IB PORTFOLIO-  MATRIX BINOMIALS

MATH SL

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10-2

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# INTRODUCTION

This project is about 2x2 matrices or matrix binomials; the aim of this work is to demonstrate a good and clear understanding of matrices and the operations that can be done with them.

At the end of the project we should have been able to generate different expressions for some different matrices powered by an integer as well as a general statement to calculate the addition of specific matrices powered to an integer.

To have a correct development of this piece of work it is essential that each question is solved in order since the result of a question will be necessary for the development of the following one.

Since there are many basic calculations that aren't really important we will use a GDC (graphic display calculator) in order to speed up the development of the project.

QUESTIONS

1)

Let $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Calcúlate $X^2, X^3, X^4; Y^2, Y^3, Y^4$.

With the help of a GDC (Graphic display calculator) the calculations for the matrices were done and registered below.

$$X^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \qquad\qquad Y^2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \qquad\qquad Y^3 = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$$

$$X^4 = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} \qquad\qquad Y^4 = \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}$$

2)

**By considering different integer powers of X and Y find an expression for $X^n$, $Y^n$ and $(X + Y)^n$.**

We have already calculated the resulting matrices for three different integer powers of X and Y in the first question, however to find an expression for $X^n$, $Y^n$ and $(X + Y)^n$ and be sure that it is true, we are also going to calculate the resulting matrices of using 5,6 and 7 as powers of X and Y.

The $X^5, X^6, X^7; Y^5, Y^6, Y^7$ shown below were calculated by using a GDC.

$$X^5 = \begin{pmatrix} 16 & 16 \\ 16 & 16 \end{pmatrix} \qquad\qquad Y^5 = \begin{pmatrix} 16 & -16 \\ -16 & 16 \end{pmatrix}$$

$$X^6 = \begin{pmatrix} 32 & 32 \\ 32 & 32 \end{pmatrix} \qquad\qquad Y^6 = \begin{pmatrix} 32 & -32 \\ -32 & 32 \end{pmatrix}$$

$$X^7 = \begin{pmatrix} 64 & 64 \\ 64 & 64 \end{pmatrix} \qquad\qquad Y^7 = \begin{pmatrix} 64 & -64 \\ -64 & 64 \end{pmatrix}$$

By analyzing the resulting matrices of different powers of X and Y we can develop an expression for $X^n$ and $Y^n$ which are shown below.

$$X^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix} \qquad\qquad Y^n = \begin{pmatrix} 2^{n-1} & -(2^{n-1}) \\ -(2^{n-1}) & 2^{n-1} \end{pmatrix}$$

Now to generate an expression for $(X + Y)^n$ we must first develop the matrix resulting from X+Y so we can then use different powers for (X+Y) and hence generate the expression for $(X + Y)^n$.

$$X + Y = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$(X + Y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$(X + Y)^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \qquad\qquad (X + Y)^5 = \begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix}$$

$$(X + Y)^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \qquad\qquad (X + Y)^6 = \begin{pmatrix} 64 & 0 \\ 0 & 64 \end{pmatrix}$$

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$$(X+Y)^4 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix} \qquad\qquad (X+Y)^7 = \begin{pmatrix} 128 & 0 \\ 0 & 128 \end{pmatrix}$$

By analyzing the resulting matrices of different powers of (X+Y) we can develop an expression for $(X+Y)^n$ which is shown below.

$$(X+Y)^n = \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$$

3)

**Let A = aX and B = bY, where a and b are constants.**

**Use different values of a and b to calculate $A^2$, $A^3$, $A^4$; $B^2$, $B^3$, $B^4$.**

The values that we are going to use for **a** are 3 and 8, and the ones for **b** are 2 and 5.

(i)Matrix **A** when **a** is 3 would be equal to **3X**.

$$A = 3X = 3\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

Using a GDC we can now calculate $A^2$, $A^3$, $A^4$; having in count that a is 3 in this case.

$$A^2 = \begin{pmatrix} 18 & 18 \\ 18 & 18 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 108 & 108 \\ 108 & 108 \end{pmatrix} \qquad A^4 = \begin{pmatrix} 648 & 648 \\ 648 & 648 \end{pmatrix}$$

(ii)Matrix **A** when **a** is 8 would be equal to **8X**.

$$A = 8X = 8\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$$

$$A = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$$

Using a GDC we can now calculate $A^2$, $A^3$, $A^4$; having in count that a is 8 in this case.

$$A^2 = \begin{pmatrix} 128 & 128 \\ 128 & 128 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 2048 & 2048 \\ 2048 & 2048 \end{pmatrix} \qquad A^4 = \begin{pmatrix} 32768 & 32768 \\ 32768 & 32768 \end{pmatrix}$$

(iii) Matrix **B** when **b** is 2 would be equal to **2Y**.

$$\mathbf{B} = 2\mathbf{Y} = 2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

Using a GDC we can now calculate $B^2, B^3, B^4$; having in count that b is 2 in this case.

$$B^2 = \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix} \qquad B^3 = \begin{pmatrix} 32 & -32 \\ -32 & 32 \end{pmatrix} \qquad B^4 = \begin{pmatrix} 128 & -128 \\ -128 & 128 \end{pmatrix}$$

(iv) Matrix **B** when **b** is 5 would be equal to **2Y**.

$$\mathbf{B} = 5\mathbf{Y} = 5\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix}$$

Using a GDC we can now calculate $B^2, B^3, B^4$; having in count that b is 5 in this case.

$$B^2 = \begin{pmatrix} 50 & -50 \\ -50 & 50 \end{pmatrix} \qquad B^3 = \begin{pmatrix} 500 & -500 \\ -500 & 500 \end{pmatrix} \qquad B^4 = \begin{pmatrix} 5000 & -5000 \\ -5000 & 5000 \end{pmatrix}$$

**By considering different integer powers of A and B find an expression for $A^n$, $B^n$ and $(A + B)^n$.**

Since we have already calculated the resulting matrices using 2, 3 and 4 as integer power values of A and B in the first part of this question we can use the previously calculated matrices to find and expression for $A^n$ and $B^n$ .

We must then remember that both matrices have the same origin since **A** by definition is **aX,** so $A^2$ is the same as $(\mathbf{aX})^2$ which can also be expressed as $(a^2X^2),$ we can check this by diving each element of both of the $A^2$ calculated previously by their respective values of $a^2$ and the result should be equal to $X^2$.

In the case of **a** being 8 , $A^2 = \begin{pmatrix} 128 & 128 \\ 128 & 128 \end{pmatrix}$

$$\frac{1}{8^2} \times A^2 = \begin{pmatrix} 128/64 & 128/64 \\ 128/64 & 128/64 \end{pmatrix} = X^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

While in the case of **a** being 3 , $A^2 = \begin{pmatrix} 18 & 18 \\ 18 & 18 \end{pmatrix}$

$$\frac{1}{3^2} \times A^2 = \begin{pmatrix} 18/9 & 18/9 \\ 18/9 & 18/9 \end{pmatrix} = X^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

So if $A^2$ can be expressed like $a^2X^2$ then $A^n$ can be expressed like $a^nX^n$ which can be expressed in a matrix form, like it is shown below.

$$A^n = a^nX^n = a^n \begin{bmatrix} (2^{n-1}) & (2^{n-1}) \\ (2^{n-1}) & (2^{n-1}) \end{bmatrix} = \begin{bmatrix} a^n(2^{n-1}) & a^n(2^{n-1}) \\ a^n(2^{n-1}) & a^n(2^{n-1}) \end{bmatrix}$$

$$A^n = \begin{bmatrix} a^n(2^{n-1}) & a^n(2^{n-1}) \\ a^n(2^{n-1}) & a^n(2^{n-1}) \end{bmatrix}$$

To find an expression for $B^n$ we can do the same process done to generate the expression for $A^n$ , so we can star by stating that **B** is **bY** by definition, so $B^3$ should be equal to $(bY)^3$ which can also be expressed as $(b^3Y^3),$ we can check this by diving each element of both of the $B^3$ calculated previously by their respective values of $b^3$ and the result should be equal to $Y^3$.

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In the case of **b** being 2 , $B^3 = \begin{pmatrix} 32 & -32 \\ -32 & 32 \end{pmatrix}$

$\frac{1}{2^3} \times B^3 = \begin{pmatrix} 32/_8 & -32/_8 \\ -32/_8 & 32/_8 \end{pmatrix} = Y^3 = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$

While in the case of **b** being 5 , $B^3 = \begin{pmatrix} 500 & -500 \\ -500 & 500 \end{pmatrix}$

$\frac{1}{5^3} \times B^3 = \begin{pmatrix} 500/_{125} & -500/_{125} \\ -500/_{125} & 500/_{125} \end{pmatrix} = Y^3 = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$

So if $B^3$ can be expressed like $b^3 Y^3$ then $B^n$ can be expressed like $b^n Y^n$ which can be expressed in a matrix form, like it is shown below.

$$B^n = b^n Y^n = b^n \begin{bmatrix} 2^{n-1} & -(2^{n-1}) \\ -(2^{n-1}) & 2^{n-1} \end{bmatrix} = \begin{bmatrix} b^n(2^{n-1}) & -b^n(2^{n-1}) \\ -b^n(2^{n-1}) & b^n(2^{n-1}) \end{bmatrix}$$

$$B^n = \begin{bmatrix} b^n(2^{n-1}) & -b^n(2^{n-1}) \\ -b^n(2^{n-1}) & b^n(2^{n-1}) \end{bmatrix}$$

The expression for $(A + B)^n$ can be found in the same way that we found the expressions for $A^n$ and $B^n$ , we know that by definition **A** is **aX** and **B** is **bY** so $(A + B)^n$ is the same as $(aX + bY)^n$. Starting from this we can develop an expression for $(A + B)^n$ as shown below.

$$(A + B)^n = (aX + bY)^n$$

$$(aX + bY)^n = \left[ a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right]^n$$

$$\left[ a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right]^n = \left[ \begin{pmatrix} a & a \\ a & a \end{pmatrix} + \begin{pmatrix} b & -b \\ -b & b \end{pmatrix} \right]^n$$

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$$\left[\begin{pmatrix} a & a \\ a & a \end{pmatrix} + \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}\right]^n = \begin{bmatrix} (a+b) & (a-b) \\ (a-b) & (a+b) \end{bmatrix}^n$$

$$(A+B)^n = \begin{bmatrix} (a+b) & (a-b) \\ (a-b) & (a+b) \end{bmatrix}^n$$

The expression for $(A+B)^n$ is then $\begin{bmatrix} (a+b) & (a-b) \\ (a-b) & (a+b) \end{bmatrix}^n$

4)

**Now consider $M = \begin{pmatrix} (a+b) & (a-b) \\ (a-b) & (a+b) \end{pmatrix}$**

**Show that $M = (A+B)$ and that $M^2 = (A^2 + B^2)$**

We know that matrix $M = \begin{pmatrix} (a+b) & (a-b) \\ (a-b) & (a+b) \end{pmatrix}$ so know we just need to add matrix $A$ to matrix $B$ and see if the resulting matrix is the same as $M$, taking in count that by definition $A$ is $aX$ and $B$ is $bY$ so $A + B = (aX + bY)$ and since in this case the value of $n$ is 1 we can use the expression developed in question 3 and we can see that certainly $M = (A+B)$. However this is also illustrated below.

$$M = \begin{pmatrix} (a+b) & (a-b) \\ (a-b) & (a+b) \end{pmatrix}$$

$$A + B = \left[ a\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] + \left[ b\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]$$

$$\left[\begin{pmatrix} a & a \\ a & a \end{pmatrix} + \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}\right] = \begin{bmatrix} (a+b) & (a-b) \\ (a-b) & (a+b) \end{bmatrix}$$

$$M = (A+B)$$

To show that $M^2 = (A^2 + B^2)$ we first have to calculate $M^2, A^2$ and $B^2$.

$A^2$ and $B^2$ can be calculated by using the previously developed expression for $A^n$ and $B^n$

$$B^2 = \begin{bmatrix} b^2(2^{2-1}) & -b^2(2^{2-1}) \\ -b^2(2^{2-1}) & b^2(2^{2-1}) \end{bmatrix} = B^2 = \begin{bmatrix} b^2(2^1) & -b^2(2^1) \\ -b^2(2^1) & b^2(2^1) \end{bmatrix}$$

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$$A^2=\begin{bmatrix} a^2(2^{2-1}) & a^2(2^{2-1}) \\ a^2(2^{2-1}) & a^2(2^{2-1}) \end{bmatrix} = A^2=\begin{bmatrix} a^2(2^1) & a^2(2^1) \\ a^2(2^1) & a^2(2^1) \end{bmatrix}$$

$$A^2 + B^2 = \begin{bmatrix} a^2(2^1) & a^2(2^1) \\ a^2(2^1) & a^2(2^1) \end{bmatrix} + \begin{bmatrix} b^2(2^1) & -b^2(2^1) \\ -b^2(2^1) & b^2(2^1) \end{bmatrix}$$

$$A^2 + B^2 = \begin{bmatrix} 2(a^2+b^2) & 2(a^2-b^2) \\ 2(a^2-b^2) & 2(a^2+b^2) \end{bmatrix}$$

$$M^2 = \begin{bmatrix} ((a+b)^2+(a-b)^2) & 2(a^2-b^2) \\ 2(a^2-b^2) & ((a+b)^2+(a-b)^2)) \end{bmatrix} =$$

$$\begin{bmatrix} (a^2+2ab+b^2+a^2-2ab+b^2) & 2(a^2-b^2) \\ 2(a^2-b^2) & a^2+2ab+b^2+a^2-2ab+b^2) \end{bmatrix} =$$

$$\begin{bmatrix} 2(a^2+b^2) & 2(a^2-b^2) \\ 2(a^2-b^2) & 2(a^2+b^2) \end{bmatrix} = M^2 = (A^2 + B^2)$$

**Hence find the general statement that expresses $M^n$ in terms of $aX$ and $bY$.**

From the previous question we can deduce that $M^n = (A^n + B^n)$ so in order to find a general statement for $M^n$ we could just add up the expressions developed for $A^n$ and $B^n$ and that should give us the general statement.

$$M^n = A^n + B^n = \begin{bmatrix} b^n(2^{n-1}) & -b^n(2^{n-1}) \\ -b^n(2^{n-1}) & b^n(2^{n-1}) \end{bmatrix} + \begin{bmatrix} a^n(2^{n-1}) & a^n(2^{n-1}) \\ a^n(2^{n-1}) & a^n(2^{n-1}) \end{bmatrix}$$

Since both $A^n$ and $B^n$ are being multiplied by $2^{n-1}$ we can take it out of both $A^n$ and $B^n$ and leave it as a scalar multiplying the adition of the matrices as shown below.

$$M^n = A^n + B^n = 2^{n-1} \times \left( \begin{bmatrix} b^n & -b^n \\ -b^n & b^n \end{bmatrix} + \begin{bmatrix} a^n & a^n \\ a^n & a^n \end{bmatrix} \right)$$

$$M^n = A^n + B^n = 2^{n-1} \times \begin{bmatrix} (a^n + b^n) & (a^n - b^n) \\ (a^n - b^n) & (a^n + b^n) \end{bmatrix}$$

$$M^n = A^n + B^n = \begin{bmatrix} 2^{n-1}(a^n + b^n) & 2^{n-1}(a^n - b^n) \\ 2^{n-1}(a^n - b^n) & 2^{n-1}(a^n + b^n) \end{bmatrix}$$

And so the general statement for $M^n$ would be:

$$M^n = \begin{bmatrix} 2^{n-1}(a^n + b^n) & 2^{n-1}(a^n - b^n) \\ 2^{n-1}(a^n - b^n) & 2^{n-1}(a^n + b^n) \end{bmatrix}$$

We can prove this by seeing that if when we replace a and b with 3 and 2 respectively it should give us the add up of **A + B** when n has a value of 1.

We have then

$$B = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$$

Now if we use our general statement it should give use the same when we replace the values of a, b and n with the ones used to develop $A + B$

$$M^1 = \begin{bmatrix} 2^{1-1}(3^1 + 2^1) & 2^{1-1}(3^1 - 2^1) \\ 2^{1-1}(3^1 - 2^1) & 2^{1-1}(3^1 + 2^1) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

For this case it worked, but to check that it is a general rule I am going to do it wi the following value for a, b and n.

a=8

b=2

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n=3

Since we have already calculated $A^3$ and $B^3$ having the previously established values we can see if the addition of $A^3$ and $B^3$ in this case is also equal to the general statement.

We have then

$$A^3 = \begin{pmatrix} 2048 & 2048 \\ 2048 & 2048 \end{pmatrix} \qquad and \qquad B^3 = \begin{pmatrix} 32 & -32 \\ -32 & 32 \end{pmatrix}$$

$$A^3 + B^3 = \begin{pmatrix} 2048 & 2048 \\ 2048 & 2048 \end{pmatrix} + \begin{pmatrix} 32 & -32 \\ -32 & 32 \end{pmatrix} = \begin{pmatrix} 2080 & 2016 \\ 2016 & 2080 \end{pmatrix}$$

$$A^3 + B^3 = \begin{pmatrix} 2080 & 2016 \\ 2016 & 2080 \end{pmatrix}$$

Now if we use our general statement it should give use the same when we replace the values of a, b and n with the ones used to develop $A^3 + B^3$

$$M^3 = \begin{bmatrix} 2^{3-1}(a^3+b^3) & 2^{3-1}(a^3-b^3) \\ 2^{3-1}(a^3-b^3) & 2^{3-1}(a^3+b^3) \end{bmatrix} = \begin{bmatrix} 2^2(8^3+2^3) & 2^2(8^3-2^3) \\ 2^2(8^3-2^3) & 2^2(8^3+2^3) \end{bmatrix}$$

$$M^3 = \begin{bmatrix} 4(520) & 4(504) \\ 4(504) & 4(520) \end{bmatrix} = \begin{bmatrix} 2080 & 2016 \\ 2016 & 2080 \end{bmatrix}$$

Once again the statement worked and therefore I can consider it a general statement.

## CONCLUSION

Nb b

- The general statement developed did worked correctly and was checked two times with different values for a, b and n.

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- The relationship between the matrix $M^n$ with the matrices $A^n$ and $B^n$ was found and it was expressed in matrix form in question 4.

- We were able to generate expressions for specific matrices powered by an unknown $z$ integer and having developed those expressions we were able to generate a general statement.

## EVALUATION

In my opinion the project was developed in a good and coherent manner, there weren't any serious problems or difficulties during the processes and the results obtained were satisfactory, since I was able to generate the general statement and the other expressions for the different matrices powered by an $z$ integer.

During the process I used all my knowledge about matrices which helped me to develop a good understanding of the questions in order to obtain the different answer.

However I think that the information in some parts is not presented in an organized way, this could have been improved with the use of tables to record the information in a more suitable place.