# Math Standard Internal Assessment

## Matrix Binomials

In this Math Internal Assessment we will be dealing with matrices. A matrix is a rectangular array of numbers arranged in rows and columns. Matrices are used for many things, such as; solving of systems and equations, linear programming, business inventories, Markov chains, strategies in games, economic modelling, graph theory, assignment problems, forestry and fisheries management, cubic spline interpolation, computer graphics, flight simulation, Computer Aided Tomography, Magnetic Resonance Imaging, Fractals, Chaos, Genetics, Cryptography, and the list goes on[1].

Let $\mathbf{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ Calculate $\mathbf{X}^2$, $\mathbf{X}^3$, $\mathbf{X}^4$; $\mathbf{Y}^2$, $\mathbf{Y}^3$, and $\mathbf{Y}^4$

$\mathbf{X}^2= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ $\mathbf{Y}^2= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$\mathbf{X}^3= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ $\mathbf{Y}^3= \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$

$\mathbf{X}^4= \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$ $\mathbf{Y}^4= \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}$

To find an expression for $\mathbf{X}^n$ and $\mathbf{Y}^n$ we must test other values for n. These values were calculated using a Texas Instrument TI-84 Plus Graphic Calculator

$\mathbf{X}^7 = \begin{pmatrix} 64 & 64 \\ 64 & 64 \end{pmatrix}$ $\mathbf{Y}^7 = \begin{pmatrix} 64 & -64 \\ -64 & 64 \end{pmatrix}$

$\mathbf{X}^{15} = \begin{pmatrix} 16384 & 16384 \\ 16384 & 16384 \end{pmatrix}$ $\mathbf{Y}^{15} = \begin{pmatrix} 16384 & -16384 \\ -16384 & 16384 \end{pmatrix}$

$\mathbf{X}^{20} = \begin{pmatrix} 52488 & 52488 \\ 52488 & 52488 \end{pmatrix}$ $\mathbf{Y}^{20} = \begin{pmatrix} 52488 & -52488 \\ -52488 & 52488 \end{pmatrix}$

$\mathbf{X}^{50} = \begin{pmatrix} 562949953421312 & 562949953421312 \\ 562949953421312 & 562949953421312 \end{pmatrix}$ $\mathbf{Y}^{50} = \begin{pmatrix} 562949953421312 & -562949953421312 \\ -562949953421312 & 562949953421312 \end{pmatrix}$

It should be noted how the elements $\mathbf{X}^n$ are equal to that of the result of 2 raised to a number.

$\mathbf{X}^2$ has the element 2 repeated. $2 = 2^1$

[1] (John Owen, Robert Haese, Sandra Haese, Mark Bruce 2004)

**X**$^3$ has the element 4 repeated. 4 = $2^2$
**X**$^4$ has the element 8 repeated. 8 =$2^3$
**X**$^7$ has the element 64 repeated. 64 = $2^6$
**X**$^{15}$ has the element 16384 repeated. 16384 = $2^{14}$
**X**$^{20}$ has the element 52488 repeated. 52488 = $2^{19}$
**X**$^{50}$ has the element 562949953421312 repeated. 562949953421312 =$2^{49}$
It should also be noted that whatever number **X** is raised to it is always 1 less than that of 2's exponent
The same applies for **Y**$^n$ except the elements [illegible] and c, these elements are negative. This is if matrix **Y**= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

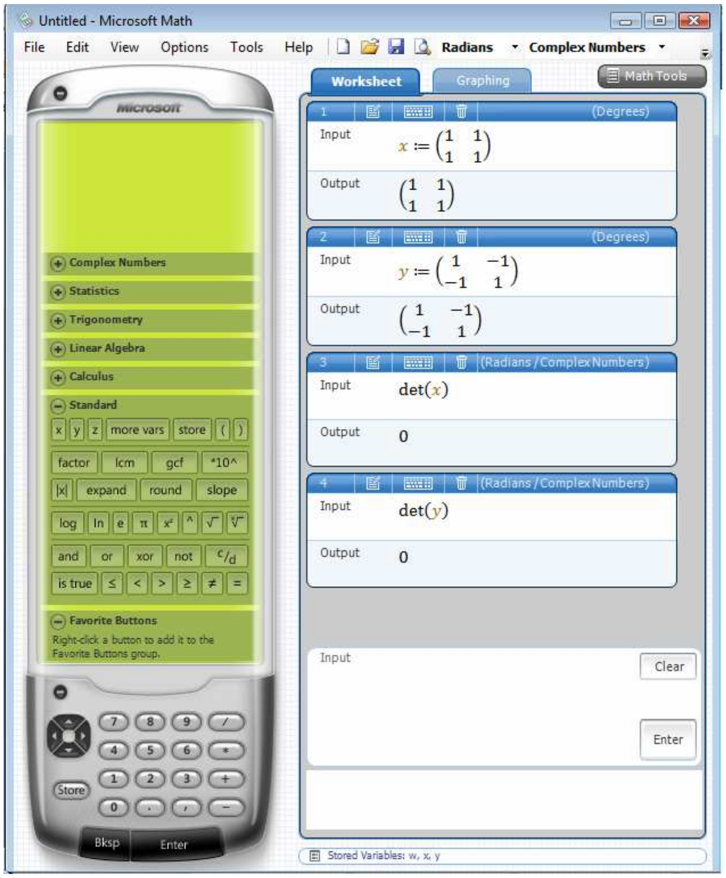
The following general statement can be made for **X**$^n$ and thus, **Y**$^n$:

**X**$^n$ = $\begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$ therefore, **Y**$^n$ = $\begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}$

To test the validity of this statement we will use different values for n:

| n | **X** | **Y** |
|---|---|---|
| 5 | $\begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} 2^4 & 2^4 \\ 2^4 & 2^4 \end{pmatrix}$ <br> $=\begin{pmatrix} 16 & 16 \\ 16 & 16 \end{pmatrix}$ √ | $\begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} 2^4 & -2^4 \\ -2^4 & 2^4 \end{pmatrix}$ <br> $=\begin{pmatrix} 16 & -16 \\ -16 & 16 \end{pmatrix}$ √ |
| □ | Undefined<br>Data type error | Undefined<br>Data type error |
| π | Undefined<br>Data type error | Undefined<br>Data type error |
| -7 | Undefined<br>Does not have an inverse | Undefined<br>Does not have an inverse |

From the results obtained from the tests outlined above we can conclude the following: **X** and **Y** do not have inverse matrices. This is because their determinant equals to 0(this was calculated using Microsoft Math)




The general statement's limitations can also be concluded with the results above; $n$ has to be a positive integer.

$n \in Z^+$

$(\mathbf{X} + \mathbf{Y}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2(I)$ where $I$ is the Identity matrix

From this we can get the following general formula

$(\mathbf{X} + \mathbf{Y})^n = (2I)^n$

To test the validity of this statement we will test out different values for $n$:

n=1

$(\mathbf{X} + \mathbf{Y})^n = (2I)^n$

$(\mathbf{X} + \mathbf{Y})^1 = (2I)^1$

$\mathbf{X} + \mathbf{Y} = (2I)$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 2 * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Or

$(\mathbf{X}+\mathbf{Y})^1 = \mathbf{X}+\mathbf{Y} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

n=2

$(\mathbf{X}+\mathbf{Y})^n = (2I)^n$

$(\mathbf{X}+\mathbf{Y})^2 = (2I)^2$

$\mathbf{X}^2 + \mathbf{Y}^2 = (2I)^2$

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = 2^2 * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

Or

$(\mathbf{X}+\mathbf{Y})^2 = (\mathbf{X}+\mathbf{Y})*(\mathbf{X}+\mathbf{Y}) = \mathbf{X}^2+\mathbf{XY}+\mathbf{YX}+\mathbf{Y}^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 4(I)$

n=3

$(\mathbf{X}+\mathbf{Y})^n = (2I)^n$

$(\mathbf{X}+\mathbf{Y})^3 = (2I)^3$

$\mathbf{X}^3 + \mathbf{Y}^3 = (2I)^3$

$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} = 2^3 * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$

Or

$(\mathbf{X}+\mathbf{Y})^3 = \mathbf{X}^3+3\mathbf{X}^2\mathbf{Y}+3\mathbf{XY}^2+\mathbf{Y}^3 =$

$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} + 3\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + 3\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} + 3 * \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} +$

$3\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = 8(I)$

Another way to approach this question is to look at how matrix binomials differ from regular binomials. Let's look at how binomial theorem works, if $(a+b)^2 = a^2+2ab+b^2$ then in that case the following should take place $(\mathbf{X}+\mathbf{Y})^2=\mathbf{X}^2+2\mathbf{XY}+\mathbf{Y}^2$. However, when matrix **X** is multiplied by matrix **Y**, their product becomes the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Anything multiplied by 0 is 0, therefore the equation will simplify to $(\mathbf{X}+\mathbf{Y})^2=\mathbf{X}^2+\mathbf{Y}^2$. The same principle can be applied to n=3, $(\mathbf{X}+\mathbf{Y})^3=\mathbf{X}^3+\mathbf{Y}^3$. It should also be noted that when matrix **X** is added to **Y** the sum is the identity matrix, $I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The results of $(\mathbf{X}+\mathbf{Y})^2$ and $(\mathbf{X}+\mathbf{Y})^3$ are all multiples of the identity matrix. We can conclude once again, $(\mathbf{X}+\mathbf{Y})^n = \mathbf{X}^n+\mathbf{Y}^n=2^{n-1}*\mathbf{X} + 2^{n-1}*\mathbf{Y} = (2I)^n$

However there are still some limitations to this general formula. Seeing that $(\mathbf{X} + \mathbf{Y})^n = \mathbf{X}^n + \mathbf{Y}^n$, we know from the general expression for $\mathbf{X}^n$ or $\mathbf{Y}^n$, that n must be a positive integer, therefore the limitation of $n$ is:

$n \in Z^+$

Let $\mathbf{A} = a\mathbf{X}$ and $\mathbf{B} = b\mathbf{Y}$, where $a$ and $b$ are constants.

$\mathbf{A} = a^*\mathbf{X}$

$\mathbf{A} = a^*\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\mathbf{A} = \begin{pmatrix} a*1 & a*1 \\ a*1 & a*1 \end{pmatrix}$

$\mathbf{A} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$

$\mathbf{B} = a^*\mathbf{Y}$

$\mathbf{B} = a^*\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$\mathbf{B} = \begin{pmatrix} a*1 & a*-1 \\ a*-1 & a*1 \end{pmatrix}$

$\mathbf{B} = \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}$

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t, j

In order to find a general formula for $\mathbf{A}^n$, $\mathbf{B}^n$, and $(\mathbf{A}+\mathbf{B})^n$, different exponents should be tried, such as the following, $\mathbf{A^2}$, $\mathbf{A^3}$, $\mathbf{A^4}$, $\mathbf{B^2}$, $\mathbf{B^3}$, and $\mathbf{B^4}$ where $a$ and $b$ are constants. The Texas Instrument TI 84 Plus calculator was used to calculate these values.

$a=3$

$\mathbf{A}^n = \mathbf{A}^2$

$\mathbf{A}^2 = a^2\mathbf{X}^2 = 3^2 * \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 9 * \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 18 & 18 \\ 18 & 18 \end{pmatrix}$

$a = \frac{1}{2}$

$\mathbf{A}^n = \mathbf{A}^3$

$\mathbf{A}^3 = a^3\mathbf{X}^3 = \left(\frac{1}{2}\right)^3 * \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \left(\frac{1}{4}\right) * \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$a = 5$

$\mathbf{A}^n = \mathbf{A}^4$

$\mathbf{A}^4 = a^4\mathbf{X}^4 = 5^4 * \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} = (625) * \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} = \begin{pmatrix} 5000 & 5000 \\ 5000 & 5000 \end{pmatrix}$

$b=3$

$\mathbf{B}^n = \mathbf{B}^2$

$\mathbf{B}^2 = b^2\mathbf{Y}^2 = 3^2 * \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = 9 * \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 18 & -18 \\ -18 & 18 \end{pmatrix}$

$b = \frac{1}{2}$

$\mathbf{B}^n = \mathbf{B}^3$

$\mathbf{B}^3 = b^3\mathbf{Y}^3 = \left(\frac{1}{2}\right)^3 * \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} = \left(\frac{1}{4}\right) * \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

b= 5

**B**$^n$ = **B**$^4$

**B**$^4$ = $\beta^4$**Y**$^4$= $5^4 * \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix} = (625) * \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix} = \begin{pmatrix} 5000 & -5000 \\ -5000 & 5000 \end{pmatrix}$

Now that more calculations have been made, we can now make a general formula. However, one should first note that when either matrix **A** or **B** is raised to the power of n both the constant $\alpha$ or $\beta$ and matrix **X** or **Y** respectively must be raised by the power of n. Using what we know about the general formula of **X**$^n$ and **Y**$^n$ and the limitation of ($n = \{Z^+\}$), we can conjure a general formula for **A**$^n$ and **B**$^n$.

**A**$^n$=$\alpha^n$ * **X** → **A**$^n$=$\alpha^n * \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$ → **A**$^n$=$\begin{pmatrix} a^n * 2^{n-1} & a^n * 2^{n-1} \\ a^n * 2^{n-1} & a^n * 2^{n-1} \end{pmatrix}$

**B**$^n$=$\beta^n$ * **Y** → **B**$^n$=$\beta^n * \begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}$ → **B**$^n$=$\begin{pmatrix} b^n * 2^{n-1} & b^n * -2^{n-1} \\ b^n * -2^{n-1} & b^n * 2^{n-1} \end{pmatrix}$

Now that **A**$^n$ and **B**$^n$ have been calculated and have been given a general formula, we can now come up with a general formula for (**A**+**B**)$^n$ , using what knowledge we've obtained from the previous problem, we now know when two composite matrices are added and then raised to a number it equals the sum of the individual matrices being raised to that number. Therefore we can conclude the following:

(**A**+**B**)$^n$=**A**$^n$ +**B**$^n$

Since the formula for **A**$^n$ and **B**$^n$ have already been formulated, we will now just plug them into the formula, from this we get:

(**A**+**B**)$^n$=**A**$^n$ +**B**$^n$=$\alpha^n$*$X^n$ +$\beta^n$*$Y^n$ = $\begin{pmatrix} a^n * 2^{n-1} & a^n * 2^{n-1} \\ a^n * 2^{n-1} & a^n * 2^{n-1} \end{pmatrix} + \begin{pmatrix} b^n * 2^{n-1} & b^n * -2^{n-1} \\ b^n * -2^{n-1} & b^n * 2^{n-1} \end{pmatrix}$

We shall now test different values for $\alpha$, $\beta$, and n. Calculations for the following problems were done using the program winmat.

$\alpha$=0, n=2

**A**$^n$ =$\alpha^n$*$X^n$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^2 = 0^2 * \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\alpha$=$\sqrt{5}$, n=8

**A**$^n$ =$\alpha^n$*$X^n$

$\begin{pmatrix} \sqrt{5} & \sqrt{5} \\ \sqrt{5} & \sqrt{5} \end{pmatrix}^8 = (\sqrt{5})^8 * \begin{pmatrix} 2^{8-1} & 2^{8-1} \\ 2^{8-1} & 2^{8-1} \end{pmatrix}$

$\begin{pmatrix} 80000 & 80000 \\ 80000 & 80000 \end{pmatrix} = \begin{pmatrix} 80000 & 80000 \\ 80000 & 80000 \end{pmatrix}$

$\alpha$=1, n=-2

**A**$^n$ =$\alpha^n$*$X^n$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{-2} = 1^{-2} * \begin{pmatrix} 2^{-2-1} & 2^{-2-1} \\ 2^{-2-1} & 2^{-2-1} \end{pmatrix}$

Can't be inversed.

$t$=10 n=8

**B** $^n$ = $t^n$ ***Y**$^n$

$\begin{pmatrix} 10 & -10 \\ -10 & 10 \end{pmatrix}^8 = (10)^8 * \begin{pmatrix} 2^{8-1} & -2^{8-1} \\ -2^{8-1} & 2^{8-1} \end{pmatrix}$

$\begin{pmatrix} 12800000000 & -12800000000 \\ -12800000000 & 12800000000 \end{pmatrix} = \begin{pmatrix} 12800000000 & -12800000000 \\ -12800000000 & 12800000000 \end{pmatrix}$

$t$=−5 n=2

**B** $^n$ = $t^n$ ***Y**$^n$

$\begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix}^2 = (-5)^2 * \begin{pmatrix} 2^{2-1} & -2^{2-1} \\ -2^{2-1} & 2^{2-1} \end{pmatrix}$

$\begin{pmatrix} 50 & -50 \\ -50 & 50 \end{pmatrix} = \begin{pmatrix} 50 & -50 \\ -50 & 50 \end{pmatrix}$

$t$=$\sqrt{3}$ n=5

**B** $^n$ = $t^n$ ***Y**$^n$

$\begin{pmatrix} \sqrt{3} & -\sqrt{3} \\ -\sqrt{3} & \sqrt{3} \end{pmatrix}^5 = (\sqrt{3})^5 * \begin{pmatrix} 2^{5-1} & -2^{5-1} \\ -2^{5-1} & 2^{5-1} \end{pmatrix}$

$\begin{pmatrix} 144\sqrt{3} & (-144)\sqrt{3} \\ (-144)\sqrt{3} & 144\sqrt{3} \end{pmatrix} = \begin{pmatrix} 144\sqrt{3} & (-144)\sqrt{3} \\ (-144)\sqrt{3} & 144\sqrt{3} \end{pmatrix}$

The examples shown above show how the constants $k$ and $t$ can take on any value and still have a solution, only if n is a positive integer. This leads us to the following statements:

$k \in Q$ , $t \in Q$ , and $n \in Z^+$

Seeing as how (**A**+**B**)$^n$= **A**$^n$+ **B**$^n$ there isn't a difference in the domain between the two formulae.

Now consider **M** = $\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$. Notice how the elements of this matrix are the sum of the elements of the matrices **A** and **B**. Explanation as follows:

**A**=$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$ and **B**=$\begin{pmatrix} b & -b \\ -b & b \end{pmatrix}$ so **A**+**B**= $\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$, therefore **M** = **A**+**B**

Now let's prove that **M**$^2$ = **A**$^2$+**B**$^2$

**A**$^2$= $k^2$**X**$^2$= $k^2$*$\begin{pmatrix} 2^{2-1} & 2^{2-1} \\ 2^{2-1} & 2^{2-1} \end{pmatrix} = \begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix}$

$$\mathbf{B}^2=\beta^2\mathbf{Y}^2=\beta^2*\begin{pmatrix} 2^{2-1} & -2^{2-1} \\ -2^{2-1} & 2^{2-1} \end{pmatrix} = \begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix}$$

$$\mathbf{A}^2+\mathbf{B}^2=\begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix}+\begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix}=\begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}=\mathbf{M}^2$$

From the results above we notice that matrices **A** and **B** are components of matrix **M**. We know this from our previous examples that when the sum of two matrices is raised to the power n, the result will equal to the sum of the individual matrices raised to the power of n. This is shown in the following example: $\mathbf{M}^2 = \mathbf{A}^2+\mathbf{B}^2$

Using evidence from previous examples and observations, we can safely conclude that:

$\mathbf{M}^n = \mathbf{A}^n+\mathbf{B}^n$

It is possible to simplify this even further using the values calculated previously for $\mathbf{A}^n$ and $\mathbf{B}^n$

$\mathbf{A}^n=\alpha^n\mathbf{X}^n=\alpha^n*\begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$ and $\mathbf{B}^n=\beta^n\mathbf{Y}^n=\beta^n*\begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}$, therefore...

$$\mathbf{M}^n=\alpha^n\mathbf{X}^n+\beta^n\mathbf{Y}^n=\alpha^n*\begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}+\beta^n*\begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}$$

We shall now test different values for $\alpha,\beta$ and n. Calculations for the following problems were done using the program Microsoft Math.

$\alpha=5, \beta=3, n=2$

$$\mathbf{M}^n=\alpha^n\mathbf{X}^n+\beta^n\mathbf{Y}^n=\alpha^n*\begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}+\beta^n*\begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}$$

$$\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}^n = a^n*\begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}+\beta^n*\begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}$$

$$\begin{pmatrix} 5+3 & 5-3 \\ 5-3 & 5+3 \end{pmatrix}^2 = 5^2*\begin{pmatrix} 2^{2-1} & 2^{2-1} \\ 2^{2-1} & 2^{2-1} \end{pmatrix}+3^2*\begin{pmatrix} 2^{2-1} & -2^{2-1} \\ -2^{2-1} & 2^{2-1} \end{pmatrix}$$

$$\begin{pmatrix} 68 & 32 \\ 32 & 68 \end{pmatrix}=\begin{pmatrix} 68 & 32 \\ 32 & 68 \end{pmatrix}$$

$\alpha=-2, \beta=4, n=0$

$$\mathbf{M}^n=\alpha^n\mathbf{X}^n+\beta^n\mathbf{Y}^n=\alpha^n*\begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}+\beta^n*\begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}$$

$$\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}^n = a^n*\begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}+\beta^n*\begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}$$

$$\begin{pmatrix}-2+4 & -2-4\\ -2-4 & -2+4\end{pmatrix}^0 = (-2)^0 * \begin{pmatrix}2^{-1} & 2^{-1}\\ 2^{-1} & 2^{-1}\end{pmatrix} + 4^2 * \begin{pmatrix}2^{-1} & -2^{-1}\\ -2^{-1} & 2^{-1}\end{pmatrix}$$

0

$a=\sqrt{4}$, $b=6$, n=1

$$M^n = aX^n + b^nY^n = a * \begin{pmatrix}2^{n-1} & 2^{n-1}\\ 2^{n-1} & 2^{n-1}\end{pmatrix} + b^n * \begin{pmatrix}2^{n-1} & -2^{n-1}\\ -2^{n-1} & 2^{n-1}\end{pmatrix}$$

$$\begin{pmatrix}a+b & a-b\\ a-b & a+b\end{pmatrix}^n = a^n * \begin{pmatrix}2^{n-1} & 2^{n-1}\\ 2^{n-1} & 2^{n-1}\end{pmatrix} + b^n * \begin{pmatrix}2^{n-1} & -2^{n-1}\\ -2^{n-1} & 2^{n-1}\end{pmatrix}$$

$$\begin{pmatrix}\sqrt{4}+6 & \sqrt{4}-6\\ \sqrt{4}-6 & \sqrt{4}+6\end{pmatrix}^1 = \sqrt{4}^1 * \begin{pmatrix}2^0 & 2^0\\ 2^0 & 2^0\end{pmatrix} + 6^1 * \begin{pmatrix}2^0 & -2^0\\ -2^0 & 2^0\end{pmatrix}$$

$$\begin{pmatrix}8 & -4\\ -4 & 8\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Now that these tests have been done, we can now state the limitations of the general formula formulated for **M<sup></sup>**$^n$. The general formula seemed to have worked for everything from negative numbers to positive number, even irrational numbers seemed to work, as well as 0. However, the formula failed to bring about results when the difference between $a$ and $b$ was less than 0. It should be noted that since the forumula **M**$^n$ = **A**$^n$+**B**$^n$ contains **A**$^n$and **B**$^n$ the same limitation exist as before where $n \epsilon Z^+$. This conclusion on limitations should be followed up with $a, b \epsilon Q$

In conclusion, by considering higher powers of the matrices of **X** and **Y**, patterns were observed and this led to the formulation of a general formula for (**X** + **Y**)$^n$ **X**$^n$ and **Y**$^n$ . These formulae were then tested for their validity by substituting several different numbers for n. It can also be concluded that he matrices **A** and **B** are actually multiples of the matrices **X** and **Y**. By testing different powers of the matrices **A** and **B** patterns were observed and this led to the formulation of a general formula for(**A** + **B**)$^n$ = **A**$^n$+ **B**$^n$. This formula was then tested for its validity by substituting different numbers for $a$, $b$, and n. Then the matrix **M** was then introduced. By proving that **M** = **A**+**B**, we were able to generate a general formula for **M**$^n$ which was expressed in the form of $a$**X** and $b$**Y**. The general formula for **M**$^n$ has the following limitations: $a$ and $b \epsilon Q$ and $n \epsilon Z^+$.