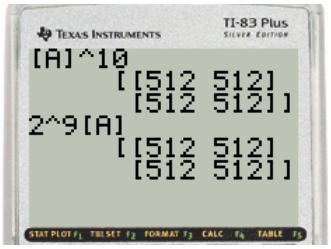
I will test this formula with a random number:

$\mathbf{X}^{10}=\begin{pmatrix}512 & 512\\ 512 & 512\end{pmatrix}$ $\qquad$ $\mathbf{X}^{10}= 2^{10-1}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}512 & 512\\ 512 & 512\end{pmatrix}$



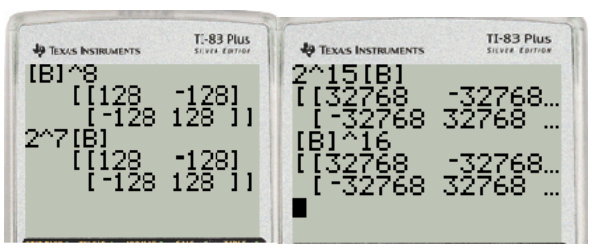
$\mathbf{X}^{19}= 2^{19-1}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}262144 & 262144\\ 262144 & 262144\end{pmatrix}$

$\mathbf{X}^{n}=2^{n-1}\mathbf{X}$ formula valid for all natural numbers ***N;*** $\mathbb{N}$ = {1,2,3,...} as it is shown in examples.

- The entries double for every higher power of **Y**, i.e.:

$\mathbf{Y}^2= 2\mathbf{Y}$
$\mathbf{Y}^3= 4\mathbf{Y}= 2^2\mathbf{Y}$
$\mathbf{Y}^4= 8\mathbf{Y} = 2^3\mathbf{Y}$ follows
$\mathbf{Y}^{n}=2^{n-1}\mathbf{Y}$

Again I will test is with random numbers using GDC:



$\mathbf{Y}^8=\begin{pmatrix}128 & -128\\ -128 & 128\end{pmatrix}$ $\qquad$ $\mathbf{Y}^8=2^{8-1}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}=\begin{pmatrix}128 & -128\\ -128 & 128\end{pmatrix}$

$\mathbf{Y}^{16}=\begin{pmatrix}32768 & -32768\\ -32768 & 32768\end{pmatrix}$ $\quad$ $\mathbf{Y}^{16}=2^{16-1}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}32768 & -32768\\ -32768 & 32768\end{pmatrix}$

$\mathbf{Y}^{n}=2^{n-1}\mathbf{Y}$ formula valid for all natural numbers ***N;*** $\mathbb{N}$ = {1,2,3,...} as it is shown in examples.

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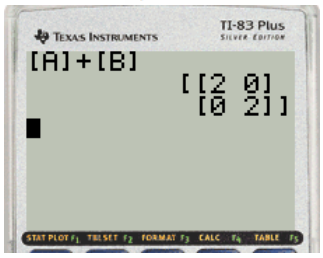
- In the case of **(X+Y)= 2I**

  (X+Y)$=\left(\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}+\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right)=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}=2\mathbf{I}$

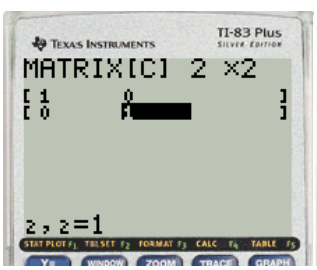
  (X+Y)$^{n}=\left(\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}+\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right)^{n}=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}^{n}=\begin{pmatrix}2^{n} & 0\\ 0 & 2^{n}\end{pmatrix}=2^{n}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

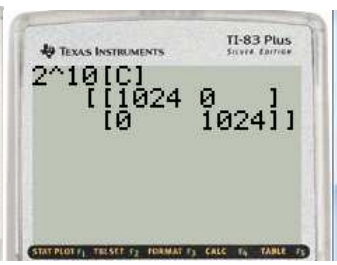


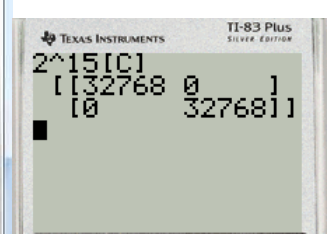
In this case we have two slightly different formulas, the first one includes the sum of **(X+Y)** and the second one includes the identify matrix, i.e. I just factorized 2^{n}.

$$\begin{pmatrix}2^{n} & 0\\ 0 & 2^{n}\end{pmatrix}=2^{n}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Testing with a random number and checking it with GDC.







(X+Y)$^{10}=\begin{pmatrix}2^{10} & 0\\ 0 & 2^{10}\end{pmatrix}=\begin{pmatrix}1024 & 0\\ 0 & 1024\end{pmatrix}$

(X+Y)$^{10}=2^{10}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1024 & 0\\ 0 & 1024\end{pmatrix}$

(X+Y)$^{15}=2^{15}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}32768 & 0\\ 0 & 32768\end{pmatrix}$

(X+Y)$^{15}=\begin{pmatrix}2^{15} & 0\\ 0 & 2^{15}\end{pmatrix}=\begin{pmatrix}32768 & 0\\ 0 & 32768\end{pmatrix}$

$2^{n}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ formula valid for all natural numbers ***N;*** $\mathbb{N}=\{1,2,3,...\}$ as it is shown in examples.

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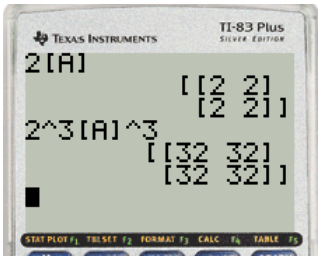
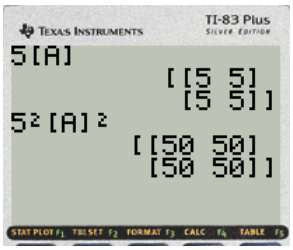
A=a**X** and **B**=b**Y**

X=$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, **Y**=$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Find **A**^{2}, **A**^{3}, **A**^{4}; **B**^{2}, **B**^{3}, **B**^{4} using different values of *a* and *b*.





a= 2

A=2**X**⇒**A**= $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

A^2=$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}^2$= $\begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$

A^3=$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}^3$= $\begin{pmatrix} 32 & 32 \\ 32 & 32 \end{pmatrix}$

A^4= $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}^4$=$\begin{pmatrix} 128 & 128 \\ 128 & 128 \end{pmatrix}$

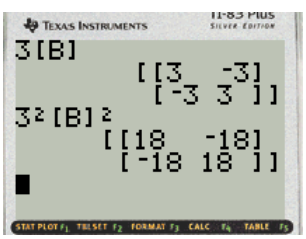
a=5

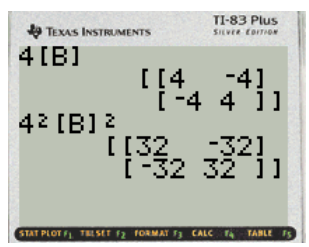
A=5**X**⇒**A**= $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$

A^2= $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}^2$= $\begin{pmatrix} 50 & 50 \\ 50 & 50 \end{pmatrix}$

A^3= $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}^3$= $\begin{pmatrix} 500 & 500 \\ 500 & 500 \end{pmatrix}$

A^4= $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}^4$=$\begin{pmatrix} 5000 & 5000 \\ 5000 & 5000 \end{pmatrix}$





b=3

B=3**Y**⇒**B**= $\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$

B^2=$\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}^2$= $\begin{pmatrix} 18 & -18 \\ -18 & 18 \end{pmatrix}$

b=4

B=4**Y**⇒**B**= $\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$

B^2= $\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}^2$= $\begin{pmatrix} 32 & -32 \\ -32 & 32 \end{pmatrix}$

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$\mathbf{B}^3=\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}^3=\begin{pmatrix} 108 & 108 \\ 108 & 108 \end{pmatrix}$ $\mathbf{B}^3=\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}^3=\begin{pmatrix} 256 & -256 \\ -256 & 256 \end{pmatrix}$

$\mathbf{B}^4=\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}^4=\begin{pmatrix} 648 & -648 \\ -648 & 648 \end{pmatrix}$ $\mathbf{B}^4=\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}^4=\begin{pmatrix} 2048 & -2048 \\ -2048 & 2048 \end{pmatrix}$

<u>Find expressions for $\mathbf{A}^n$, $\mathbf{B}^n$, $(\mathbf{A}+\mathbf{B})^n$</u>

- As I'm raising powers by one, the entries increase double the constant is. For example:

$\mathbf{A}= 6\mathbf{X} \Rightarrow \mathbf{A}=\begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix}$

$\mathbf{A}^2=\begin{pmatrix} 72 & 72 \\ 72 & 72 \end{pmatrix}$ $\mathbf{A}^3=\begin{pmatrix} 864 & 864 \\ 864 & 864 \end{pmatrix}$ $\mathbf{A}^4=\begin{pmatrix} 10368 & 10368 \\ 10368 & 10368 \end{pmatrix}$

As it is visible, every entry increases by 12, i.e. double the constant *a* is.

From this pattern I derived formula that:

$\mathbf{A}^n= a^n\mathbf{X}^n$

Testing with random numbers:

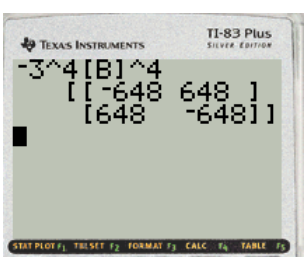
$\mathbf{A}^2=6^2\mathbf{X}^2$

$\mathbf{A}^2=36\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}=\begin{pmatrix} 72 & 72 \\ 72 & 72 \end{pmatrix}$

$\mathbf{A}=10\mathbf{X} \Rightarrow \mathbf{A}=\begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix}$ $\mathbf{A}^3=10^3\mathbf{X}^3$

$\mathbf{A}^3=\begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix}^3=\begin{pmatrix} 4000 & 4000 \\ 4000 & 4000 \end{pmatrix}$ $\mathbf{A}^3=\begin{pmatrix} 4000 & 4000 \\ 4000 & 4000 \end{pmatrix}$

The same pattern goes for **B**=b**Y**.



B=-3**Y**

$\mathbf{B}^2=\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}^2=\begin{pmatrix} -18 & 18 \\ 18 & -18 \end{pmatrix}$

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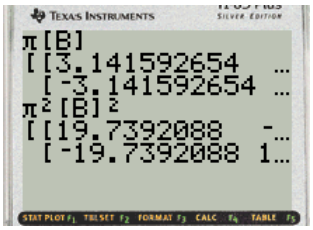
$\mathbf{B}^3=\begin{pmatrix}-3 & 3\\ 3 & -3\end{pmatrix}^3=\begin{pmatrix}-108 & 108\\ 108 & -108\end{pmatrix}$

$\mathbf{B}^4=\begin{pmatrix}-3 & 3\\ 3 & -3\end{pmatrix}^4=\begin{pmatrix}-648 & 648\\ 648 & -648\end{pmatrix}$ The entries increase by 6, i.e. the double b is.



$\mathbf{B}=\pi\mathbf{Y}$

$\mathbf{B}\approx\begin{pmatrix}3.14 & -3.14\\ -3.14 & 3.14\end{pmatrix}$

$\mathbf{B}^2\approx\begin{pmatrix}3.14 & -3.14\\ -3.14 & 3.14\end{pmatrix}^2\approx\begin{pmatrix}19.7 & -19.7\\ -19.7 & 19.7\end{pmatrix}$

The entries increase by app. 6.26, i.e. the double π is.

Therefore the derived formula is

$\mathbf{B}^n= b^n\mathbf{Y}^n$

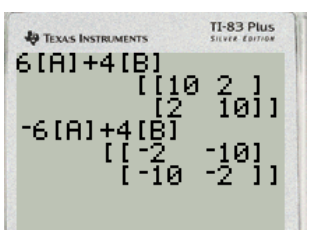
$\mathbf{B}^5= 3^5\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}^5$

$\mathbf{B}^5=\begin{pmatrix}3888 & -3888\\ -3888 & 3888\end{pmatrix}$

$\mathbf{B}=10\mathbf{Y} \Rightarrow \mathbf{B}=\begin{pmatrix}10 & -10\\ -10 & 10\end{pmatrix}$ $\mathbf{B}^3=10^3\mathbf{Y}^3$

$\mathbf{B}^3=\begin{pmatrix}10 & -10\\ -10 & 10\end{pmatrix}^3=\begin{pmatrix}4000 & -4000\\ -4000 & 4000\end{pmatrix}$ $\mathbf{A}^3=\begin{pmatrix}4000 & -4000\\ -4000 & 4000\end{pmatrix}$

- **<u>(A+B)^n</u>**



$\mathbf{(A+B)}= 6\mathbf{X} + 4\mathbf{Y}$

$\mathbf{(A+B)}=\begin{pmatrix}6 & 6\\ 6 & 6\end{pmatrix}+\begin{pmatrix}4 & -4\\ -4 & 4\end{pmatrix}=\begin{pmatrix}10 & 2\\ 2 & 10\end{pmatrix}$

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(A+B)= **-6X** + 4**Y**

(A+B)= $\begin{pmatrix} -6 & -6 \\ -6 & -6 \end{pmatrix} + \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -10 \\ -10 & -2 \end{pmatrix}$

From the examples solved, I derived formulas

(A+B)^n= $\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}^n$

(A+B)^n= (a**X** + bY)^n= a^n**X**^{n} + b^{n}Y^n

Testing with different constant values:

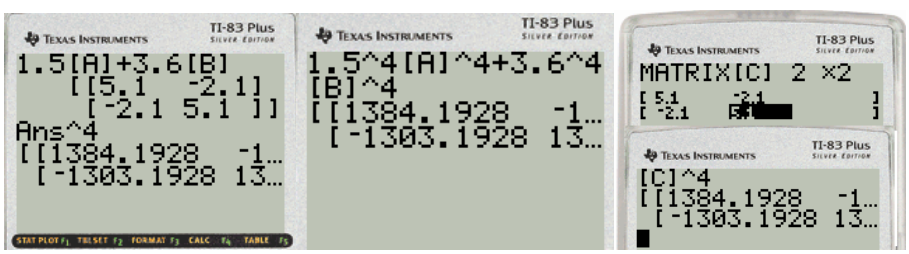
a	b	n
2	-4	5

(A+B)^5= $\left(\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\right)^5 = \begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix}^5 = \begin{pmatrix} -15872 & 16896 \\ 16896 & -15872 \end{pmatrix}$

(A+B)^5= $2^5\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^5 + (-4)^5\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^5 = \begin{pmatrix} -15872 & 16896 \\ 16896 & -15872 \end{pmatrix}$

(A+B)^5= $\begin{pmatrix} 2+(-4) & 2-(-4) \\ 2-(-4) & 2+(-4) \end{pmatrix}^5 = \begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix}^5 = \begin{pmatrix} -15872 & 16896 \\ 16896 & -15872 \end{pmatrix}$

a	b	n
1.5	3.6	4



(A+B)^4≈ $1.5^4\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^4 + 3.6^4\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^4 \approx \begin{pmatrix} 1384.2 & -1303.2 \\ -1303.2 & 1384.2 \end{pmatrix}$

(A+B)^4≈ $\begin{pmatrix} 1.5+3.6 & 1.5-3.6 \\ 1.5-3.6 & 1.5+3.6 \end{pmatrix}^4 \approx \begin{pmatrix} 5.1 & -2.1 \\ -2.1 & 5.1 \end{pmatrix}^4 \approx \begin{pmatrix} 1384.2 & -1303.2 \\ -1303.2 & 1384.2 \end{pmatrix}$

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$\mathbf{M}=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$

$\mathbf{M=A+B}$, $\mathbf{M^n=A^n+B^n}$

$\mathbf{M=A+B}$ and $\mathbf{M^2=A^2+B^2}$ I showed in the previous exercise and I will prove $\mathbf{M^n=A^n+B^n}$ by induction that I've learnt at HL Math classes.

$\mathbf{XY}=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}=0$

Therefore we know $\mathbf{AB=BA}=0$

$\mathbf{M^n=A^n+B^n}$

If conjecture is true it must work for <u>x=1</u>

$\mathbf{(A+B)^1=A+B}$

If it is true it needs to work for any real number k, <u>x=k</u>

$\mathbf{*(A+B)^k=A^k+B^k}$

Again it should also work for any real number <u>k + 1</u>

$\mathbf{*(A+B)^k(A+B)= (A^k+B^k)(A+B)= A^kA + B^kB + A^kB + B^kA}$

We showed that $\mathbf{AB=BA}=0$, therefore $\mathbf{A^kB + B^kA}= 0$

It follows that

$\mathbf{A^kA + B^kB + A^kB + B^kA= A^kA + B^kB}$

$\mathbf{A^kA + B^kB= A^{k+1} + B^{k+1}}$

$\mathbf{M^{k+1}= A^{k+1} + B^{k+1}}$

By doing and deriving formulas (especially in the second exercise) I found that the general formula is:

$$\mathbf{M}^n=a^n\mathbf{X}^n + b^nY^n$$
$$\mathbf{M}^n=(a\mathbf{X}+ b\mathbf{Y})^n$$

		n
5	2	2

$\mathbf{M}=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix} \Rightarrow \mathbf{M}=\begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}$

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$\mathbf{M}^2 = \mathbf{A}^2 + \mathbf{B}^2$

$$\begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}^2 = \left(5\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right)^2$$

$$\begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}^2 = \begin{pmatrix} 5+2 & 5-2 \\ 5-2 & 5+2 \end{pmatrix}^n$$

$$\begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}^2 = \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}^2$$

Using GDC:

$$\mathbf{M}^2 = \begin{pmatrix} 58 & 42 \\ 42 & 58 \end{pmatrix}$$

a	b	n
4	-1.5	5

$$\mathbf{M} = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix} \Rightarrow \mathbf{M} = \begin{pmatrix} 2.5 & 5.5 \\ 5.5 & 2.5 \end{pmatrix}$$

$$\begin{pmatrix} 2.5 & 5.5 \\ 5.5 & 2.5 \end{pmatrix}^5 = \left(4\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (-1.5)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right)^5$$

$$\begin{pmatrix} 2.5 & 5.5 \\ 5.5 & 2.5 \end{pmatrix}^5 = \begin{pmatrix} 2.5 & 5.5 \\ 5.5 & 2.5 \end{pmatrix}^5$$

$$\mathbf{M}^5 = \begin{pmatrix} 36.5 & 27.5 \\ 27.5 & 36.5 \end{pmatrix}$$

In the conclusion I can say that I found the scope of the statement.

$|\mathbf{A}| = ab\text{-}cd$

$|\mathbf{X}| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ $\qquad$ $|\mathbf{Y}| = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$

$|\mathbf{X}| = 1\text{-}1 = 0$ $\qquad$ $|\mathbf{Y}| = 1\text{-}1 = 0$

Since the starting matrices have determinant 0 and thus they don't have inverse, in my conclusion I can say that $n \in \mathbb{N}$. Also I showed in examples above that constants a and b can be rational and irrational numbers (π), therefore a, $b \in \boldsymbol{R}$. I would limit my general statement on the set of real numbers this is the

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only set of number that we learnt in matrix unit.