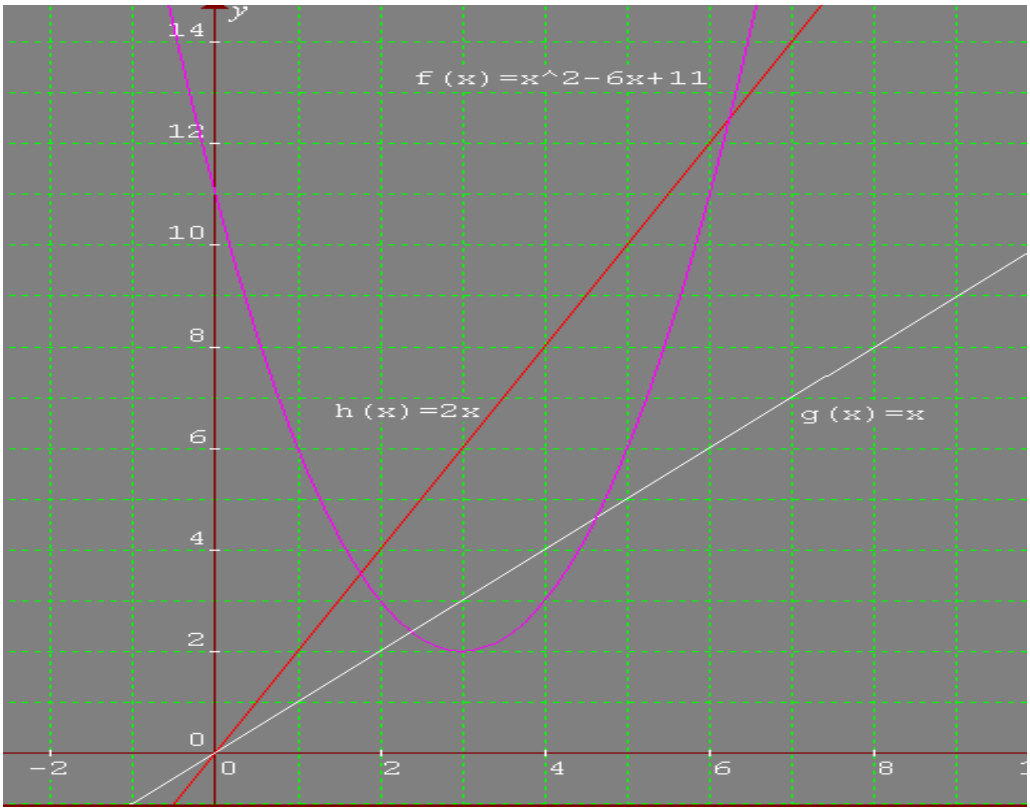
PARABOLA INVESTIGATION

Description

In this task, we will investigate the patterns in the intersections of parabola and the lines y =x and y=2x. Then we will prove and find the conjectures and to broaden the scope of the investigation to include other lines and other types of polynomials.

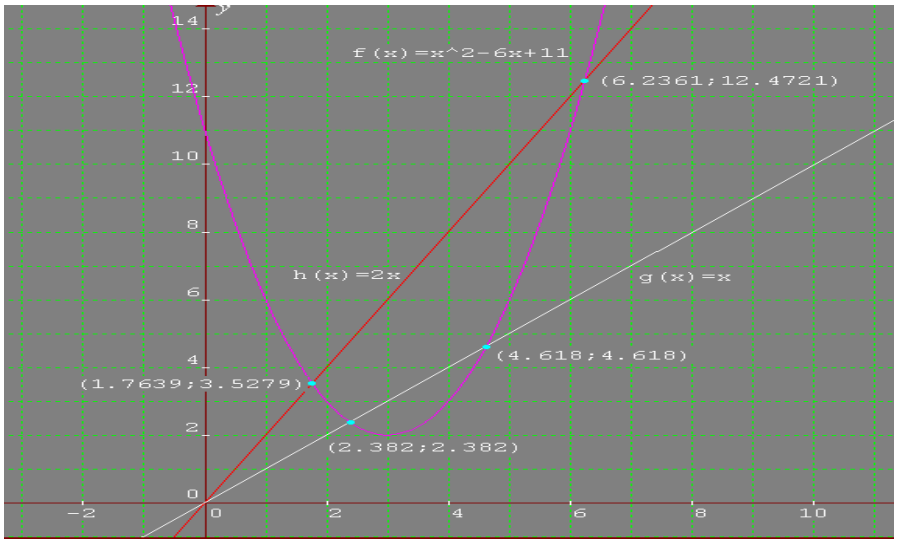
1. Consider the parabola $y=(x-3)^2+2=x^2-6x+11$ and the line $y=x$ and $y=2x$.



- Using Graphmatica software, we can find the four intersections. Below these points are illustrated.

  We find out 2 intersection points of f(x) and h(x) which are (1.7639; 3.5279) and (6.2361; 12.4721).

The other 2 intersection points between f(x) and g(x) are (2.382; 2.382) and (4.618; 4.618).



- The x-values of these intersections as they appear from the left to right on the x-axis as x_1, x_2, x_3, x_4.

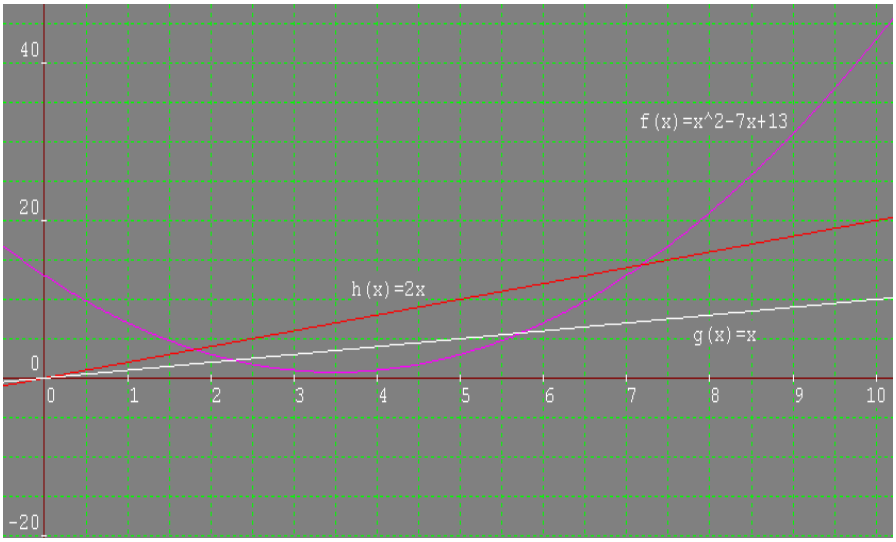
- $x_1 \approx 1.764$
- $x_2 \approx 2.382$
- $x_3 \approx 4.618$
- $x_4 \approx 6.236$

- Find the values of $x_2 - x_1$ and $x_4 - x_3$ and name them respectively S_L and S_R.
  $S_L = x_2 - x_1 \approx 2.382 - 1.764 \approx 0.618$
  $S_R = x_4 - x_3 \approx 6.236 - 4.618 \approx 1.618$
- Finally, calculate $D = |S_L - S_R|$.

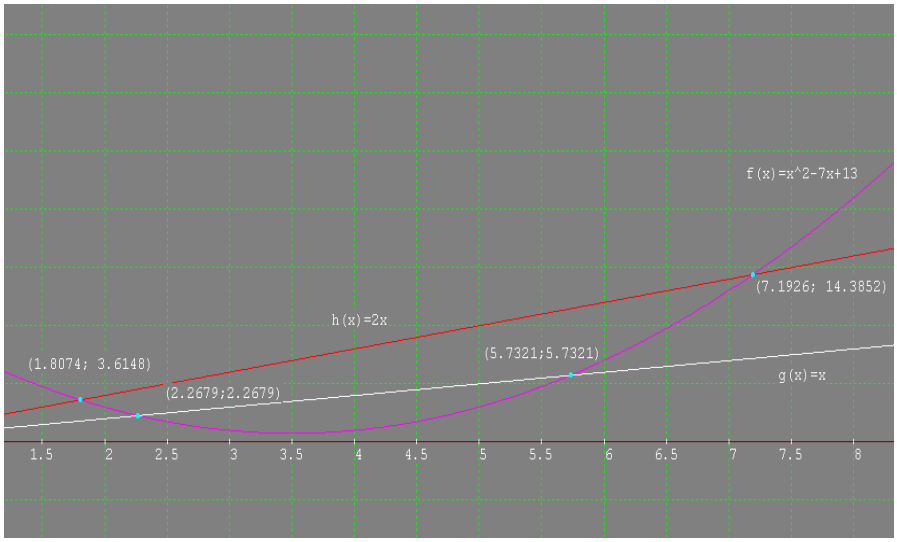
$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.618 - 1.618| \\ &= |-1| \\ &= 1 \end{aligned}$$

2. Find values of D for other parabolas of the form $y = ax^2 + bx + c$, $a > 0$ with vertices in quadrant 1, intersected by the lines $y = x$ and $y = 2x$. Consider various values of a, beginning with $a = 1$. Make a conjecture about the value of D for these parabolas.

At first, we consider the parabola $y = x^2 - 7x + 13$, the lines $y = x$ and $y = 2x$.



Again, we use the Graphmatica software to obtain four intersection points, repeat from step a to step c. Then label the intersections on the graph shown below.



The x-values of these intersections from the left to the right on the x-axis:

- $x_1 \approx 1.807$
- $x_2 \approx 2.268$
- $x_3 \approx 5.732$
- $x_4 \approx 7.193$

Find the values of $x_2 - x_1$ and $x_4 - x_3$ and name them respectively S_L and S_R.

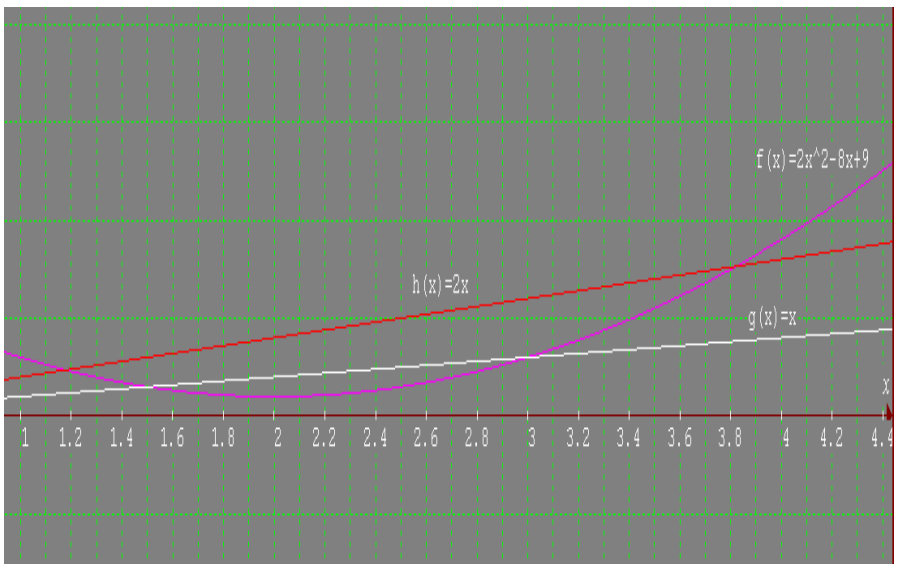
$S_L = x_2 - x_1 \approx 2.268 - 1.807 \approx 0.461$

$S_R = x_4 - x_3 \approx 7.193 - 5.732 \approx 1.461$

Finally, calculate $D = |S_L - S_R|$.

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.461 - 1.461| \\ &= |-1| \\ &= 1 \end{aligned}$$

Secondly, we consider the parabola $y = 2x^2 - 8x + 9$, the lines $y = x$ and $y = 2x$.



By using the manual calculation, we can calculate the four intersections.

Calculate the intersections between f(x) and h(x):

$y = 2x^2 - 8x + 9 \quad (1)$

$y = 2x \quad (2)$

Substitute (2) into (1):

$2x^2 - 8x + 9 = 2x$

$2x^2 - 10x + 9 = 0$

$x = \dfrac{5+\sqrt{7}}{2}$ or $x = \dfrac{5-\sqrt{7}}{2}$.

Using calculator to obtain the approximate values of x

$x \approx 3.823$ or $x \approx 1.177$

- Substitute $x \approx 3.823$ into (2):

  $y \approx 7.646$

  (x, y) = (3.823; 7.646)

- Substitute $x \approx 1.177$ into (2):
  $y \approx 2.354$
  (x, y) = (1.177; 2.354)

Calculate the intersections between f(x) and g(x):

$y = 2x^2 - 8x + 9$ (3)

$y = x$ (4)

Substitute (4) into (3):

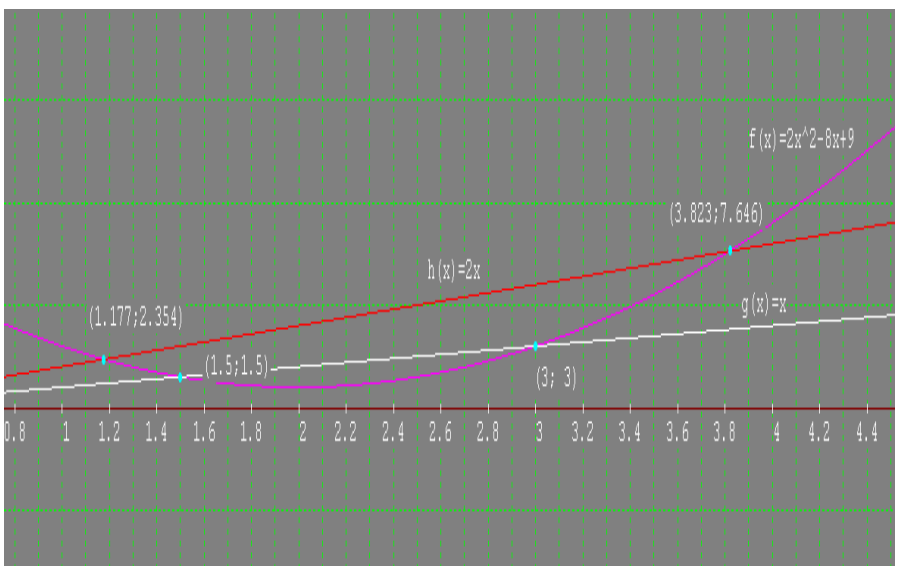
$2x^2 - 8x + 9 = x$

$2x^2 - 9x + 9 = 0$

$x = 3$ or $x = \frac{3}{2}$

- Substitute $x = 3$ into (4):
  $y = 3$
  (x, y) = (3; 3)
- Substitute $x = 1.5$ into (4):
  $y = 1.5$
  (x, y) = (1.5; 1.5)
  Hence, the x-values from left to right are:

➢ $x_1 \approx 1.177$
➢ $x_2 = 1.500$
➢ $x_3 = 3.000$
➢ $x_4 \approx 3.823$



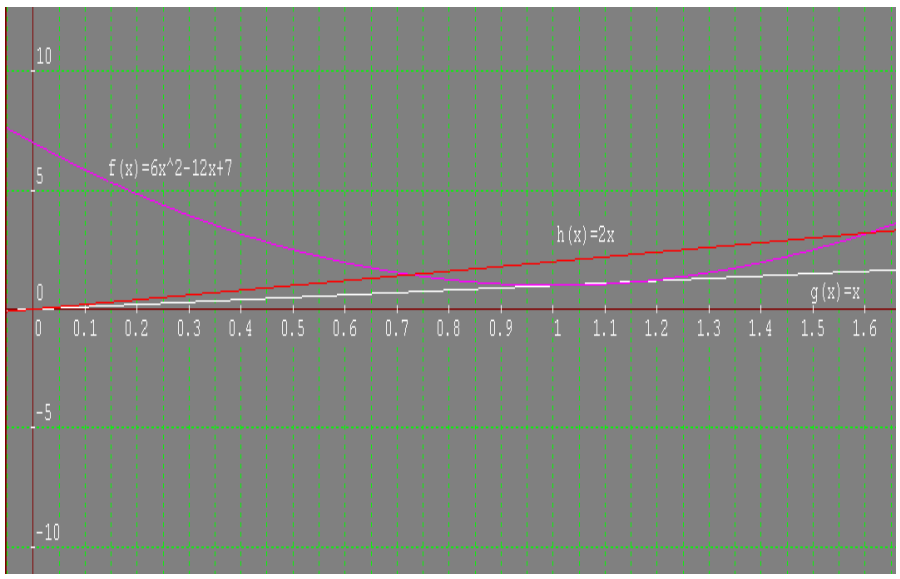
Calculation of S_L and S_R:
$S_L = x_2 - x_1 \approx 1.500 - 1.177 \approx 0.323$

$S_R = x_4 - x_3 \approx 3.823 - 3.000 \approx 0.823$

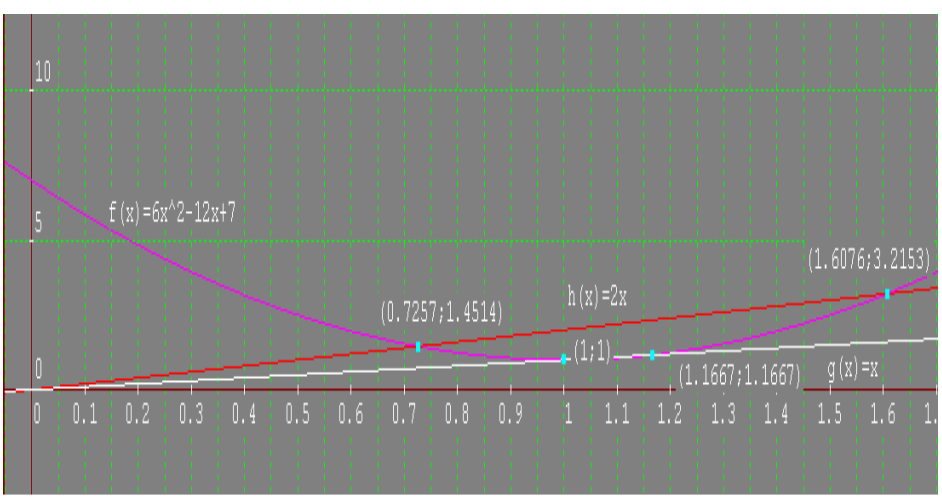
Calculate $D = |S_L - S_R|$.

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.323 - 0.823| \\ &= \left|-\frac{1}{2}\right| \\ &= \frac{1}{2} \end{aligned}$$

Thirdly, consider parabola $y = 6x^2 - 12x + 7$, the lines $y = x$ and $y = 2x$.



Using the same method, we apply the Graphmatica software to determine the four intersections which are illustrated below:



The x-values of these intersections from the left to the right on the x-axis:

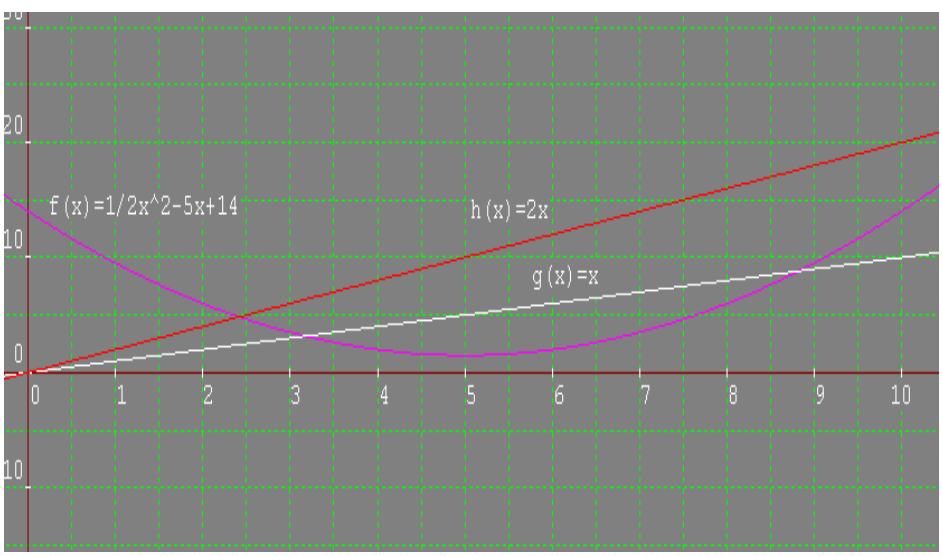
- $x_1 \approx 0.726$
- $x_2 = 1.000$
- $x_3 \approx 1.167$
- $x_4 \approx 1.608$

Calculation of S_L and S_R:
$S_L = x_2 - x_1 \approx 1.000 - 0.726 \approx 0.274$
$S_R = x_4 - x_3 \approx 1.608 - 1.167 \approx 0.441$

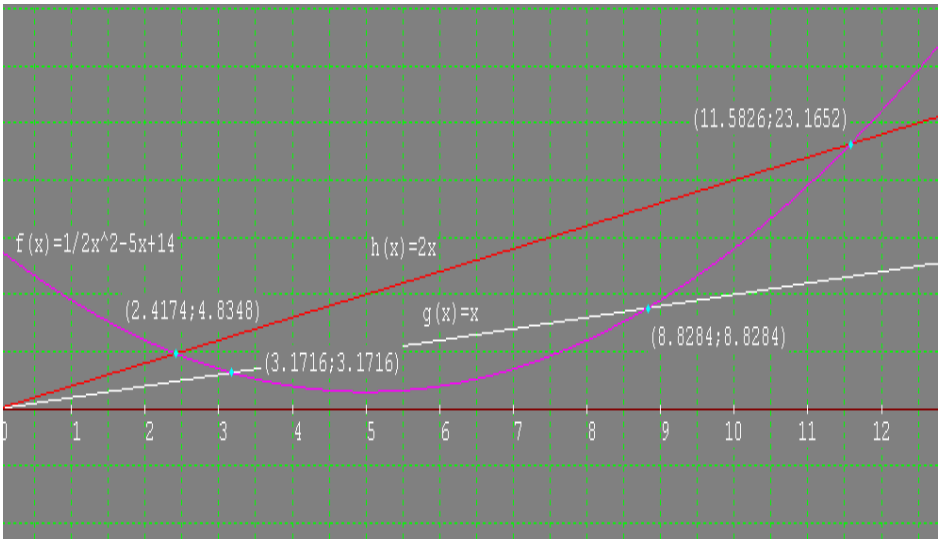
Calculate $D = |S_L - S_R|$.

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.274 - 0.441| \\ &= |-0.167| \\ &= 0.167 \\ &\approx \frac{1}{6} \end{aligned}$$

Next, consider parabola $y = \frac{1}{2}x^2 - 5x + 14$, the lines $y = x$ and $y = 2x$.



By using Graphmatica software, we can obtain the four intersections of parabola $y = \frac{1}{2}x^2 - 5x + 14$, the lines $y = x$ and $y = 2x$.



The x-values of these intersections from the left to the right on the x-axis:

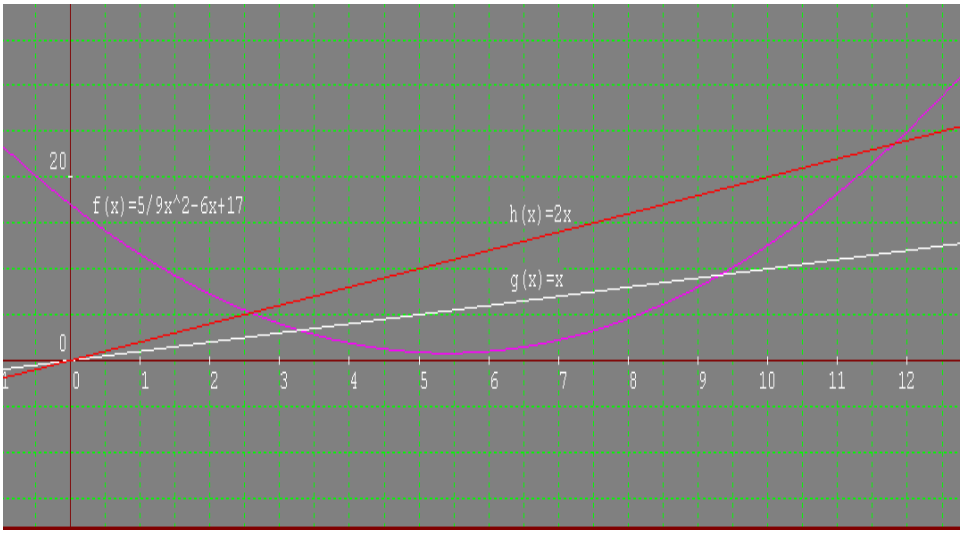
- $x_1 \approx 2.417$
- $x_2 \approx 3.172$
- $x_3 \approx 8.828$
- $x_4 \approx 11.583$

Calculation of S_L and S_R:
$S_L = x_2 - x_1 \approx 3.172 - 2.417 \approx 0.755$
$S_R = x_4 - x_3 \approx 11.583 - 8.828 \approx 2.755$

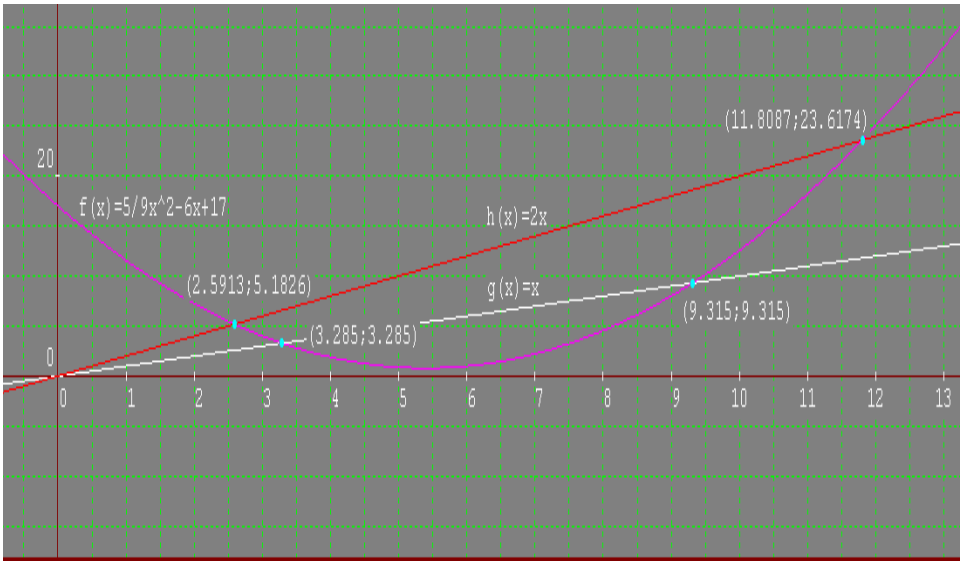
Calculate $D = |S_L - S_R|$.

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.755 - 2.755| \\ &= |-2| \\ &= 2 \end{aligned}$$

Consider parabola $y = \frac{5}{9}x^2 - 6x + 17$, the lines $y = x$ and $y = 2x$.



By using Graphmatica software, we can obtain the four intersections of parabola $y = \frac{5}{9}x^2 - 6x + 17$, the lines $y = x$ and $y = 2x$.



The x-values of these intersections from the left to the right on the x-axis:

- $x_1 \approx 2.591$
- $x_2 \approx 3.285$
- $x_3 \approx 9.315$
- $x_4 \approx 11.809$

Calculation of S_L and S_R:
$S_L = x_2 - x_1 \approx 3.285 - 2.591 \approx 0.694$
$S_R = x_4 - x_3 \approx 11.809 - 9.315 \approx 2.494$
Calculate $D = |S_L - S_R|$.

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.694 - 2.494| \\ &= |-1.8| \\ &= \frac{9}{5} \end{aligned}$$

Table shows the values of D for parabolas of the form $y = ax^2 + bx + c$, $a > 0$ with vertices in quadrant 1, intersected by the lines $y = x$ and $y = 2x$.

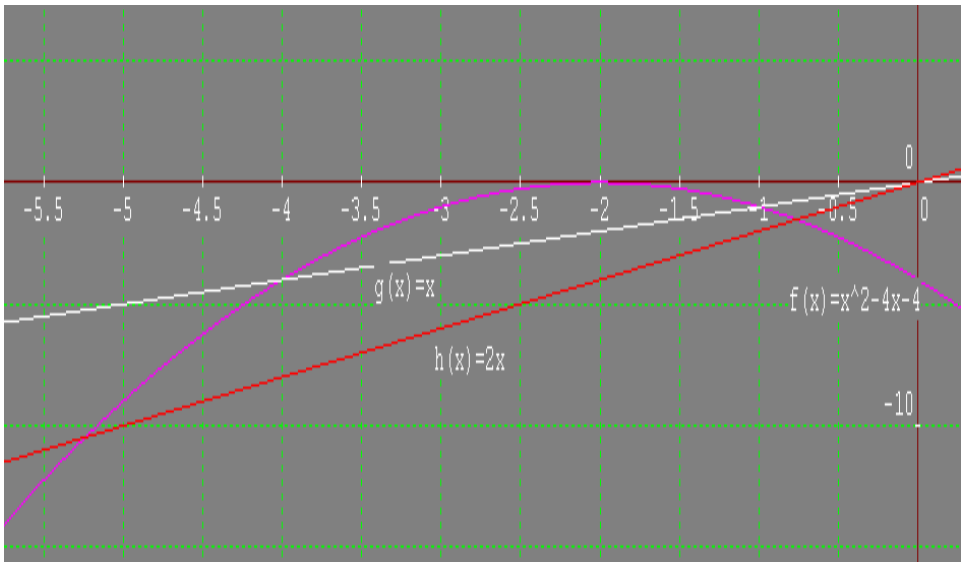
Parabolas	a	b	c	D
$y = x^2 - 6x + 11$	1	-6	11	1
$y = x^2 - 7x + 13$	1	-7	13	1
$y = 2x^2 - 8x + 9$	2	-8	9	$\frac{1}{2}$
$y = 6x^2 - 12x + 7$	6	-12	7	$\frac{1}{6}$
$y = \frac{1}{2}x^2 - 5x + 14$	$\frac{1}{2}$	-5	14	2
$y = \frac{5}{9}x^2 - 6x + 17$	$\frac{5}{9}$	-6	17	$\frac{9}{5}$

As can be seen from the table D is inversely proportional to the value of a.
The values of D for parabolas of the form $y = ax^2 + bx + c$, $a > 0$ with vertices in quadrant 1, intersected by the lines $y = x$ and $y = 2x$, are inversely proportional to the values of a.

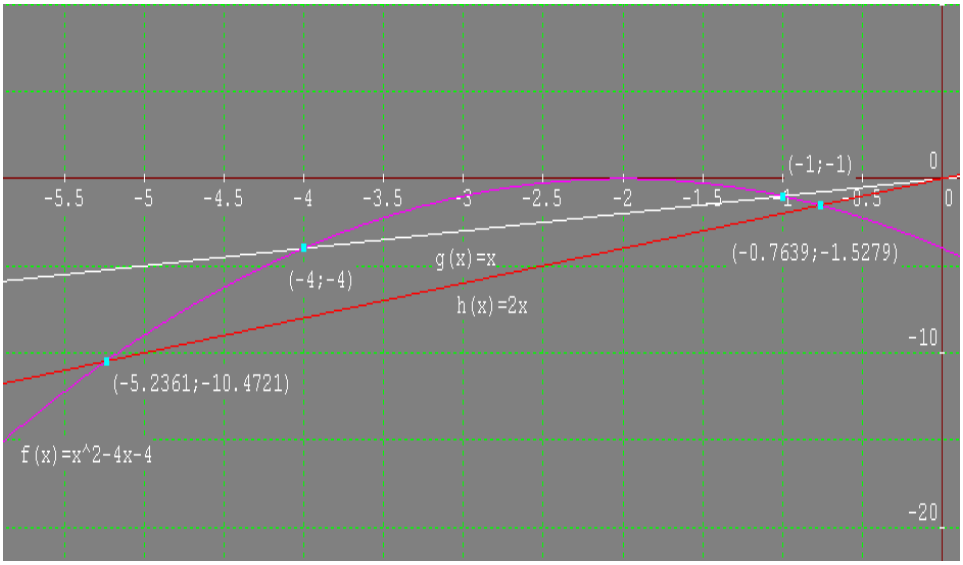
$$D = \frac{1}{a}$$

3. Investigate your conjecture for any real value of a and any placement of the vertex. Maintain the labeling convention used in parts 1 and 2 by having the intersections of the first line to be x_2 and x_3 and the intersections with the second line to be x_1 and x_4.

Consider parabola $y = -x^2 - 4x - 4$, the lines $y = x$ and $y = 2x$.



Determine the four intersections which are illustrated below:



The x-values of these intersections from the left to the right on the x-axis:

- $x_1 \approx -5.236$
- $x_2 \approx -4.000$
- $x_3 \approx -1.000$
- $x_4 \approx -0.764$

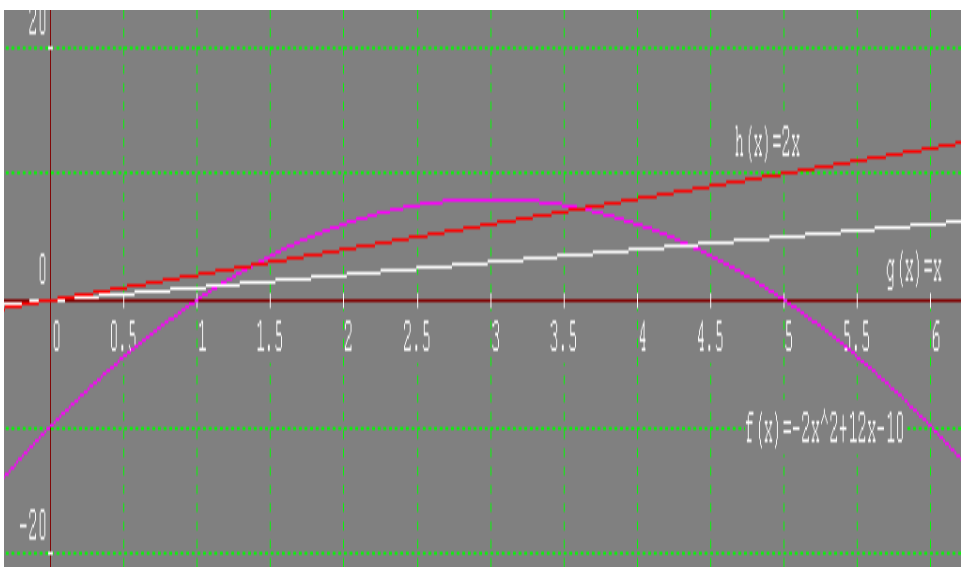
Calculation of S_L and S_R:
$S_L = x_2 - x_1 \approx -4.000 - (-5.236) \approx 1.236$
$S_R = x_4 - x_3 \approx -0.794 - (-1.000) \approx 0.206$
Calculate $D = |S_L - S_R|$.

$$\begin{aligned} D &= |S_L - S_R| \\ &= |1.236 - 0.206| \\ &= |-1.03| \\ &\approx 1 \end{aligned}$$

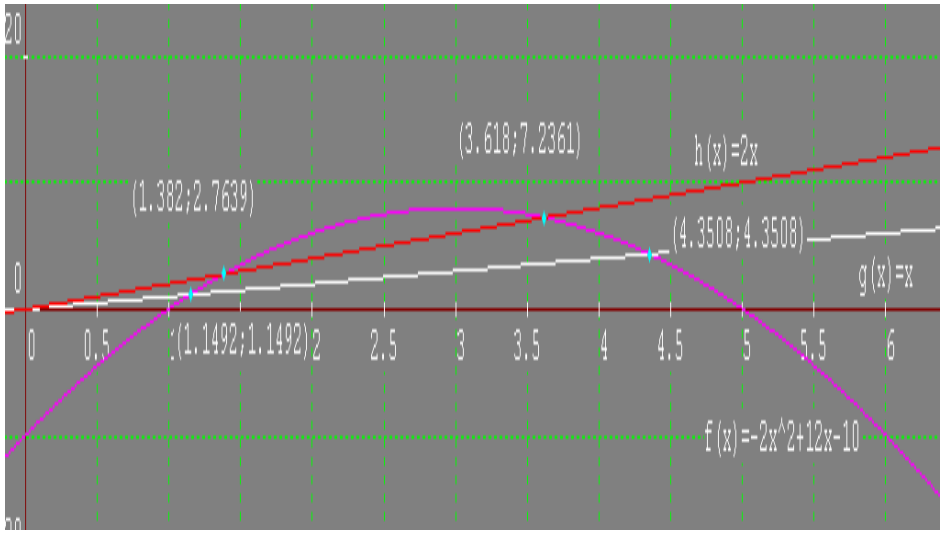
The conjecture does not hold true because due to part 2, $D = \frac{1}{a}$ so in this case when a = -1, D should be: $D = \frac{1}{a} = \frac{1}{-1} = -1$.

Therefore, the conjecture should be $D = \frac{1}{|a|}, a \neq 0$.

Consider the parabola $y = -2x^2 + 12x - 10$, the lines $y = x$ and $y = 2x$.



The four intersections between the parabola $y = -2x^2 + 12x - 10$, the lines $y = x$ and $y = 2x$ can be found by again using Graphmatica software.



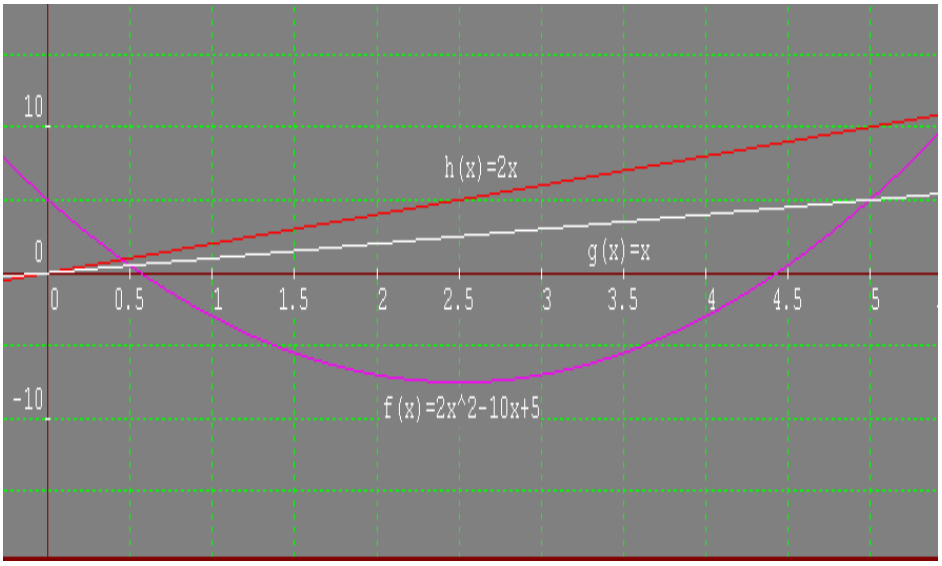
The x-values of these intersections from the left to the right on the x-axis:

- $x_1 \approx 1.149$
- $x_2 \approx 1.382$
- $x_3 \approx 3.618$
- $x_4 \approx 4.351$

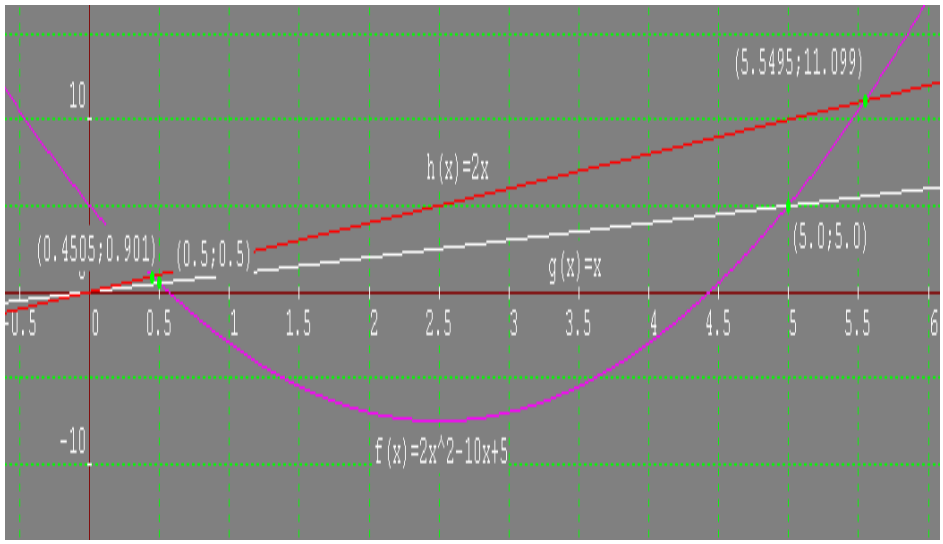
Calculation of S_L and S_R:
$S_L = x_2 - x_1 \approx 1.382 - 1.149 \approx 0.233$
$S_R = x_4 - x_3 \approx 4.351 - 3.618 \approx 0.733$
Calculate $D = |S_L - S_R|$.

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.233 - 0.733| \\ &= \left|-\frac{1}{2}\right| \\ &= 0.5 \end{aligned}$$

Consider the parabola $y = -2x^2 + 12x - 10$, the lines $y = x$ and $y = 2x$.



By using Graphmatica software, we can obtain the four intersections of parabola $y = 2x^2 - 10x + 5$, the lines $y = x$ and $y = 2x$.



The x-values of these intersections from the left to the right on the x-axis:

- $x_1 \approx 0.451$
- $x_2 \approx 0.500$
- $x_3 \approx 5.000$
- $x_4 \approx 5.550$

Calculation of S_L and S_R:
$S_L = x_2 - x_1 \approx 0.500 - 0.451 \approx 0.049$
$S_R = x_4 - x_3 \approx 5.550 - 5.000 \approx 0.550$
Calculate $D = |S_L - S_R|$.

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.049 - 0.550| \\ &= |-0.501| \\ &\approx 0.5 \end{aligned}$$

The results of investigating different real values of *a* and placement of the vertex :

Parabolas	a	b	c	D
$y = -x^2 - 4x - 4$	-1	-4	-4	1
$y = -2x^2 + 12x - 10$	-2	12	-10	0.5
$y = 2x^2 - 10x + 5$	2	-10	5	0.5

Proof:
Find the two intersections between parabola f(x) and g(x):

$$y = ax^2 + bx + c$$
$$y = x$$
$$ax^2 + bx + c = x$$
$$ax^2 + (b-1)x + c = 0$$
$$x = \frac{(1-b) \pm \sqrt{(b-1)^2 - 4ac}}{2a}$$
$$x_2 = \frac{(1-b) - \sqrt{(b-1)^2 - 4ac}}{2a} \text{ or } x_3 = \frac{(1-b) + \sqrt{(b-1)^2 - 4ac}}{2a}$$
$$x_2 + x_3 = \frac{2(1-b)}{2a} = \frac{(1-b)}{a}$$

Find the two intersections between parabola f(x) and h(x):

$$y = ax^2 + bx + c$$
$$y = 2x$$
$$ax^2 + bx + c = 2x$$
$$ax^2 + (b-2)x + c = 0$$
$$x = \frac{(2-b) \pm \sqrt{(b-2)^2 - 4ac}}{2a}$$
$$x_1 = \frac{(2-b) - \sqrt{(b-2)^2 - 4ac}}{2a} \text{ or } x_4 = \frac{(2-b) + \sqrt{(b-2)^2 - 4ac}}{2a}$$

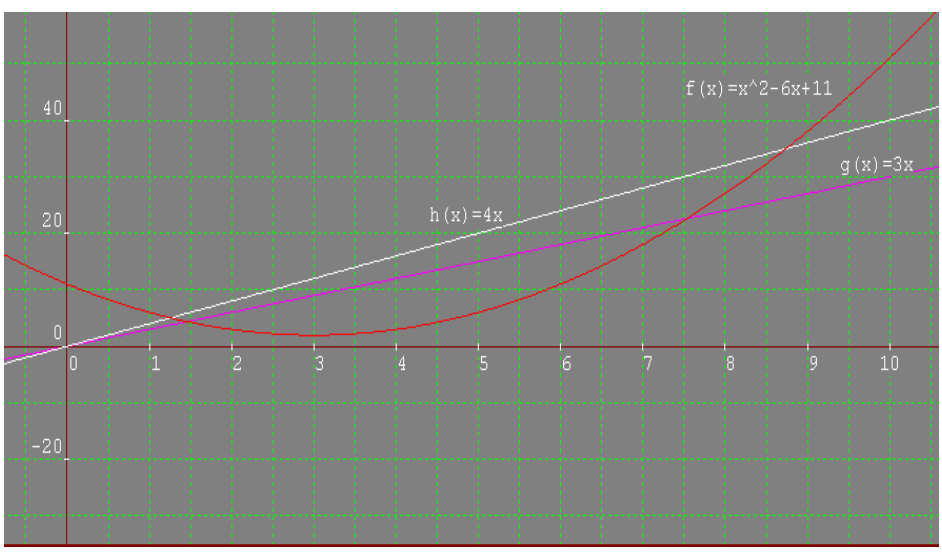
$$x_1 + x_4 = \frac{2(2-b)}{2a} = \frac{(2-b)}{a}$$
$$S_L = x_2 - x_1$$
$$S_R = x_4 - x_3$$
$$D = |S_L - S_R| = |(x_2 - x_1) - (x_4 - x_3)|$$
$$= |(x_2 + x_3) - (x_1 + x_4)|$$
$$= \left|-\frac{1}{a}\right| = \frac{1}{|a|}, a \neq 0$$

Hence, the conjecture about the values of D, for all real values of a, $a \neq 0$.

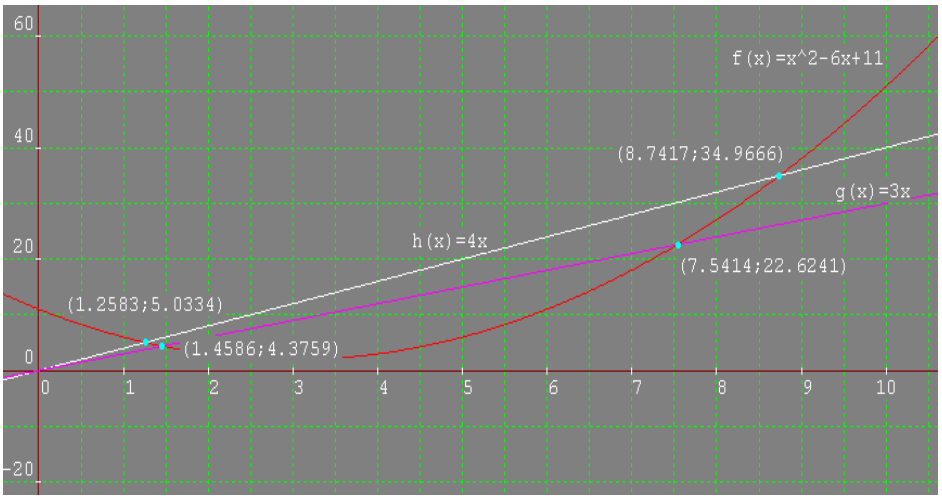
$$D = \frac{1}{|a|}$$

4. Investigating the conjecture when the intersecting lines are changed.
   We will still use the same parabola but the intersecting lines will be varied
   The general equation of intersecting lines is $y = mx + c$
   As there are two intersecting lines, they should be written as the following equations:
   $y = m_1 x + c_1$
   $y = m_2 x + c_2$

Consider parabola $y = x^2 - 6x + 11$ and the lines $y = 4x$ and $y = 3x$.



The intersection points then can be found by using Graphmatica software.



- $x_1 \approx 1.258$
- $x_2 \approx 1.459$
- $x_3 \approx 7.541$
- $x_4 \approx 8.742$

Calculation of S_L and S_R:
$S_L = x_2 - x_1 \approx 1.459 - 1.258 \approx 0.201$
$S_R = x_4 - x_3 \approx 8.742 - 7.541 \approx 1.201$
Calculate $D = |S_L - S_R|$.

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.201 - 1.201| \\ &= |-1| \\ &= 1 \end{aligned}$$

In this case, the conjecture holds true. $D = \frac{1}{|a|}$

Consider parabola $y = 4x^2 - 16x + 8$ and the lines $y = 6x$ and $y = 8$

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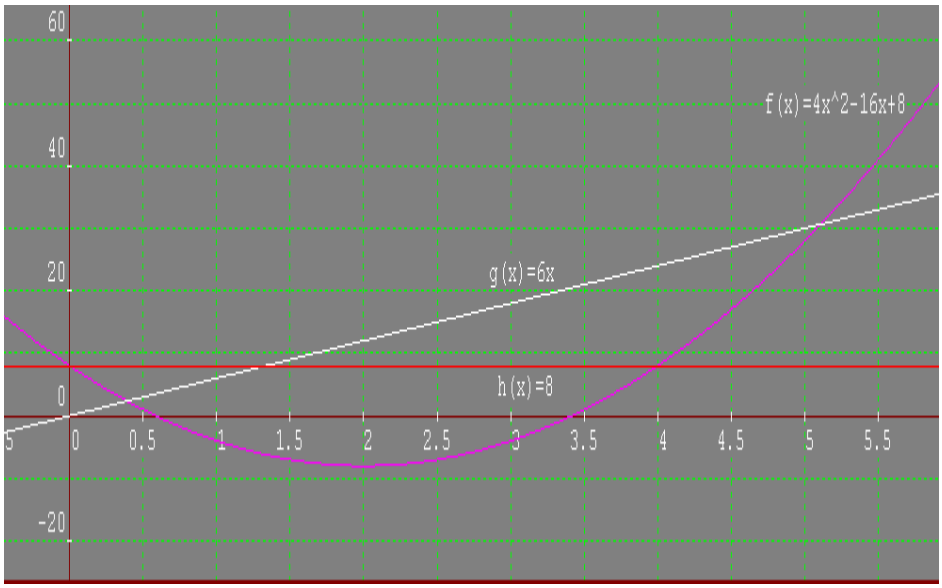
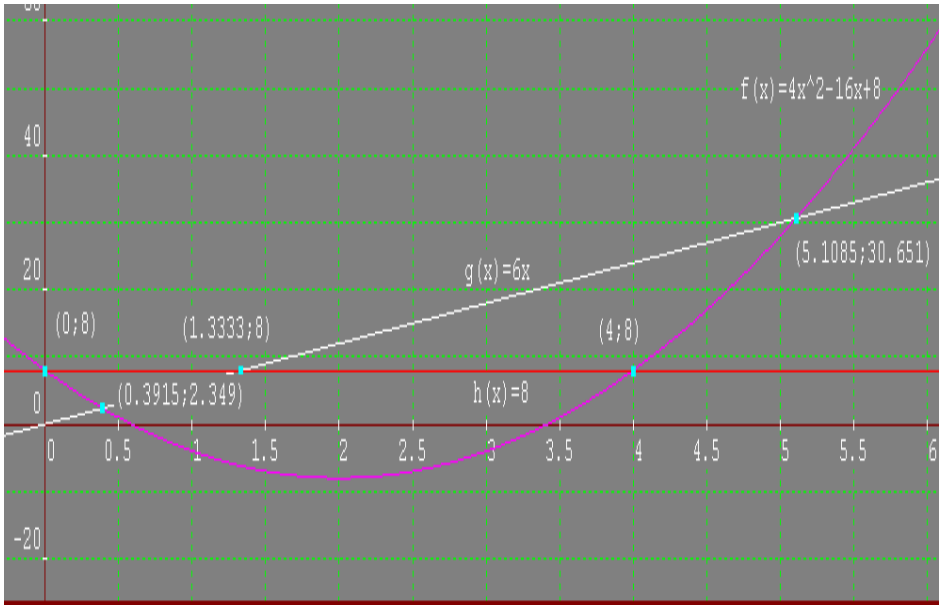


By using Graphmatica software, we can obtain the four intersections of parabola $y = 4x^2 - 16x + 8$, the lines $y = 6x$ and $y = 8$.

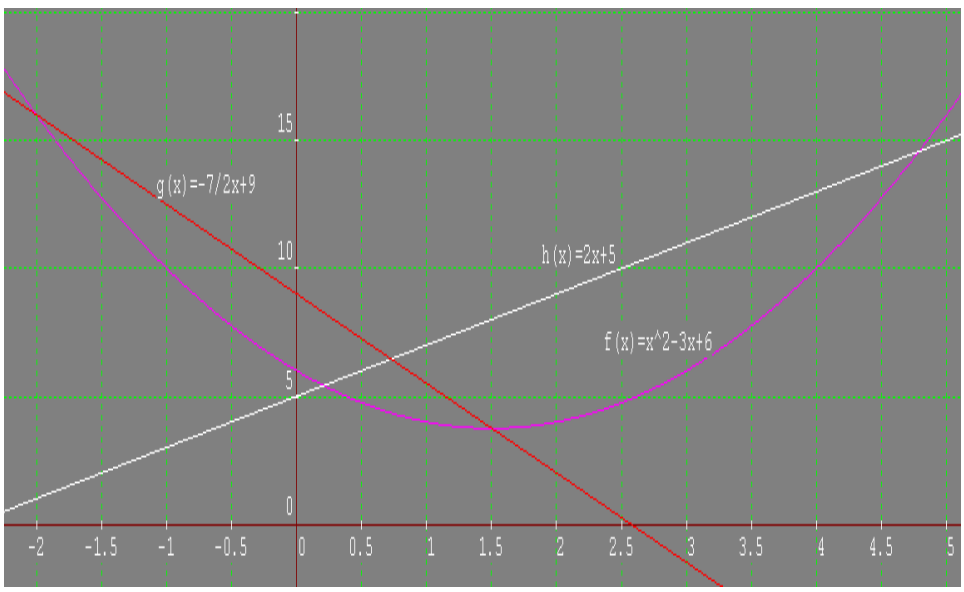


- $x_1 \approx 0$
- $x_2 \approx 0.392$
- $x_3 \approx 4.000$
- $x_4 \approx 5.109$

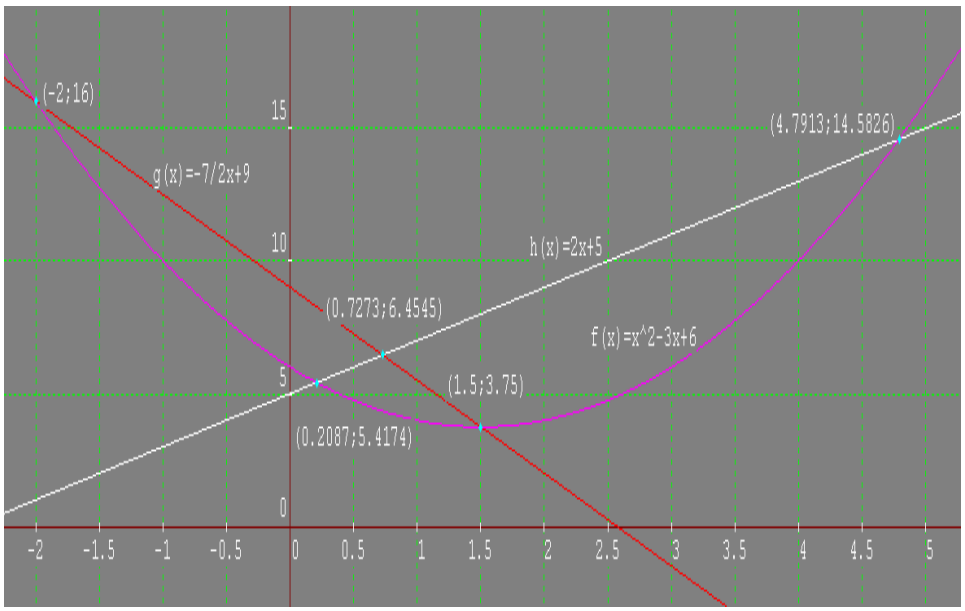
Calculation of S_L and S_R:
$S_L = x_2 - x_1 \approx 0.392 - 0 \approx 0.392$
$S_R = x_4 - x_3 \approx 5.109 - 4.000 \approx 1.109$
Calculate $D = |S_L - S_R|$.

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.392 - 1.109| \\ &= |-0.717| \\ y &= 0.717 \end{aligned}$$

Consider parabola $y = x^2 - 3x + 6$ and the lines $y = -\frac{7}{2}x + 9$ and $y = 2x + 5$



Again, using Graphmatica software, we can obtain the four intersection points.

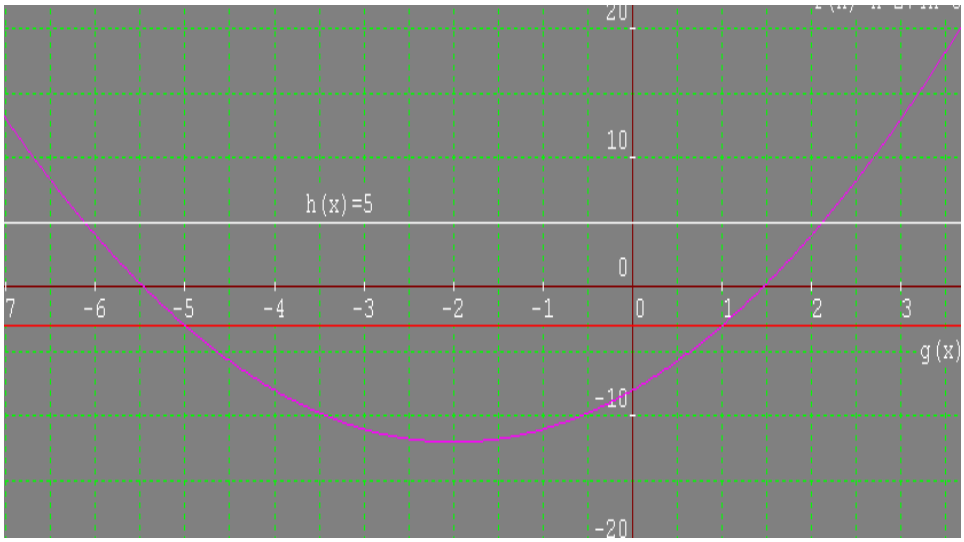


- $x_1 \approx -2.000$
- $x_2 \approx 0.209$
- $x_3 \approx 1.500$
- $x_4 \approx 4.791$

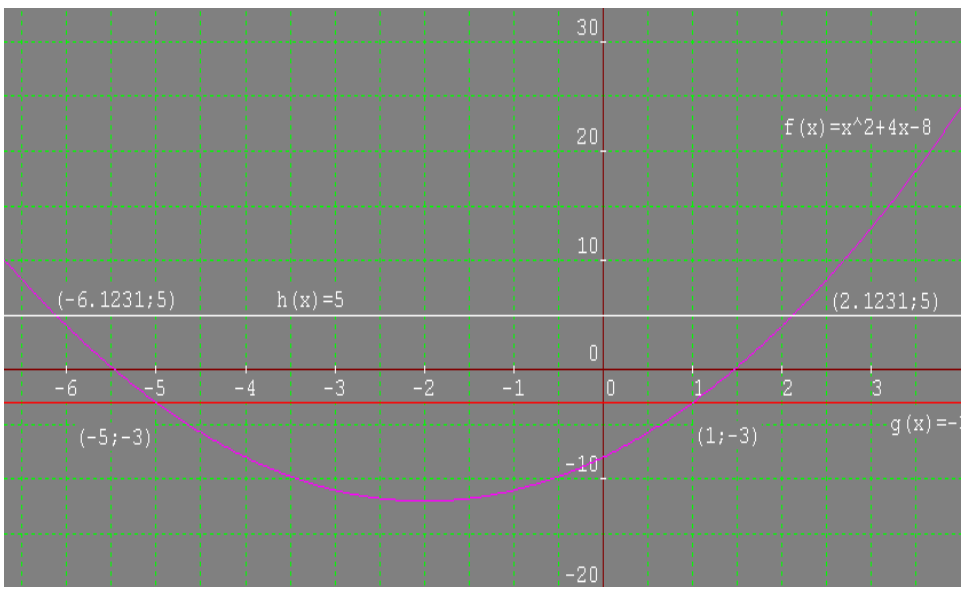
Calculation of S_L and S_R:
$S_L = x_2 - x_1 \approx 0.209 + 2.000 \approx 2.209$
$S_R = x_4 - x_3 \approx 4.791 - 1.500 \approx 3.291$
Calculate $D = |S_L - S_R|$.

$$\begin{aligned} D &= |S_L - S_R| \\ &= |2.209 - 3.291| \\ &= |-1.082| \\ &= 1.082 \end{aligned}$$

Consider parabola $y = x^2 + 4x - 8$ and the line y = 5 and y =-3.



By using Graphmatica software, we can obtain the four intersection points.



- $x_1 \approx -6.123$
- $x_2 \approx -5.000$
- $x_3 \approx 1.000$
- $x_4 \approx 2.123$

Calculation of S_L and S_R:
$S_L = x_2 - x_1 \approx (-5.000) - (-6.123) \approx 1.123$
$S_R = x_4 - x_3 \approx 2.123 - 1.000 \approx 1.123$
Calculate $D = |S_L - S_R|$.

$$D = |S_L - S_R|$$
$$= |1.123 - 1.123|$$
$$= |0|$$

Proof:

$ax^2 + (b - m_1)\,x + c = 0$

$ax^2 + (b - m_2)\,x + c = 0$

$$x = -\frac{b - m_1 \pm \sqrt{b - m_1}^{\,2} - 4ac}{2a}$$

$$x = -\frac{b - m_2 \pm \sqrt{b - m_2}^{\,2} - 4ac}{2a}$$

$D = |S_L - S_R| = |(x_2 - x_1) - (x_4 - x_3)| = |x_2 - x_1 - x_4 + x_3|$

$= |x_2 + x_3 - x_1 - x_4| = |(x_2 + x_3) - (x_1 + x_4)|$

$x_2 + x_3 = 2[-(b - m_1)/2a] = -(b - m_1)/a$

$x_1 + x_4 = 2[-(b - m_2)/2a] = -(b - m_2)/a$

$$D = \left| \left(\frac{-\ b - m_1}{a} \right) - \left(\frac{-\ b - m_2}{a} \right) \right| = \left| \frac{m_2 - m_1}{a} \right|, \text{ with } a \neq 0, \forall a \in \mathbf{R}.$$