When parallel lines are intersected by parallel transversals to form parallelograms, relationships can be drawn between the number of parallelograms, transversals, and parallel lines. The number of parallelograms follow a triangular sequence when two parallel lines are intersected by transverals. In contrast, when three parallel lines are intersected by transversals, the number of parallelograms follow a tetrahedral number sequence.

Figure 1 below shows a pair of horizontal parallel lines and four parallel transversals. Six parallelograms are formed: A_1, A_2, A_3, $A_1 \cup A_2$, $A_2 \cup A_3$, $A_1 \cup A_3$.

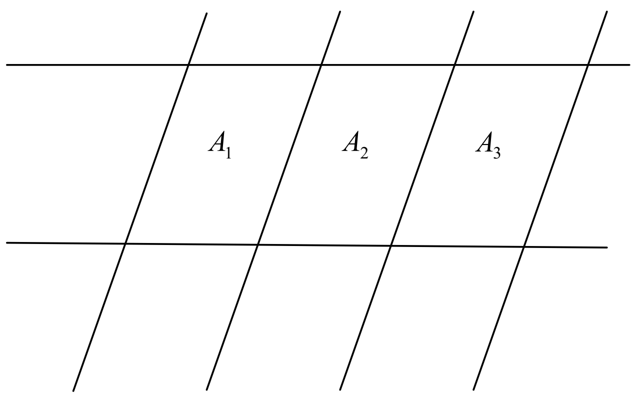

Figure 1. Fourth parallel transversal crossing two parallel lines

Sum of Parallelogram(s)	Evidence
1	A_1, A_2, A_3
2	$A_1 \cup A_2$, $A_2 \cup A_3$
3	$A_1 \cup A_3$

A fifth parallel transversal is added to the diagram as shown in Figure 2. 10 parallelograms are formed: A_1, A_2, A_3, A_4, $A_1 \cup A_3$, $A_2 \cup A_4$, $A_1 \cup A_2$, $A_2 \cup A_3$, $A_3 \cup A_4$, $A_1 \cup A_4$

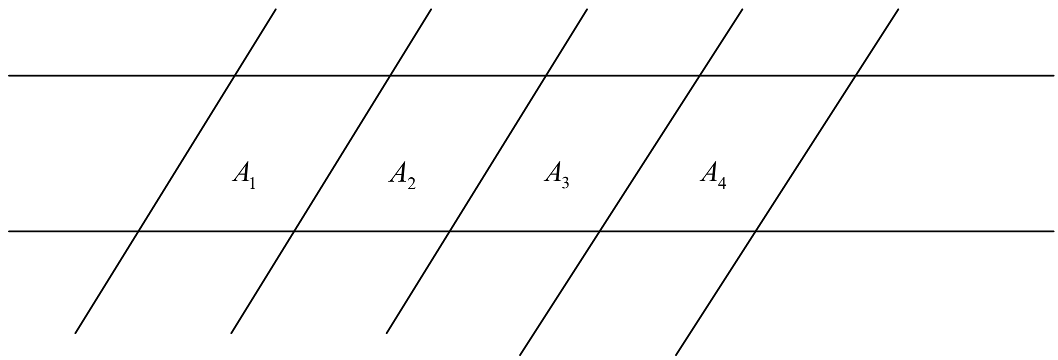

Figure 2. Fifth parallel transversal crossing two parallel lines.

Sum of Parallelogram(s)	Evidence
1	A_1, A_2, A_3, A_4
2	$A_1 \cup A_2, A_2 \cup A_3, A_3 \cup A_4$
3	$A_1 \cup A_3, A_2 \cup A_4$
4	$A_1 \cup A_4$

A sixth parallel transversal is added to the diagram as shown in Figure 3. 15 parallelograms are formed: $A_1, A_2, A_3, A_4, A_5, A_1 \cup A_4,\ A_1 \cup A_5, A_2 \cup A_5,$

$A_1 \cup A_3, A_2 \cup A_4,\ A_3 \cup A_5, A_1 \cup A_2, A_2 \cup A_3, A_3 \cup A_4,\ A_4 \cup A_5.$

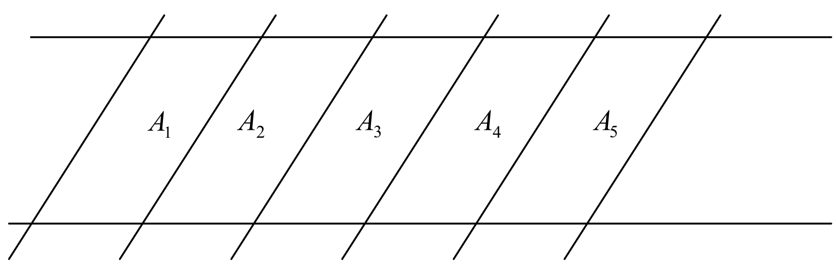

Figure 3. Sixth transversal crossing two parallel lines

Sum of Parallelogram(s)	Evidence
1	A_1, A_2, A_3, A_4, A_5
2	$A_1 \cup A_2, A_2 \cup A_3, A_3 \cup A_4,\ A_4 \cup A_5$.
3	$A_1 \cup A_3, A_2 \cup A_4,\ A_3 \cup A_5$
4	$A_1 \cup A_4, A_2 \cup A_5$
5	$A_1 \cup A_5$

A seventh parallel transversal is added to the diagram as shown in Figure 4. Fifteen parallelograms are formed: $A_1, A_2, A_3, A_4, A_5, A_6, A_1 \cup A_6, A_1 \cup A_5, A_2 \cup A_6, A_1 \cup A_4,$

$A_2 \cup A_5,\ A_3 \cup A_6, A_1 \cup A_3, A_2 \cup A_4,\ A_3 \cup A_5, A_4 \cup A_6, A_1 \cup A_2, A_2 \cup A_3, A_3 \cup A_4,$

$A_4 \cup A_5, A_5 \cup A_6$

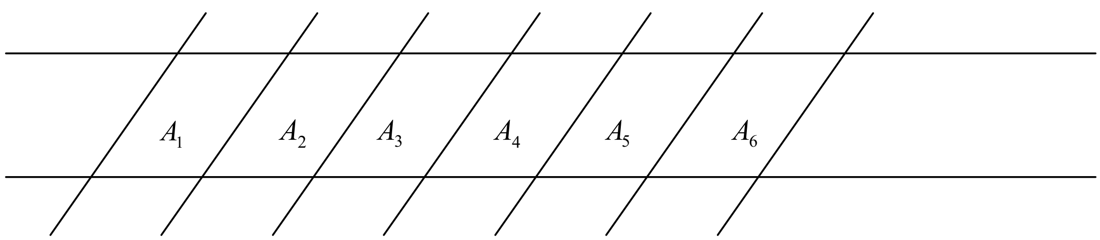

Figure 4. Seventh parallel transversal crossing two parallel lines

Sum of Parallelogram(s)	Evidence
1	$A_1, A_2, A_3, A_4, A_5, A_6$
2	$A_1 \cup A_2, A_2 \cup A_3, A_3 \cup A_4,$ $A_4 \cup A_5, A_5 \cup A_6$.
3	$A_1 \cup A_3, A_2 \cup A_4,\ A_3 \cup A_5, A_4 \cup A_6$
4	$A_1 \cup A_4, A_2 \cup A_5,\ A_3 \cup A_6,$
5	$A_1 \cup A_5, A_2 \cup A_6,$
6	$A_1 \cup A_6$

After repeating the process of adding transversals consecutively, Figure 5 shows the conclusions that were made between the relationship between transversals and number of parallelograms formed between two parallel lines.

Number of Transversal(s)	Number of Parallelograms	Process:
1	1	1
2	3	1 + 2
3	6	1 + 2 + 3
4	10	1 + 2 + 3 + 4
5	15	1 + 2 + 3 + 4 + 5
6	21	1 + 2 + 3 + 4 + 5 + 6
7	28	1 + 2 + 3 + 4 + 5 + 6 + 7

Figure 5. Relationship between parallelograms and transversals between two parallel lines

The evidence from Figure 6 illustrates that the number of parallelograms are a general representation of trianglular numbers, or in other words, the sum of consecutive numbers. The third diagonal of Pascal`s triangle, starting at the third row (Figure 6; shown in red), represents the number of parallelograms formed. The number of transversals can be calculated from the Pascal's triangle by subtracting the first number from the second number in each row.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

Figure 6: Triangular numbers in Pascal's triangle

From the inference that the parallelograms formed between two parallel lines follow the triangular number sequence, a general equation is:

Suppose that:

i=final value

$$\sum$$

i=starting value

Then:

$$T_n + T_n = \sum_{i=1}^{i=n} i + \sum_{i=1}^{i=n} (n+1-i)$$

(There are two T_n because the binomial coefficients are repetitive. For example, in the seventh row it contains **1 7 21 35 35 21 7 1**)

$$2T_n = n(n+1)$$

$$T_n = \binom{n}{2} = \frac{n(n+1)}{2}$$

Next, considering the number of parallelograms formed by three horizontal parallel lines intersected by parallel transversals, I repeated the same process with the two parallel lines. Figure 7 shows the number sequence that can be be generated by adding up triangular numbers as shown in the table below:

Numbers of Transversal(s)	Number of Parallelogram(s)	Process:
1	1	1
2	4	1 + 3
3	10	1 + 3 + 6
4	20	1 + 3 + 6 + 10
5	35	1 + 3 + 6 + 10 + 15

6	56	1 + 3 + 6 + 10 + 15 + 21
7	84	1 + 3 + 6 + 10 + 15 +21 + 28

Figure 7:. Relationship between parallelograms and transversals between three parallel lines

The number of parallelograms follow the tetrahedral number sequence which can be found by adding the consecutive triangular numbers. Beginning from the fourth diagonal of Pascal's triangle, the number of parallelograms are shown in red in Figure 8. The number of transversals are represented starting from the third row (1 transversal) and so on. For example the 2^{nd} transversal would be in the fourth row while the 3^{rd} transversal would be in the fifth row.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

The *n*th tetrahedral number can be given by $\binom{n+2}{3}$, which is the binomial coefficient.

Adding 2 to *n* is for the reason that the first tetrahedral number is actually found in the third row. The tetrahedral formula can be expanded by:

$$T_n = \binom{n+2}{3} = \frac{(n+2)!}{3!(n-1)!} = \frac{(n+2)(n+1)n}{3 \cdot 2 \cdot 1}$$

We know that the triangular number sequence (which represents number of parallelograms formed between two parallel lines) is represented by g(n) = n(n+1)/2. Thus, to prove that the tetrahedral formula works for the parallelograms formed when transversal(s) cross three parallel lines, I used induction reasoning. If it's true for n (n=1), then it's true for n+1, which means that it's true for all n >= 1. We know the formula for the sum of the first n squares (n)(n+1)(2n+1)/6.

To prove that tetrahedral numbers are the sum of the triangular numbers:

$$\sum_{k=1}^{t} k = \frac{t(t+1)}{2} = \text{ t-th triangular number}$$

$$\sum_{t=1}^{n} \frac{t(t+1)}{2} = \text{summation of a t-th triangular number(s) from 1 to n}$$

$$\sum_{t=1}^{n} t = \frac{n(n+1)}{2}$$

$$\sum_{t=1}^{n} t^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{2}\sum_{t=1}^{n} t^2 + \frac{1}{2}\sum_{t=1}^{n} t$$

$$= \frac{1}{2}\left[\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{1}{2}\left[\frac{3n(n+1)}{6} + \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3}{6} + \frac{2n+1}{6}\right]$$

$$\frac{n(n+1)}{2}\left[\frac{2n+4}{6}\right] =$$

$$\frac{n(n+1)(2n+4)}{12} =$$

$$\frac{n(n+1)(n+2)}{6}$$

Thus this shows that if it's true for n, it's true for n + 1. Since we showed it was true for n = 1, we now know it's also true for n = 1 + 1 = 2, and then for n = 2 + 1 = 3, and so on, for all n >= 1.

Then, a general statement statement that satisfies both transversal and parallel lines can be drawn in that *m* represents horizontal parallel lines in *n* represents the intersected parallel transversals:

Suppose that:

$$C^{n}_{2} = \frac{n(n-1)}{2} \text{ and } C^{m}_{2} = \frac{m(n-1)}{2}$$

Then:

$$C^{n}_{2} * C^{m}_{2} = \frac{mn(m-1)(n-1)}{4}$$

If m=2 in that there are 2 horizontal parallel lines and n=3 in that there are 3 parallel transversals

Then,

$$C^{n}_{2} * C^{m}_{2} = \frac{2(3)(2-1)(3-1)}{4}$$

$$= \frac{6(1)(2)}{4}$$

$$= \frac{12}{4}$$

$$=3$$

The conclusion that 3 parallelograms formed when 2 horizontal parallel lines are intersected by 3 parallel transversals is valid. Therefore, the general statement validity is true. The limitations of the equation $C^{n}_{2} * C^{m}_{2} = \frac{mn(m-1)(n-1)}{4}$ is that it only considers the number of parallelograms formed by intersecting parallel lines. However, the equation gives an accurate result of when m represents horizontal parallel lines in n represents the intersected parallel transversals.