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Math SL Matrix Binomials Portfolio

This portfolio will investigate the properties of matrix binomials in order to determine a general statement for M^n where n is a real number and an integer, and M is the 2×2 matrix $\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$

Let:

$X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\qquad$ $Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$X^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $\qquad$ $X^3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

$X^4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$

$Y^2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ $\qquad$ $Y^3 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$

$Y^4 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$

$(X+Y) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$(X+Y)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ $\qquad$ $(X+Y)^3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$

$(X+Y)^4 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$

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Expressions for X^n, Y^n and $(X+Y)^n$

$X^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$ $\quad Y^n = \begin{bmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{bmatrix}$ $\quad (X+Y)^n = \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}$

$n > 0$,

let: W = any 2x2 matrix, $\begin{bmatrix} w & w \\ w & w \end{bmatrix}$

$W^{-n} = {}^{1}/_{W^n}$ → It is not possible to divide an integer by a matrix, $n < 0$ does not exist

$n \neq 0$

For any matrix where n=0 $W^n = I$ $\quad W^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A=aX → $a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$

B=bX → $b\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}$

Let: a = 3 and b = 4

$A^2 = \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix} \rightarrow \begin{bmatrix} 2\times 3^2 & 2\times 3^2 \\ 2\times 3^2 & 2\times 3^2 \end{bmatrix} = \begin{bmatrix} 18 & 18 \\ 18 & 18 \end{bmatrix}$

$A^3 = \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} a^2 & a^2 \\ a^2 & a^2 \end{bmatrix} = \begin{bmatrix} 2a^3 & 2a^3 \\ 2a^3 & 2a^3 \end{bmatrix} \rightarrow \begin{bmatrix} 2\times 3^3 & 2\times 3^3 \\ 2\times 3^3 & 2\times 3^3 \end{bmatrix} = \begin{bmatrix} 54 & 54 \\ 54 & 54 \end{bmatrix}$

$A^4 = \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} a^3 & a^3 \\ a^3 & a^3 \end{bmatrix} = \begin{bmatrix} 2a^4 & 2a^4 \\ 2a^4 & 2a^4 \end{bmatrix} \rightarrow \begin{bmatrix} 2\times 3^4 & 2\times 3^4 \\ 2\times 3^4 & 2\times 3^4 \end{bmatrix} = \begin{bmatrix} 162 & 162 \\ 162 & 162 \end{bmatrix}$

$B^2 = \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}\begin{bmatrix} b & -b \\ -b & b \end{bmatrix} = \begin{bmatrix} 2b^2 & 2(-b)^2 \\ 2(-b)^2 & 2b^2 \end{bmatrix} \rightarrow \begin{bmatrix} 2\times 4^2 & 2(-4)^2 \\ 2(-4)^2 & 2\times 4^2 \end{bmatrix} = \begin{bmatrix} 32 & 32 \\ 32 & 32 \end{bmatrix}$

$B^3 = \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}\begin{bmatrix} b^2 & (-b)^2 \\ (-b)^2 & b^2 \end{bmatrix} = \begin{bmatrix} 2b^3 & 2(-b)^3 \\ 2(-b)^3 & 2b^3 \end{bmatrix} \rightarrow \begin{bmatrix} 2\times 4^3 & 2(-4)^3 \\ 2(-4)^3 & 2\times 4^3 \end{bmatrix} = \begin{bmatrix} 128 & 128 \\ 128 & 128 \end{bmatrix}$

$B^4 = \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}\begin{bmatrix} b^3 & (-b)^3 \\ (-b)^3 & b^3 \end{bmatrix} = \begin{bmatrix} 2b^4 & 2(-b)^4 \\ 2(-b)^4 & 2b^4 \end{bmatrix} \rightarrow \begin{bmatrix} 2\times 4^4 & 2(-4)^4 \\ 2(-4)^4 & 2\times 4^4 \end{bmatrix} = \begin{bmatrix} 512 & 512 \\ 512 & 512 \end{bmatrix}$

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Therefore:

(A+B) = $\begin{bmatrix} a & a \\ a & a \end{bmatrix} + \begin{bmatrix} b & -b \\ -b & b \end{bmatrix} = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} \rightarrow \begin{bmatrix} 3+4 & 3-4 \\ 3-4 & 3+4 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix}$

$(A+B)^2 = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$

$= \begin{bmatrix} a^2+2ab+b^2+a^2-2ab+b^2 & a^2-b^2+a^2-b^2 \\ a^2-b^2+a^2-b^2 & a^2-2ab+b^2+a^2+2ab+b^2 \end{bmatrix}$

$= \begin{bmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{bmatrix} \rightarrow \begin{bmatrix} 2(3)^2+2(4)^2 & 2(3)^2-2(4)^2 \\ 2(3)^2-2(4)^2 & 2(3)^2+2(4)^2 \end{bmatrix} = \begin{bmatrix} 50 & -14 \\ -14 & 50 \end{bmatrix}$

$(A+B)^3 = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} \begin{bmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{bmatrix} = \begin{bmatrix} 4a^3+4b^3 & 4a^3-4b^3 \\ 4a^3-4b^3 & 4a^3+4b^3 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 4(3)^3+4(4)^3 & 4(3)^3-4(4)^3 \\ 4(3)^3-4(4)^3 & 4(3)^3+4(4)^3 \end{bmatrix} = \begin{bmatrix} 364 & -148 \\ -148 & 364 \end{bmatrix}$

$(A+B)^4 = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} \begin{bmatrix} 4a^3+4b^3 & 4a^3-4b^3 \\ 4a^3-4b^3 & 4a^3+4b^3 \end{bmatrix} = \begin{bmatrix} 8a^4+8b^4 & 8a^4-8b^4 \\ 8a^4-8b^4 & 8a^4+8b^4 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 8(3)^4+8(4)^4 & 8(3)^4-8(4)^4 \\ 8(3)^4-8(4)^4 & 8(3)^4+8(4)^4 \end{bmatrix} = \begin{bmatrix} 2696 & -1400 \\ -1400 & 2696 \end{bmatrix}$

Expressions for A^n, B^n and $(A+B)^n$

$A^n = \begin{bmatrix} 2a^n & 2a^n \\ 2a^n & 2a^n \end{bmatrix}$ $\quad B^n = \begin{bmatrix} 2b^n & 2(-b)^n \\ 2(-b)^n & 2b^n \end{bmatrix}$

$(A+B)^n = \begin{bmatrix} a(2a^{n-1})+b(2b^{n-1}) & a(2a^{n-1})-b(2b^{n-1}) \\ a(2a^{n-1})-b(2b^{n-1}) & a(2a^{n-1})+b(2b^{n-1}) \end{bmatrix}$

$n > 0$,

let: W = any 2×2 matrix, $\begin{bmatrix} w & w \\ w & w \end{bmatrix}$

$W^{-n} = {}^{1}/_{W^n}$ → It is not possible to divide an integer by a matrix, $n < 0$ does not exist

$n \neq 0$

For any matrix where n=0 $W^n = I$ $\quad W^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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Let: M = $\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$, M = A+B and $M^2 = A^2+B^2$

A = aX = $a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$ B = bY = $b\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} b & -b \\ -b & b \end{bmatrix}$

A+B = $\begin{bmatrix} a & a \\ a & a \end{bmatrix} + \begin{bmatrix} b & -b \\ -b & b \end{bmatrix} = \begin{bmatrix} a+b & a+(-b) \\ a+(-b) & a+b \end{bmatrix} = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$

Therefore: <u>M = A+B</u>

M= (A+B) → $M^2 = (A+B)^2 = \begin{bmatrix} a(2a)^{2-1}+b(2b)^{2-1} & a(2a)^{2-1}-b(2b)^{2-1} \\ a(2a)^{2-1}-b(2b)^{2-1} & a(2a)^{2-1}+b(2b)^{2-1} \end{bmatrix}$

$= \begin{bmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{bmatrix}$

$A^n = \begin{bmatrix} 2a^n & 2a^n \\ 2a^n & 2a^n \end{bmatrix} \rightarrow A^2 = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix}$ $B^n = \begin{bmatrix} 2b^n & 2(-b)^n \\ 2(-b)^n & 2b^n \end{bmatrix} \rightarrow B^2 = \begin{bmatrix} 2b^2 & 2(-b)^2 \\ 2(-b)^2 & 2b^2 \end{bmatrix}$

$A^2+B^2 = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix} + \begin{bmatrix} 2b^2 & 2(-b)^2 \\ 2(-b)^2 & 2b^2 \end{bmatrix} = \begin{bmatrix} 2a^2+2b^2 & 2a^2+-2b^2 \\ 2a^2+-2b^2 & 2a^2+2b^2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{bmatrix}$

Therefore: <u>$M^2 = A^2+B^2$</u>

A= aX = $a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$, $2a^2X = 2a^2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix}$

$A^2 = \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix}$

B= bY = $b\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}$,

$2b^2Y = 2b^2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{bmatrix}$

$B^2 = \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}\begin{bmatrix} b & -b \\ -b & b \end{bmatrix} = \begin{bmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{bmatrix}$: M^2 in terms of aX and bY = $M^2 = 2a^2X + 2b^2Y$

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$M^2 = 2a^2X + 2b^2Y = a(2a)^{2-1}X + b(2b)^{2-1}Y$

$$\rightarrow \begin{bmatrix} a(2a)^1 & a(2a)^1 \\ a(2a)^1 & a(2a)^1 \end{bmatrix} + \begin{bmatrix} b(2b)^1 & -b(2b)^1 \\ -b(2b)^1 & b(2b)^1 \end{bmatrix} = \begin{bmatrix} 2a^2 + 2b^2 & 2a^2 - 2b^2 \\ 2a^2 - 2b^2 & 2a^2 + 2b^2 \end{bmatrix}$$

M^n in terms of aX and bY

$M = (A+B) \rightarrow M^n = (A+B)^n =$

$M^n = a(2a)^{n-1}X + b(2b)^{n-1}Y$, $M^n = \begin{bmatrix} a(2a)^{n-1} + b(2b)^{n-1} & a(2a)^{n-1} - b(2b)^{n-1} \\ a(2a)^{n-1} - b(2b)^{n-1} & a(2a)^{n-1} + b(2b)^{n-1} \end{bmatrix}$

$a(2a)^{n-1}X = a(2a)^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a(2a)^{n-1} & a(2a)^{n-1} \\ a(2a)^{n-1} & a(2a)^{n-1} \end{bmatrix}$

$b(2b)^{n-1}X = b(2b)^{n-1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} b(2b)^{n-1} & -b(2b)^{n-1} \\ -b(2b)^{n-1} & b(2b)^{n-1} \end{bmatrix}$

$M^n = \begin{bmatrix} a(2a)^{n-1} & a(2a)^{n-1} \\ a(2a)^{n-1} & a(2a)^{n-1} \end{bmatrix} + \begin{bmatrix} b(2b)^{n-1} & -b(2b)^{n-1} \\ -b(2b)^{n-1} & b(2b)^{n-1} \end{bmatrix}$

$$= \begin{bmatrix} a(2a)^{n-1} + b(2b)^{n-1} & a(2a)^{n-1} - b(2b)^{n-1} \\ a(2a)^{n-1} - b(2b)^{n-1} & a(2a)^{n-1} + b(2b)^{n-1} \end{bmatrix} = (A+B)^n$$

$M^2 = a(2a)^{2-1}X + b(2b)^{2-1}Y$

$$\rightarrow \begin{bmatrix} a(2a)^1 & a(2a)^1 \\ a(2a)^1 & a(2a)^1 \end{bmatrix} + \begin{bmatrix} b(2b)^1 & -b(2b)^1 \\ -b(2b)^1 & b(2b)^1 \end{bmatrix} = \begin{bmatrix} 2a^2 + 2b^2 & 2a^2 - 2b^2 \\ 2a^2 - 2b^2 & 2a^2 + 2b^2 \end{bmatrix}$$

Therefore **the** general statement for M^n: <u>$M^n = a(2a)^{n-1}X + b(2b)^{n-1}Y$</u>

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Testing the statement $\underline{M^n = a(2a)^{n-1}X + b(2b)^{n-1}Y}$ with real numbers

Let: a=2

b=3

n=4

$$M = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$$

$$M^4 = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} =$$

$$\rightarrow \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & -10 \\ -10 & 26 \end{bmatrix}\begin{bmatrix} 26 & -10 \\ -10 & 26 \end{bmatrix} = \begin{bmatrix} 776 & -520 \\ -520 & 776 \end{bmatrix}$$

$M^4 = a(2a)^3X + b(2b)^3Y$

$$\rightarrow 2(2\times 2)^3\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 3(2\times 3)^3\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 128 & 128 \\ 128 & 128 \end{bmatrix} + \begin{bmatrix} 648 & -648 \\ -648 & 648 \end{bmatrix}$$

$$= \begin{bmatrix} 776 & -520 \\ -520 & 776 \end{bmatrix}$$

General statement is valid

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Scope and limitations:

The preceding calculations determined general expressions for the defined matrices A^n, B^n, $(A+B)^n$ and M^n. The mentioned matrices however, were all 2×2 square matrices. Although the determined expressions are valid for 2×2 matrices, they may not be valid for square matrices with more columns and rows, or non-square matrices. The value of *n* must also be greater than zero. It cannot equal zero because any matrix raised to the power of zero would equal *I* the identity matrix, and it cannot be less than zero, because that would be the same as one divided by the matrix raised to the power of *n*. Since it is not possible to divide an integer by a matrix, it is not possible to raise any matrix to a negative exponent and this *n* must be greater than zero. Therefore the expressions determined in this portfolio are valid for 2×2 matrices, where the value of *n* is greater than zero