

John Vu  
000948-111  
1

## Matrix Binomials

A matrix is a rectangular array of numbers arranged in rows and columns. Numbers or letters inside the brackets in matrices are called entries. Matrices can be added, subtracted, multiplied, divided, and also raised to a power. A common matrix can look like this  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  where a, b, c, and d are the entries.

Given the matrices  $X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  and  $Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$  I calculated  $X^2, X^3, X^4, Y^2, Y^3, Y^4$ .

Using my GDC (graphic display calculator) I evaluated these matrices.

$$X^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$Y^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$X^3 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$Y^3 = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

$$X^4 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

$$Y^4 = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$$

After calculating the powers of X and Y, I observed my solutions. Looking at this I saw a trend that emerged as I increased the power of the matrix. The trend was increasing the power of the matrix by one, caused the product to double its previous solution. When X is to the second power, the entries of the solution are all 2's; when X is to the third power the entries are all 4's; when X is to the fourth power the entries are all 8's, and so on. As for Y, the pattern is similar but some entries are negative (-). Using this information I produced an expression to solve for X to a certain power, and it is  $X^n =$  letting "n" represent the power. For Y the expression is the same for the numbers in position of a and d. The expression for numbers of b and c is also the same but it is negative. As a matrix the expression is  $Y^n =$  . I also calculated integer powers of  $(X + Y)^n$ .

John Vu  
000948-111  
2

Using my GDC I found

$$(X + Y)^1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(X + Y)^3 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$(X + Y)^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(X + Y)^4 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$$

When I finished with these calculations, I saw another pattern that was developed. Seeing this pattern I formed an expression to solve for the matrix  $(X + Y)^n$ . For the numbers of a and d the expression is  $2^n$  and for c and d it is 0. As a matrix the expression is  $(X + Y)^n = \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}$ .

To make sure all my expressions worked correctly, I calculated each matrix to the fifth power which was the next integer power in the sequence.

$$X^5 = \begin{bmatrix} 2^{(5-1)} & 2^{(5-1)} \\ 2^{(5-1)} & 2^{(5-1)} \end{bmatrix}$$

$$Y^5 = \begin{bmatrix} 2^{(5-1)} & -(2^{(5-1)}) \\ -(2^{(5-1)}) & 2^{(5-1)} \end{bmatrix}$$

$$(X + Y)^5 = \begin{bmatrix} 2^5 & 0 \\ 0 & 2^5 \end{bmatrix}$$

$$= \begin{bmatrix} 2^4 & 2^4 \\ 2^4 & 2^4 \end{bmatrix}$$

$$= \begin{bmatrix} 2^4 & -(2^4) \\ -(2^4) & 2^4 \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 16 \\ 16 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -16 \\ -16 & 16 \end{bmatrix}$$

After these calculations rose to the fifth power, the pattern still continued for each matrix. Letting  $A = aX$  and  $B = bY$ , where a and b are constants; I used different values of a and b to calculate  $A^2, A^3, A^4; B^2, B^3, B^4$ . For the value of a, I used the number 2 and the value of b, I used 3. With these values I used my GDC to solve for A and B.

John Vu  
000948-111  
3

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 12 & 12 \\ 12 & 12 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 27 & -27 \\ -27 & 27 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 16 & 16 \\ 16 & 16 \end{bmatrix}$$

$$B^4 = \begin{bmatrix} 81 & -81 \\ -81 & 81 \end{bmatrix}$$

By considering integer powers of these matrices, I came up with an expression to also solve these matrices. For matrix A the expression is  $A^n = (a^n) \begin{matrix} 2^{(n-1)} & 2^{(n-1)} \\ 2^{(n-1)} & 2^{(n-1)} \end{matrix}$  and matrix B the expression  $B^n = (b^n) \begin{matrix} 2^{(n-1)} & -(2^{(n-1)}) \\ -(2^{(n-1)}) & 2^{(n-1)} \end{matrix}$

Using my GDC I solved these matrices.

$$(A + B)^1 = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

$$(A + B)^3 = \begin{bmatrix} 19 & -7 \\ -7 & 19 \end{bmatrix}$$

$$(A + B)^2 = \begin{bmatrix} 26 & -10 \\ -10 & 26 \end{bmatrix}$$

$$(A + B)^4 = \begin{bmatrix} 76 & -50 \\ -50 & 76 \end{bmatrix}$$

After solving these equations, I generated an expression to solve the matrix for  $(A + B)$ . The expression to solve the matrix is  $(A + B)^n = (a^n \ X^n) + (b^n \ Y^n)$ . To make sure the expression worked I plugged in number 2 and 4 for the value of n.

John Vu  
000948-111  
4

$$(A + B)^2 = (2^2 X^2) + (3^2 Y^2)$$

$$(A + B)^4 = (2^4 X^4) + (3^4 Y^4)$$

$$\text{Using GDC } (A + B)^2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\text{Using GDC } (A + B)^4 = \begin{bmatrix} 16 & -8 \\ -8 & 16 \end{bmatrix}$$

Considering that  $M = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$ , this expression can also be derived into

$M = A + B$  and  $M^2 = A^2 + B^2$ . By proving the expression  $M = A + B$ , A and B needs to be substituted for  $aX$  and  $bY$  as well as keeping the constants as a variable. This will

prove the expression  $M = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$  through  $M = A + B$ .

$$M = aX + bY$$

$$\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} = a \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix} + \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}$$

$$\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$$

$$\therefore M = A + B$$

We can also prove that  $M^2 = A^2 + B^2$  to later help find a general statement for  $M^n$ , in terms of  $aX$  and  $bY$ .

John Vu  
000948-111  
5

$$M^2 = A^2 + B^2$$

$$\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} = (aX)^2 + (bX)^2$$

$$\begin{bmatrix} (a+b)(a+b) + (a-b)(a-b) & (a+b)(a-b) + (a-b)(a+b) \\ (a-b)(a+b) + (a+b)(a-b) & (a-b)(a-b) + (a+b)(a+b) \end{bmatrix} = a^2 \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + b^2 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\frac{(a^2 + 2ab + b^2) + (a^2 + 2ab + b^2)}{(a^2 - b^2) + (a^2 - b^2)} \quad \frac{(a^2 - b^2) + (a^2 - b^2)}{(a^2 + 2ab + b^2) + (a^2 + 2ab + b^2)} = \frac{2a^2}{2a^2} \frac{2a^2}{2a^2} +$$

$$\frac{2b^2}{-2b^2} \frac{-2b^2}{2b^2}$$

$$\frac{2a^2 + 2b^2}{2a^2 - 2b^2} \frac{2a^2 - 2b^2}{2a^2 + 2b^2} = \frac{2a^2 + 2b^2}{2a^2 - 2b^2} \frac{2a^2 - 2b^2}{2a^2 + 2b^2}$$

∴ Hence, the general statement that expresses  $M^n$  in terms of  $aX$  and  $bY$  can be expressed as:

$$M^n = (aX)^n + (bY)^n$$

To test the validity of my general statement by using different values of  $a$ ,  $b$ , and  $n$ .

First I will use the general statement  $M^n = (aX)^n + (bY)^n$  and verify it by using the

John Vu  
000948-111  
6

$$\text{expression } M^n = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}^n$$

Let  $a = -1$ ,  $b = 0.5$ , and  $n = 2$

$M^n = (aX)^n + (bY)^n$ $M^2 = (-1X)^2 + (0.5Y)^2$ $= \left(-1^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) + \left(0.5^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)$ $= \left(1 \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right) + \left(0.25 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\right)$ $= \begin{bmatrix} 2.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix}$ $= 5 \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{bmatrix}$	$M^n = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}^n$ $M^2 = \begin{pmatrix} -1+0.5 & -1-0.5 \\ -1-0.5 & -1+0.5 \end{pmatrix}^2$ $= \begin{pmatrix} -0.5 & -1.5 \\ -1.5 & -0.5 \end{pmatrix}^2$ $= \begin{pmatrix} 2.5 & 1.5 \\ 1.5 & 2.5 \end{pmatrix}$ $= 5 \begin{pmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{pmatrix}$
--	--

By finding if there are any limitations within this expression:  $M^n = (aX)^n + (bY)^n$ , I am going to change the constants 'a' and 'b' as well the power raised to 'n' into , decimals, fractions and negative exponents.

The first example I am going to prove that this general statement has some limitations:

Let  $a=0.3$ ,  $b=0.32$ , and  $n=-3$

$$M^{-3} = (0.3X)^{-3} + (0.32Y)^{-3}$$

$$M^{-3} = \left(0.3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)^{-3} + \left(0.32 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)^{-3}$$

John Vu  
000948-111  
7

As I plug this equation into my calculator it comes up with: (error domain). As a result this equation cannot be solved because the matrix cannot be raised to a negative power. Therefore the limitation in this expression is that 'n' cannot be a negative number.

In this second example I am going to let 'n' equal to a positive integer and the constants a and b equal zero:

Let  $a = 0$ ,  $b = 0$ ,  $n = 3$

$$M^3 = (0X)^3 + (0Y)^3$$

$$M^3 = \left( 0 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) + \left( 0 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The limitation for the expression  $M^n = (aX)^n + (bY)^n$  is that 'n' cannot contain a negative exponent nor a decimal value or a fraction because if we multiply an exponent raised to a negative number it would make the value flip. However, both of the constants 'a' and 'b' can equal to any set of real numbers. Therefore the limitations and scope are:

$$n \in \mathbb{Z}^+$$

$$a \text{ \& } b \in \mathbb{R}.$$

To conclude this assignment, last I will explain how I arrived at my general statement using an algebraic method.

John Vu  
000948-111  
8

$$\text{Let } n = 2, a = a, b = b, X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$(aX)^n + (bY)^n = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$$

$$a^2 \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + b^2 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$$

$$\begin{bmatrix} 2a^2 & 2a^2 + 2b^2 & -2b^2 \\ 2a^2 & 2a^2 - 2b^2 & 2b^2 \end{bmatrix} = \begin{bmatrix} (a+b)(a+b) + (a-b)(a-b) & (a+b)(a-b) + (a-b)(a+b) \\ (a-b)(a+b) + (a+b)(a-b) & (a-b)(a-b) + (a+b)(a+b) \end{bmatrix}$$

$$\begin{bmatrix} 2a^2 + 2b^2 & 2a^2 - 2b^2 \\ 2a^2 - 2b^2 & 2a^2 + 2b^2 \end{bmatrix} = \begin{bmatrix} (a^2 + 2ab + b^2) + (a^2 + 2ab + b^2) & (a^2 - b^2) + (a^2 - b^2) \\ (a^2 - b^2) + (a^2 - b^2) & (a^2 + 2ab + b^2) + (a^2 + 2ab + b^2) \end{bmatrix}$$

$$\begin{bmatrix} 2a^2 + 2b^2 & 2a^2 - 2b^2 \\ 2a^2 - 2b^2 & 2a^2 + 2b^2 \end{bmatrix} = \begin{bmatrix} 2a^2 + 2b^2 & 2a^2 - 2b^2 \\ 2a^2 - 2b^2 & 2a^2 + 2b^2 \end{bmatrix}$$

$$\text{Therefore the equation } M^n = (aX)^n + (bY)^n \text{ equals with } M^n = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$$

Arriving at this general statement, it can help solve for the value of M in any equation.



John Vu  
000948-111  
9