Description

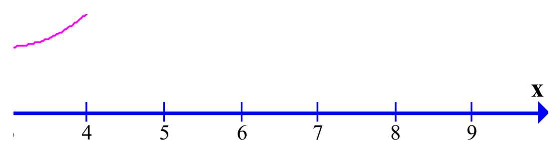
In this task, you will investigate the patterns in the intersections of parabolas and the lines y=x and y=2x. Then you will be asked to prove your conjectures and to broaden the scope of the investigation to include other lines and other types of polynomials.

Method

1. Consider the parabola $y=(x-3)^2+2 = x^2-6x+11$ and the lines y=x and y=2x.
    - The four points of intersections are illustrated in the graph below. Using a graphing program and a quadratic program, one can solve for these points.

The four points of intersection are:
$x_1 \approx (1.764, 5.528)$
$x_2 \approx (2.381, 2.381)$
$x_3 \approx (4.618, 4.618)$
$x_4 \approx (6.236, 12.272)$

- The x-values of these intersections are labeled as they appear from left to right on the x-axis as x_1, x_2, x_3, and x_4.



- The values of S_L and S_R are defined and solved.

  $S_L= x_2-x_1 \approx 2.382-1.764 \approx 0.618$

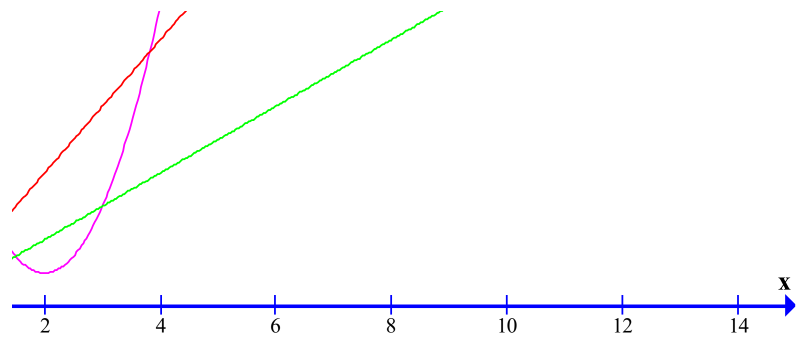
  $S_R = x_4-x_3 \approx 6.236-4.618 \approx 1.618$

- Calculate $D=|S_L- S_R|$.

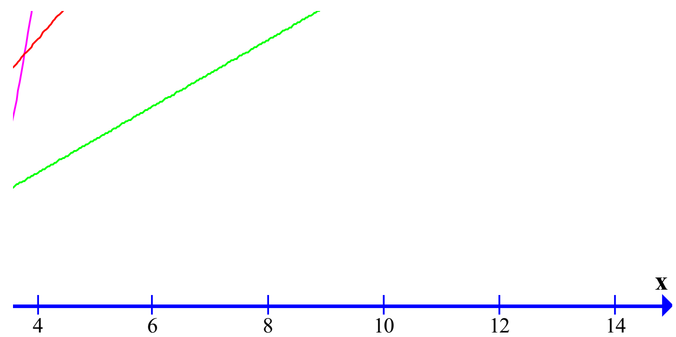
  $D=|S_L- S_R| \approx |0.618-1.618|= |-1|= 1$

2. Find values of D for other parabolas of the form $y= ax^2+bx+c$, $a>0$, with vertices in quadrant 1, intersected by the lines $y=x$ and $y=2x$. Consider various values of *a*, beginning with *a*=1. Make a conjecture about the value of D for these parabolas.
   - I'm going to consider three different parabolas with vertices in quadrant 1 in the form of form $y= ax^2+bx+c$, $a>0$.

$y= (x-4)^2+2 = x^2-8x+18$



$y = 2x^2 - 8x + 9$



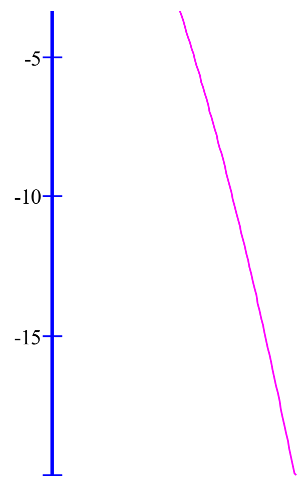
$y= 4x^2-20x+26$

To summarize, the results are listed in the chart below:

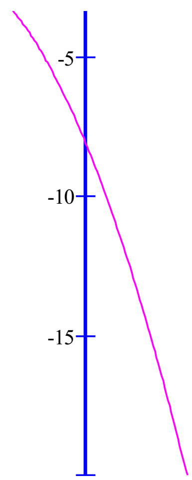
Formula	x_1	x_2	x_3	x_4	D
$y= x^2-8x+18$	2.354	3	6	7.646	1
$y= 2x^2-8x+9$	1.177	1.5	3	3.823	0.5
$y= 4x^2-20x+26$	1.719	2	3.25	3.781	0.25

Conjecture:
The relationship of D and a looks like it should be:
$D=|-1/a|$

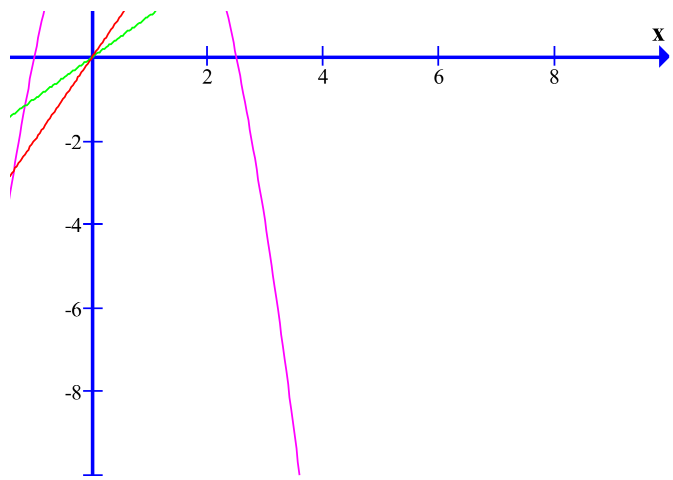
3. Investigating this conjecture further for any real value of *a* and any placement of the vertex, I try different values. The labeling convention used in parts 1 and 2 by having the intersections of the first line to be x_2 and x_1 and the intersections with the second line to be x_1 and x_4 will be maintained.



$y=-x^2-2x+6$



$y= -x^2-5x-8$



$y = -2x^2+3x+5$

The results of investigating different real values of *a* and placement of the vertex is shown below:

Formula	x_1	x_2	x_3	x_4	D
$y = -x^2-2x+6$	-5.162	-4.372	1.372	1.162	1
$y = -x^2-5x-8$	-5.562	-4	-2	-1.438	1
$y = -2x^2+3x+5$	-1.351	-1.158	2.158	1.851	0.5

The conjecture still holds and the results of D fit the conjecture found in part 2.

Proof:

$y = ax^2-bx+c$

$y = x$

$ax^2-(b-1)x+c=0$

$$x = \frac{(1-b)\pm\sqrt{((b-1)^2-4ac)}}{2a}$$

$$x_{2,3} = \frac{(1-b)\pm\sqrt{((b-1)^2-4ac)}}{2a}$$

$y = ax^2-bx+c$

$y = 2x$

$ax^2-(b-2)x+c=0$

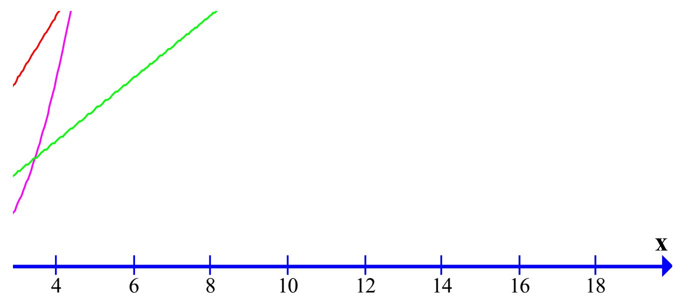
$$x = \frac{(2-b)\pm\sqrt{((b-2)^2-4ac)}}{2a}$$

$$X_{1,4} = \frac{(2-b)\pm\sqrt{((b-2)^2-4ac)}}{2a}$$

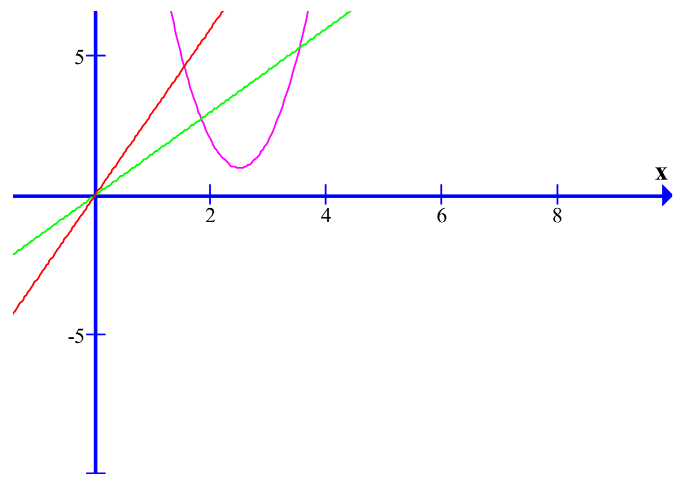
$S_L= x_2-x_1$
$S_R= x_4-x_3$
$D=|S_L-S_R| = |(x_2-x_1) - (x_4-x_3)|= |(x_2+x_3)-(x_1+x_4)|= |(1-b)2/2a - (2-b)2/2a|$
$= 1/a|1-b-2+b|=1/a$

4. To prove that the conjecture will still hold with the lines are changed, I will use the same examples from part 2. Now, the two intersecting lines will be y=1.5x and y=3x.

$y= x^2-8x+18$



$y = 2x^2 - 8x + 9$



$y= 4x^2-20x+26$

Formula	x_1	x_2	x_3	x_4	D
$y= x^2-8x+18$	2	2.614	6.886	9	1.5
$y= 2x^2-8x+9$	1	1.307	3.443	4.5	0.75
$y= 4x^2-20x+26$	1.546	1.837	3.538	4.204	0.375

Some modifications had to be made to the conjecture. The new conjecture is:
$y=b_1x$ (these are the equations of the lines)
$y=b_2x$
$D=|b_2- b_1| /a$
So D is actually the absolute value of the difference of the slopes of the intersecting lines divided by the *a* of the parabola.

Proof:

$ax^2+(b- b_1)x+c= 0$

$ax^2+(b- b_2)x+c= 0$

$$x= \frac{-(b- b_1)\pm\sqrt{((b- b_1)^2-4ac)}}{2a}$$

$$x= \frac{-(b- b_2)\pm\sqrt{((b- b_2)^2-4ac)}}{2a}$$

$D=|S_L-S_R| = |(x_2-x_1) - (x_4-x_3)|= |x_2-x_1- x_4+x_3|= |x_2+x_3-x_1- x_4|= |(x_2+x_3)-(x_1+x_4)|$

$x_2+x_3=[-(b- b_1)/2a]2= -(b- b_1)/a$

$x_1+x_4=[-(b- b_2)/2a]2= -(b- b_2)/a$

$D=|(-(b- b_1)/a)-(-(b- b_2)/a)|= |b_2- b_1|/a$

5. A similar conjecture can be made for cubic polynomials.

   According to the fundamental theorem of algebra:

   $ax^3+bx^2+cx+d= a(x-x_1)(x-x_2)(x-x_3)$

   $(x-x_1)(x-x_2)(x-x_3)= x^2-xx_1-xx_2+x_1x_2\ (xx_3)$

   $=x^3-x^2x_1-x^2x_2-xx_1x_2- x^2x_3+xx_1x_3+xx_2x_3-x_1x_2x_3$

   $= a(x^3-(x_1+x_2+x_3)x^2+(x_1x_2+x_2x_3+x_1x_3)x-(x_1x_2x_3))$

   $= ax^3-a(x_1+x_2+x_3)x^2+a(x_1x_2+x_2x_3+x_1x_3)x-a(x_1x_2x_3)$

   From the proof we can see what each of the coefficient equals:

   $a=a$

   $b= -a(x_1+x_2+x_3)$

   $c= a(x_1x_2+x_2x_3+x_1x_3)$

   $d= a(x_1x_2x_3)$

   From the expression for b, we can find the sum of the roots:

   $b= -a(x_1+x_2+x_3)$

   $x_1+x_2+x_3= b/-a = -b/a$

   From the conjecture and proof for the parabola,
   We know that:

   $D=|(x_2+x_3)-(x_1+x_4)|$

   For a cubic polynomial, we've found that the sum of the roots is –b/a so

   $D= |(-b/a)-(-b/a)|=0$

6. The conjecture can be modified to include higher order polynomials and it would very similar to the cubic one. For higher order polynomials, the roots will cancel out so D=0 will always be true.

   $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0= a_n(x-x_n)(x-x_{n-1})\ldots(x-x_1)$

   $a_nx^3-a_n(x_n+x_{n-1}+\ldots+x_1)+ a_n(x_nx_{n-1}+x_{n-1}x_{n-2}+\ldots+x_nx_1)+a_n(x_nx_{n-1}\ldots x_1)$

   $x_n+x_{n-1}+\ldots x_1=(a_{n-1})/-a_n$