Muhammad Faiz M.Ismail 12B

Matrix Powers

1. Consider the matrix **M** = $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

   a. Calculate $\boldsymbol{M}^n$ for n = 2, 3, 4, 5, 10, 20, 50.

   $\boldsymbol{M}^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

   $\boldsymbol{M}^3 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$

   $\boldsymbol{M}^4 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$

   $\boldsymbol{M}^5 = \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix}$

   $\boldsymbol{M}^{10} = \begin{bmatrix} 1024 & 0 \\ 0 & 1024 \end{bmatrix}$

   $\boldsymbol{M}^{20} = \begin{bmatrix} 1048576 & 0 \\ 0 & 1048576 \end{bmatrix}$

   $\boldsymbol{M}^{50} = \begin{bmatrix} 1.12589907 \times 10^{15} & 0 \\ 0 & 1.12589907 \times 10^{15} \end{bmatrix}$

   b. Describe in words any pattern you observe.

   Using a TI-83 Plus to calculate each of these power matrices , we are able to find the nth matrix. The matrix provided is an identity matrix, which is uniquely defined by the property $IM = M$ thus the value of M^n is equated to two to the power of n as well; the value zero stays constant for other values of n. The matrix M^n can also be calculated using geometric sequence: $U_n = U_1 r^{n-1}$;as the common ratio between each matrix is a factor of two.

   c. Use this pattern to find a general expression for the matrix $\boldsymbol{M}^n$ in terms of n.

   $\therefore M^n = I \times 2^n$

2. Consider the matrices $\boldsymbol{N} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and $\boldsymbol{S} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

   $\boldsymbol{N}^2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^2 = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} = 2\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ $\boldsymbol{S}^2 = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}^2 = \begin{bmatrix} 20 & 16 \\ 16 & 20 \end{bmatrix} = 2\begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}$

   a. Calculate $\boldsymbol{N}^n$ and $\boldsymbol{S}^n$ for other values of n and describe any pattern(s) you observe.

   $\boldsymbol{N}^3 = \begin{bmatrix} 36 & 28 \\ 28 & 36 \end{bmatrix} = 4\begin{bmatrix} 9 & 7 \\ 7 & 9 \end{bmatrix}$ $\boldsymbol{S}^3 = \begin{bmatrix} 112 & 112 \\ 104 & 104 \end{bmatrix} = 4\begin{bmatrix} 28 & 26 \\ 26 & 28 \end{bmatrix}$

$P^4 = \begin{bmatrix} 136 & 120 \\ 120 & 136 \end{bmatrix} = 8\begin{bmatrix} 17 & 15 \\ 15 & 17 \end{bmatrix}$ $S^4 = \begin{bmatrix} 656 & 640 \\ 640 & 656 \end{bmatrix} = 8\begin{bmatrix} 82 & 80 \\ 80 & 82 \end{bmatrix}$

$P^5 = \begin{bmatrix} 528 & 496 \\ 496 & 528 \end{bmatrix} = 16\begin{bmatrix} 33 & 31 \\ 31 & 33 \end{bmatrix}$ $S^5 = \begin{bmatrix} 3904 & 3872 \\ 3872 & 3904 \end{bmatrix} = 16\begin{bmatrix} 244 & 242 \\ 242 & 244 \end{bmatrix}$

$P^7 = \begin{bmatrix} 8256 & 8128 \\ 8128 & 8256 \end{bmatrix} = 64\begin{bmatrix} 129 & 127 \\ 127 & 129 \end{bmatrix}$ $S^7 = \begin{bmatrix} 140032 & 139904 \\ 139904 & 140032 \end{bmatrix} = 64\begin{bmatrix} 2188 & 2186 \\ 2186 & 2188 \end{bmatrix}$

$P^{10} = \begin{bmatrix} 524800 & 523776 \\ 523776 & 524800 \end{bmatrix} = 512\begin{bmatrix} 1025 & 1023 \\ 1023 & 1025 \end{bmatrix}$

$S^{10} = \begin{bmatrix} 30233600 & 30232576 \\ 30232576 & 30233600 \end{bmatrix} = 512\begin{bmatrix} 59050 & 59048 \\ 59048 & 59050 \end{bmatrix}$

From a standard matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can see that the difference between a and b, c and d is two. Therefore we could infer that it follows the pattern $\begin{bmatrix} k+1 & k-1 \\ k-1 & k+1 \end{bmatrix}$, and that k in matrix P is two whereas k in matrix S is three.

However, a coefficient had to be factorised in order to have that difference of two. A pattern for the coefficient was found as 2^{n-1}.

3. Now consider matrices of the form $\begin{bmatrix} k+1 & k-1 \\ k-1 & k+1 \end{bmatrix}$.

   Steps 1 and 2 contain examples of these matrices for k= 1, 2 and 3.

   Consider other values of k, and describe any pattern(s) you observe.

   Generalize these results in terms of k and n.

If k=6; **M** = $\begin{bmatrix} 7 & 5 \\ 5 & 7 \end{bmatrix}$; **M**^{2} = $2\begin{bmatrix} 37 & 35 \\ 35 & 37 \end{bmatrix}$; **M**^{3} = $4\begin{bmatrix} 217 & 215 \\ 215 & 217 \end{bmatrix}$

If k=10; **M** = $\begin{bmatrix} 11 & 9 \\ 9 & 11 \end{bmatrix}$; **M**^{2} = $2\begin{bmatrix} 101 & 99 \\ 99 & 101 \end{bmatrix}$; **M**^{3} = $4\begin{bmatrix} 1001 & 999 \\ 999 & 1001 \end{bmatrix}$

If k=-1; **M** = $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$; **M**^{2} = $2\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$; **M**^{3} = $4\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$

If k=-2; **M** = $\begin{bmatrix} -1 & -3 \\ -3 & -1 \end{bmatrix}$; **M**^{2} = $2\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$; **M**^{3} = $4\begin{bmatrix} -7 & -9 \\ -9 & -7 \end{bmatrix}$

If k=½; **M** = $\begin{bmatrix} 1½ & -½ \\ -½ & 1½ \end{bmatrix}$; **M**^{2} = $2\begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix}$; **M**^{3} = $4\begin{bmatrix} 1.125 & -0.875 \\ -0.875 & 1.125 \end{bmatrix}$

There is still a common difference of two between a and b, c and d according to 2^{n-1}, ranging from whole values such as six or ten, to negative and rational values. From these results we can generalise the pattern as $2^{n-1}\begin{bmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{bmatrix}$ to incorporate other values of n.

4. Use technology to investigate what happens with further valu es of k and n. State the scope or limitations of k and n.

If k=20; $\mathbf{M} = \begin{bmatrix} 21 & 19 \\ 19 & 21 \end{bmatrix}$; $\mathbf{M}^{10} = 512\begin{bmatrix} 1.024\times10^{13} & 1.024\times10^{13} \\ 1.024\times10^{13} & 1.024\times10^{13} \end{bmatrix}$

If k=-10; $\mathbf{M} = \begin{bmatrix} -9 & -11 \\ -11 & -9 \end{bmatrix}$; $\mathbf{M}^{-5}$ = ERR:DOMAIN if put in calculator

$\mathbf{M}^{-5} = 0.015265\begin{bmatrix} 0.9999830649 & -1.000016935 \\ -1.000016935 & 0.9999830649 \end{bmatrix}$ if put in formula

If k=-10; $\mathbf{M} = \begin{bmatrix} -9 & -11 \\ -11 & -9 \end{bmatrix}$; $\mathbf{M}^{0.5}$ = ERR:DOMAIN if put in calculator

$\mathbf{M}^{0.5} = 0.7071067812\begin{bmatrix} -2.16227766 & -4.16227766 \\ -4.16227766 & -2.16227766 \end{bmatrix}$ if put in calculator

The value of k has no limitations, any number plugged into k, be it negative, rational, whole, will be coherent with the pattern. The values of n however, is limited to only whole numbers. It can go to the extent of whole numbers but it can never be a negative value or a rational value as the err or DOMAIN would appear on the calculator. Theoretically, if these disallowed n values were to be solely plugged into the pattern $2^{n-1}\begin{bmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{bmatrix}$ and not $\mathbf{M}^n$, then similar to k the pattern would also function with any rational or negative numbers.

5. Explain why your results hold true in general.

Since we have proven the general expression for $\boldsymbol{M}$ and all those examples are part of the pattern, the results holds true for all examples in the pattern except for the restrictions that have been proven .