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Period 7   Pre Calculus

$Matrix$ Powers: $Type$ I

$Matrices$ are useful mathematical tools that help us to interpret, represent, and ultimately understand information. By comparing matrices in their original form, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can observe specific patterns that are helpful in interpreting information demonstrated using matrices. But, in some specific cases, the determinant of said matrices must be calculated in order to see a specific correlation. This investigation analyzes relationships between both matrices and determinates of differing powers. It further presents a general rule that can be used to calculate a specific pattern for any power.

When considering the matrix: $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, an expression for the matrix of M^n in terms of n can be found by simply formalizing an observed pattern of the matrix M^n where n=1,2,3,4,5,6,7,8,9,10,20,50. After plugging the matrix M into a graphing calculator, M to the previously stated powers can be easily calculated using the "^" key. After making these calculations, the results obtained can be observed below in Table 1.1.

Table 1.1

Power	$Matrix$
n = 1	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
n = 2	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$
n = 3	$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$
n = 4	$\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$
n = 5	$\begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix}$
n = 10	$\begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix}$
n = 20	$\begin{pmatrix} 1048576 & 0 \\ 0 & 1048576 \end{pmatrix}$
n = 50	$\begin{pmatrix} 8.88*10^{34} & 0 \\ 0 & 8.88*10^{34} \end{pmatrix}$

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An obvious pattern can be seen above, in which as the power n increases, so do the numbers in the top left portion (a) and lower right portion (d) of the matrix, while the other two values stay constant at zero. In fact, the numbers in the top left and lower right increase exponentially. Therefore, an equation can be derived to calculate M^n in terms of n:

$M^n = \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$. This equation accurately represents the exponential increase in the upper left and lower right quadrants of the matrix and, and is congruent with the pattern witnessed above.

Again, when considering the matrix M, another interesting relationship between the exponent, n, and the determinant of M can be observed and expressed in a general formula found by formalizing the observed pattern of the determinant of matrix M^n where n=1,2,3,4,5,6,7,8,9,10,20,50. In order to find the determinants of the matrices listed in the second column of Table 1.1, they must first be entered into a graphing calculator. Next, in the "Math" tab, found in the "Matrix" menu, the "det ( )" button can be used in conjunction with the matrix. The results of this process for the matrices seen in Table 1.1 are replicated below in Table 1.2.

Table 1.2

Power	Matrix	Determinant
n = 1	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	4
n = 2	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	16
n = 3	$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$	64
n = 4	$\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$	256
n = 5	$\begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix}$	1024
n = 10	$\begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix}$	1048576
n = 20	$\begin{pmatrix} 1048576 & 0 \\ 0 & 1048576 \end{pmatrix}$	$1.099*10^{12}$
n = 50	$\begin{pmatrix} 8.88*10^{\wedge}34 & 0 \\ 0 & 8.88*10^{\wedge}34 \end{pmatrix}$	$7.88*10^{69}$

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Again, there is an obvious pattern that can be seen between n, the exponent, and the determinant of the matrix. The determinant increases exponentially by the nth power, similarly to the top left and bottom right portions of the matrix. But, instead of having a base of two like the matrix, the determinant of the matrix will have a base of four. Therefore, the determinant of M^n can be represented with the equation $Det(M^n)=4^n$.

Although the matrix M shows very obvious patterns when it is exponentially increased, if the zeroes in the matrix are replaced by numbers greater than zero, the patterns observed become far more complex and scale factors must be utilized in order to simplify them. Consider the matrix S=$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. Because the matrix itself is so complex, there will be no evident pattern unless the matrix is factored to a point where a relationship can be observed and formalized into a general equation. First, S^n must be found where n=1,2,3,4,5,10,20,50 using a calculator. Then, a correlation between the scale factor and n must be found, in which the pattern observed in the matrix of S^1, is observed in all matrices of S^n. Therefore, the scale factor for the first matrix must be one. In this case, the equation for the scale factor in terms of n is 2^{n-1}, which makes sense because the numbers in the upper-left/lower-right and upper-right/lower-left have a difference of two. A chart representing the original matrix, scale factor, and factored matrix can be seen below in Table 2.1.

Table 2.1 (continues on page 4)

Power	Matrix	Scale Factor	Factored Matrix
n = 1	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$	1	$1\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$
n = 2	$\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$	2	$2\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$
n = 3	$\begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix}$	4	$4\begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix}$
n = 4	$\begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix}$	8	$8\begin{pmatrix} 17 & 15 \\ 15 & 17 \end{pmatrix}$
n = 5	$\begin{pmatrix} 528 & 496 \\ 496 & 528 \end{pmatrix}$	16	$16\begin{pmatrix} 33 & 31 \\ 31 & 33 \end{pmatrix}$

n = 10	$\begin{pmatrix} 52480 & 52376 \\ 52376 & 52480 \end{pmatrix}$	512	$512\begin{pmatrix} 105 & 103 \\ 103 & 105 \end{pmatrix}$
n = 20	$\begin{pmatrix} 549756338200 & 549755289600 \\ 549755289600 & 549756338200 \end{pmatrix}$	524288	$524288\begin{pmatrix} 104857 & 104855 \\ 104855 & 104857 \end{pmatrix}$

*"n=50" is not represented because the graphing calculator will not calculate out to enough decimal places to accurately depict the value (all values in the matrix are the same)

It can be seen that after the matrix is factored using the scale factor found by the formula, the pattern again becomes apparent. The values within the matrix increase exponentially while continuing to have a difference of two in their values. Therefore, this pattern can be turned into the equation

$$2^{n-1}\begin{pmatrix} 2^n+1 & 2^n-1 \\ 2^n-1 & 2^n+1 \end{pmatrix}$$, which represents both the matrix itself as well as the scale factor.

The relationship between the exponent, n, and the determinant of S is much simpler. The determinants of S^n were calculated using a graphing calculator, the same way that the determinants of M^n were calculated. The results are represented below in Table 2.2.

Table 2.2 (continues on page 5)

Power	Matrix	Determinant
n = 1	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$	8
n = 2	$\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$	64
n = 3	$\begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix}$	512
n = 4	$\begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix}$	4096
n = 5	$\begin{pmatrix} 528 & 496 \\ 496 & 528 \end{pmatrix}$	32768

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n = 10	$\begin{pmatrix} 52480 & 52376 \\ 52376 & 52480 \end{pmatrix}$	1073741824
n = 20	$\begin{pmatrix} 549756338200 & 549755289600 \\ 549755289600 & 549756338200 \end{pmatrix}$	$1.15*10^{18}$

*"n=50" is not represented because the graphing calculator will not calculate out to enough decimal places to accurately depict the value (all values in the matrix are the same)

By merely calculating the determinants, the relationship between the determinant of S^n and n is clear. The determinant undoubtedly increases exponentially with a base of eight as n increases linearly. The base eight is evident because when $n=1$ the determinant is eight. Therefore, the equation that represents the determinant relative to n is $det(S^n)=8^n$.

When considering matrices of the $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^{\wedge n}$ form where k and n are all real numbers greater or equal to one, a general equation can be developed in terms of k and n for both the factored matrix as well as the derivative of the original matrix. The results for $k=1$ and 2 have already been represented in the previous paragraphs. Therefore, by using the preceding calculations it is possible to come up with these general equations. First, the equation for the factored matrix is calculated by observing the previous pattern of the scale factor, which had already been calculated as 2^{n-1}. Then, it must be observed that all parts of the matrix increase exponentially by the power of n, as seen in the equations for $k=1$ and $k=2$. Furthermore, the base is always k. Therefore, the general, derived equation for the factored form of the above matrix is: $2^{n-1}\begin{pmatrix} K^n+1 & K^n-1 \\ K^n-1 & K^n+1 \end{pmatrix}$. Finding the general equation for the determinant is far easier. It is clear from the past examples that the equation is always to a power of n. Furthermore, the base of the exponent is always a factor of four. Therefore the equation for determinant is $(4k)^n=det(M^n)$. Using a graphing calculator to calculate both negative and integer values for k and plugging them into the derived equation, gives proof that the value for k is all real numbers. Furthermore, n can be all real numbers, including negative numbers as well. This was proved similarly by use of a calculator. Although, the calculator gave a "domain error" when attempting to put a negative value in for n, putting the negative value in by hand into the equation, resulted in a similar pattern previously witnessed. Therefore, both k and n can be all real numbers, including negative numbers and integers. The proof that both k and n can be all real numbers is witnessed in the chart below, Table 3.1.

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*Ta*ble 3.1

K	N	$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix} \wedge n$	Factored Matrix	Determinant
-6	-2	Err. Domain	$\frac{1}{8}\begin{pmatrix} -35 & -33 \\ -33 & -33 \end{pmatrix}$	0.001736
-6	0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	1
0	3	$\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$	$4\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	0
$\sqrt{16}$	3	$\begin{pmatrix} 260 & 252 \\ 252 & 260 \end{pmatrix}$	$4\begin{pmatrix} 65 & 63 \\ 63 & 65 \end{pmatrix}$	4096

Although the calculator was not able to verify the expression with a negative two, a similar pattern is seen in which the top-left/bottom-right values and bottom-left/top-right values have a difference of two.

In conclusion, the relationship between n and both the determinant and matrix to the power of n, is possible to find. But, the relationship between n and the determinant of the matrix to the power of n is far simpler and therefore easier to recognize. Therefore, determinants are helpful in visualizing relationships between matrices that otherwise would seem completely divergent of one another. Furthermore, using the equation for the derivative (($4k$)^n=det(M^n)) to represent patterns is far less confusing than attempting to show patterns through the equation for the factored matrix to the power of n ($2^{n-1}\begin{pmatrix} K^n+1 & K^n-1 \\ K^n-1 & K^n+1 \end{pmatrix}$). But, despite the obvious benefits of using one equation to show patterns over the use of the other, both generalized equations make the calculation of matrix powers and their derivatives without a calculator extremely efficient and nearly effortless.