MATHEMATICS STANDARD LEVEL INTERNAL ASSESSMENT

MATRIX BINOMIALS

Summary of Investigation:

A matrix can be defined as a rectangular array of numbers of information or data that is arranged in rows and columns. There are a number of operations in which these matrices can perform (i.e., addition, multiplication, etc). In this investigation, we will identify a general statement by examining the patterns of the matrices.

Investigation:

To begin with, we are given the two matrices, X and Y.

We let X=$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$and Y=$\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$, and calculated X^2, X^3, X^4; Y^2, Y^3, Y^4

$$X^2=\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}=\begin{pmatrix}1+1 & 1+1\\1+1 & 1+1\end{pmatrix}=\begin{pmatrix}2 & 2\\2 & 2\end{pmatrix}$$

$$X^3=\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}=\begin{pmatrix}2 & 2\\2 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}=\begin{pmatrix}2+2 & 2+2\\2+2 & 2+2\end{pmatrix}=\begin{pmatrix}4 & 4\\4 & 4\end{pmatrix}$$

$$X^4=\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}=\begin{pmatrix}2 & 2\\2 & 2\end{pmatrix}\begin{pmatrix}2 & 2\\2 & 2\end{pmatrix}=\begin{pmatrix}4+4 & 4+4\\4+4 & 4+4\end{pmatrix}=\begin{pmatrix}8 & 8\\8 & 8\end{pmatrix}$$

$$Y^2=\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}=\begin{pmatrix}1+1 & -1+(-1)\\-1+(-1) & 1+1\end{pmatrix}=\begin{pmatrix}2 & -2\\-2 & 2\end{pmatrix}$$

$$Y^3=\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}=\begin{pmatrix}2 & -2\\-2 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}=\begin{pmatrix}2+2 & -2+(-2)\\-2+(-2) & 2+2\end{pmatrix}=\begin{pmatrix}4 & -4\\-4 & 4\end{pmatrix}$$

$$Y^4=\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}=\begin{pmatrix}4 & -4\\-4 & 4\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}=\begin{pmatrix}4+4 & -4+(-4)\\-4+(-4) & 4+4\end{pmatrix}$$

$$=\begin{pmatrix}8 & -8\\-8 & 8\end{pmatrix}$$

And therefore,

$$X^2=\begin{pmatrix}2 & 2\\2 & 2\end{pmatrix},\ X^3=\begin{pmatrix}4 & 4\\4 & 4\end{pmatrix},\ X^4=\begin{pmatrix}8 & 8\\8 & 8\end{pmatrix}$$

$$Y^2=\begin{pmatrix}2 & -2\\-2 & 2\end{pmatrix},\ Y^3=\begin{pmatrix}4 & -4\\-4 & 4\end{pmatrix},\ Y^4=\begin{pmatrix}8 & -8\\-8 & 8\end{pmatrix}$$

Here, it seems reasonable to suggest a pattern for the X and Y values.

And so, by considering integer powers of X and Y, we can find the expressions for X^n, Y^n:

$$X^n=\begin{pmatrix}2^{n-1} & 2^{n-1}\\2^{n-1} & 2^{n-1}\end{pmatrix},\ Y^n=\begin{pmatrix}2^{n-1} & -2^{n-1}\\-2^{n-1} & 2^{n-1}\end{pmatrix},$$

With the aforementioned expressions for the value of X^n and Y^n, we will now determine the value for $(X+Y)^n$. This can be done through substituting the value of *n* to find a pattern for the matrices, as done so when determining the value of X^n and Y^n.

$$(X+Y)^2=\left[\begin{pmatrix}1&1\\1&1\end{pmatrix}+\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\right]^2=\left[\begin{pmatrix}2&0\\0&2\end{pmatrix}\right]^2=\begin{pmatrix}2^2&0\\0&2^2\end{pmatrix}=\begin{pmatrix}4&0\\0&4\end{pmatrix}$$

$$(X+Y)^3=\left[\begin{pmatrix}1&1\\1&1\end{pmatrix}+\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\right]^3=\left[\begin{pmatrix}2&0\\0&2\end{pmatrix}\right]^3=\begin{pmatrix}4&0\\0&4\end{pmatrix}\begin{pmatrix}2&0\\0&2\end{pmatrix}=\begin{pmatrix}8&0\\0&8\end{pmatrix}$$

$$(X+Y)^4=\left[\begin{pmatrix}1&1\\1&1\end{pmatrix}+\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\right]^4=\left[\begin{pmatrix}2&0\\0&2\end{pmatrix}\right]^4=\begin{pmatrix}4&0\\0&4\end{pmatrix}\begin{pmatrix}4&0\\0&4\end{pmatrix}=\begin{pmatrix}16&0\\0&16\end{pmatrix}$$

Thus, with these patterns, the following expression can be suggested:

$$(X+Y)^n=\begin{pmatrix}2^n&0\\0&2^n\end{pmatrix}$$

The matrices X and Y can now be used to form two new matrices A and B. Here, we will use *a* and *b* as constants for the matrices A and B, respectively. And hence the following:

$$A=aX=a\begin{pmatrix}1&1\\1&1\end{pmatrix},\ B=bX=b\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$$

Now, the different values of *a and b* can be used to calculate the values of $A^2, A^3, A^4; B^2, B^3, B^4$

$$A^2=\begin{pmatrix}a&a\\a&a\end{pmatrix}\begin{pmatrix}a&a\\a&a\end{pmatrix}=\begin{pmatrix}2a^2&2a^2\\2a^2&2a^2\end{pmatrix}$$

$$A^3=\begin{pmatrix}a&a\\a&a\end{pmatrix}\begin{pmatrix}a&a\\a&a\end{pmatrix}\begin{pmatrix}a&a\\a&a\end{pmatrix}=\begin{pmatrix}2a^2&2a^2\\2a^2&2a^2\end{pmatrix}\begin{pmatrix}a&a\\a&a\end{pmatrix}=\begin{pmatrix}4a^3&4a^3\\4a^3&4a^3\end{pmatrix}$$

$$A^4=\begin{pmatrix}a&a\\a&a\end{pmatrix}\begin{pmatrix}a&a\\a&a\end{pmatrix}\begin{pmatrix}a&a\\a&a\end{pmatrix}\begin{pmatrix}a&a\\a&a\end{pmatrix}=\begin{pmatrix}2a^2&2a^2\\2a^2&2a^2\end{pmatrix}\begin{pmatrix}2a^2&2a^2\\2a^2&2a^2\end{pmatrix}=\begin{pmatrix}8a^4&8a^4\\8a^4&8a^4\end{pmatrix}$$

$$B^2=\begin{pmatrix}b&-b\\-b&b\end{pmatrix}\begin{pmatrix}b&-b\\-b&b\end{pmatrix}=\begin{pmatrix}2b^2&-2b^2\\-2b^2&2b^2\end{pmatrix}$$

$$B^3=\begin{pmatrix}b&-b\\-b&b\end{pmatrix}\begin{pmatrix}b&-b\\-b&b\end{pmatrix}\begin{pmatrix}b&-b\\-b&b\end{pmatrix}=\begin{pmatrix}2b^2&-2b^2\\-2b^2&2b^2\end{pmatrix}\begin{pmatrix}b&-b\\-b&b\end{pmatrix}=\begin{pmatrix}4b^3&-4b^3\\-4b^3&4b^3\end{pmatrix}$$

$$B^4=\begin{pmatrix}b&-b\\-b&b\end{pmatrix}\begin{pmatrix}b&-b\\-b&b\end{pmatrix}\begin{pmatrix}b&-b\\-b&b\end{pmatrix}\begin{pmatrix}b&-b\\-b&b\end{pmatrix}=\begin{pmatrix}2b^2&-2b^2\\-2b^2&2b^2\end{pmatrix}\begin{pmatrix}2b^2&-2b^2\\-2b^2&2b^2\end{pmatrix}=\begin{pmatrix}8b^4&-8b^4\\-8b^4&8b^4\end{pmatrix}$$

And therefore,

$$A^2=\begin{pmatrix}2a^2&2a^2\\2a^2&2a^2\end{pmatrix},\ A^3=\begin{pmatrix}4a^3&4a^3\\4a^3&4a^3\end{pmatrix},\ A^4=\begin{pmatrix}8a^4&8a^4\\8a^4&8a^4\end{pmatrix}$$

$$B^2=\begin{pmatrix}2b^2&-2b^2\\-2b^2&2b^2\end{pmatrix},\ B^3=\begin{pmatrix}4b^3&-4b^3\\-4b^3&4b^3\end{pmatrix},\ B^4=\begin{pmatrix}8b^4&-8b^4\\-8b^4&8b^4\end{pmatrix}$$

With the patterns from these matrices, we can determine the expressions for matrices A and B by considering its integer powers:

$$A^n=\begin{pmatrix} 2^{n-1}a^n & 2^{n-1}a^n \\ 2^{n-1}a^n & 2^{n-1}a^n \end{pmatrix}=2^{n-1}a^nX$$

$$B^n=\begin{pmatrix} 2^{n-1}b^n & -2^{n-1}b^n \\ -2^{n-1}b^n & 2^{n-1}b^n \end{pmatrix}=2^{n-1}b^nY$$

We will now investigate a new matrix, $M=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$

Here, we will prove how M = A + B and that $M^2 = A^2 + B^2$

$$A=\begin{pmatrix} a & a \\ a & a \end{pmatrix},\ B=\begin{pmatrix} b & -b \\ -b & b \end{pmatrix}$$

$$A+B=aX+bY=a\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}+b\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}=\begin{pmatrix} a & a \\ a & a \end{pmatrix}+\begin{pmatrix} b & -b \\ -b & b \end{pmatrix}=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$$

Using the Transitive Property (which states that if *x=y* and *y=z*, then x=z), we know that M=A+B.

$$M=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix} \text{and } A+B=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}, \text{ therefore } M=A+B$$

And to prove that $M^2 = A^2 + B^2$, the following can be done:

$$M^2=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$$

$$=\begin{pmatrix} (a+b)^2+(a-b)^2 & (a+b)(a-b)+(a-b)(a+b) \\ (a-b)(a+b)+(a+b)(a-b) & (a-b)^2+(a+b)^2 \end{pmatrix}$$

$$=\begin{pmatrix} a^2+2ab+b^2+a^2-2ab+b^2 & a^2-b^2+a^2+b^2 \\ a^2-b^2+a^2-b^2 & a^2-2ab+b^2+a^2+2ab+b^2 \end{pmatrix}$$

$$=\begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}$$

$$A^2=\begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} a & a \\ a & a \end{pmatrix}=\begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix}$$

$$B^2=\begin{pmatrix} b & -b \\ -b & b \end{pmatrix}\begin{pmatrix} b & -b \\ -b & b \end{pmatrix}=\begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix}$$

$$A^2 + B^2 = \begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix} + \begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix} = \begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}$$

The Transitive Property can be applied here as well. Thus, we know that $M^2=A^2+B^2$

$$A^2+B^2=\begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix} \text{and } M=\begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}, \text{ therefore } M^2=A^2+B^2$$

Here, the general statement that expresses M^n in terms of aX and bY can be suggested:

$$M^n = (aX)^n + (bY)^n$$

In order to test the validity of this general statement, we substituted different values for variables *a, b* and *n*. We substituted values of *a and b* with positive and negative integers, fractions and zero.

For example,

Let a=1, b=2 and n=3 for the value of M^n

$$M^3=\begin{pmatrix} 1+2 & 1-2 \\ 1-2 & 1+2 \end{pmatrix}^3 = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 36 & -28 \\ -28 & 36 \end{pmatrix}$$

Using the value for attained for M^3, the general statement was proved:

$$M^3=\begin{pmatrix} 2^2\times1^3 & 2^2\times1^3 \\ 2^2\times1^3 & 2^2\times1^3 \end{pmatrix} + \begin{pmatrix} 2^2\times2^3 & -(2^2)\times2^3 \\ -(2^2)\times2^3 & 2^2\times2^3 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 32 & -32 \\ -32 & 32 \end{pmatrix} = \begin{pmatrix} 36 & -28 \\ -28 & 36 \end{pmatrix}$$

The two values attained when calculating the value of M^3 through multiplying the matrices and through using the general statement proved to be the same. Therefore, we know that when the numbers of the matrices are positive integers, the general statement is valid.

Now, let *a=-1*, *b=-2* and *n=3* (*n* cannot be a negative value, for the negative value in a matrices does not exist) for the value of M^n

$$M^3=\begin{pmatrix} -1+(-2) & -1-(-2) \\ -1-(-2) & -1+(-2) \end{pmatrix}^3 = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}^3 = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} -8 & -6 \\ -6 & -8 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} -36 & 28 \\ 28 & -36 \end{pmatrix}$$

Using the value attained for M^3, we can prove the general statement:

$$M =\begin{pmatrix} 2^2\times(-1)^3 & 2^2\times(-1)^3 \\ 2^2\times(-1)^3 & 2^2\times(-1)^3 \end{pmatrix} + \begin{pmatrix} 2^2\times(-2)^3 & -(2^2)\times(-2)^3 \\ -(2^2)\times(-2)^3 & 2^2\times(-2)^3 \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix} + \begin{pmatrix} -32 & 32 \\ 32 & -32 \end{pmatrix} = \begin{pmatrix} -36 & 28 \\ 28 & -36 \end{pmatrix}$$

Again, the two values attained when calculating the value of M^3 through multiplying the matrices and through using the general statement were the same. Therefore, we know that when the numbers of the matrices are negative integers, the general statement is valid.

And lastly, let a=0, b= $\frac{1}{2}$ and n=3for the value of M^n

$$M^3=\begin{pmatrix}0+\frac{1}{2} & 0-\frac{1}{2}\\ 0-\frac{1}{2} & 0+\frac{1}{2}\end{pmatrix}^3=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

And using the value attained for the above, we can prove the general statement:

$$M^3=\begin{pmatrix}2^2\times 0^3 & 2^2\times 0^3\\ 2^2\times 0^3 & 2^2\times 0^3\end{pmatrix}+\begin{pmatrix}2^2\times\left(\frac{1}{2}\right)^3 & -(2^2)\times\left(\frac{1}{2}\right)^3\\ -(2^2)\times\left(\frac{1}{2}\right)^3 & 2^2\times\left(\frac{1}{2}\right)^3\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

Again, the two values attained are the same. So it seems reasonable to suggest that even when the value of the matrices is a zero or fraction, the general statement can still be proven.

Scope and Limitations:

Although the general statement would work for most matrices, there are still scope and limitations that must be looked into. For instance, for the scope in this investigation, we assumed the value of *a* and *b* to be all rational numbers, for the reason that irrational numbers could arise uncertainties when validating the general statement. In other words, the values of the matrices would become too complex to validate. On the other hand, we limited all values of *n* to be natural numbers for this investigation. This is because in matrices, not all exponents may be valid in a matrix. For instance, negative values for *n* were not included because it lacked its purpose when in a matrices. Furthermore, although a matrix may be raised to the power of -1, it does not identify an exponent; but rather, the inverse of the matrix. In this case, if the matrix is multiplied with another matrix, the value will still be equivalent to the original matrix. Therefore, it seems reasonable to suggest that the general statement can be applied when the determinant is not equal to zero. However, because there may be possible abnormalities, such as the identity matrix, there seemed to be a limit when investigating the general statement.

The Algebraic Method:

Lastly, we will investigate the use of an algebraic method to explain how the general statement was reached.

To begin with, we let A=aX and B=bY, where X=$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ and Y=$\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

$$A=a\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}a & a\\ a & a\end{pmatrix},\ B=b\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}=\begin{pmatrix}b & -b\\ -b & b\end{pmatrix}$$

Now, we let $M=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$, and A=$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$ and B= $\begin{pmatrix} b & -b \\ -b & b \end{pmatrix}$

$$A+B=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix},$$

$$M=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix} \text{and } A+B=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$$

And therefore,

M=A+B

And, if $M^n = (A+B)^n$,

$$(A+B)^n=\begin{pmatrix} |2^{n-1}|\,|a^n+b^n| & |2^{n-1}|\,|a^n-b^n| \\ |2^{n-1}|\,|a^n-b^n| & |2^{n-1}|\,|a^n+b^n| \end{pmatrix},$$

$$\begin{pmatrix} |2^{n-1}|\,|a^n+b^n| & |2^{n-1}|\,|a^n-b^n| \\ |2^{n-1}|\,|a^n-b^n| & |2^{n-1}|\,|a^n+b^n| \end{pmatrix}=\begin{pmatrix} 2^{n-1}a^n & 2^{n-1}a^n \\ 2^{n-1}a^n & 2^{n-1}a^n \end{pmatrix}+\begin{pmatrix} 2^{n-1}b^n & -(2^{n-1})b^n \\ -(2^{n-1})b^n & 2^{n-1}b^n \end{pmatrix}$$

And given that,

$$A^n=\begin{pmatrix} 2^{n-1}\cdot a^n & 2^{n-1}\cdot a^n \\ 2^{n-1}\cdot a^n & 2^{n-1}\cdot a^n \end{pmatrix} \text{and } B^n=\begin{pmatrix} 2^{n-1}\cdot b^n & -|2^{n-1}|\cdot b^n \\ -|2^{n-1}|\cdot b^n & 2^{n-1}\cdot b^n \end{pmatrix}$$

It seems reasonable to suggest the general statements,

$(A+B)^n = A^n + B^n$

$M^n = A^n + B^n = (aX)^n+(bY)^n=a^nX^n + b^nY^n$