Maths Portfolio Standard Level

International Baccalaureate

Matrix Binomials

The main aim of this portfolio is to investigate the matrix binomials and observe and determine a general expression from the patterns that we obtain through the workings. Throughout the project, I shall be using solely matrices of 2 x 2 formations, and investigate the patterns I find.

1. To begin with, we consider the matrices **X =** $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and **Y** = $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

The values of these matrices, each raised to the power of 2, 3 and 4 are calculated, as shown below;

$\mathbf{X}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ $\mathbf{Y}^2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$\mathbf{X}^3 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ and $\mathbf{Y}^3 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$

$\mathbf{X}^4 = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$ $\mathbf{Y}^4 = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}$

It can be observed that all the matrices calculated above are in the form of 2 x 2, they are all square matrices. The corresponding diagonal elements are also observed to be the same. Since the matrices of each n^{th} power can be seen to be the value of 1 less than the n^{th} term, the general expression for the matrix $\mathbf{X}^n$ in terms of n is -

$$\mathbf{X}^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$$

And the general expression for $\mathbf{Y}^n$ is –

$$\mathbf{Y}^n = \begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}$$

Likewise, the values of the matrix **(X + Y),** raised to the power 2, 3, and 4 is calculated to find its general expression.

The matrix: **(X + Y)** = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

So, $(\mathbf{X} + \mathbf{Y})^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

$(\mathbf{X} + \mathbf{Y})^3 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$

$(\mathbf{X} + \mathbf{Y})^4 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$

And from the above, we can infer that the general expression for **(X + Y)** ^{n} is as follows,

$$(\mathbf{X}+\mathbf{Y})^n = \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$$

Proof:

Taking n as 3, the value is substituted in the above expression –

$$(\mathbf{X}+\mathbf{Y})^3 = \begin{pmatrix} 2^3 & 0 \\ 0 & 2^3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

So since **(X + Y)** ^{3} is equal to $\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ as calculated previously, therefore the general expression **(X + Y)** ^{n} = $\begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$ is true and consistent with all values of n.

2. Consider ***A*** = a***X*** and ***B*** = b***Y*** where a and b are constants. In order to find the general expressions of **A**^{n}, **B**^{n} and **(A + B)**^{n}, different values of a and b are used to calculate **A**^{2}, **A**^{3}, **A**^{4} and **B**^{2}, **B**^{3}, **B**^{4} below.

Let a = 2 and b = -2

A = a**X** **B** = b**Y**

$$\mathbf{A} = 2X = 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \qquad \mathbf{B} = -2Y = -2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$\mathbf{A}^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}^2 = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} \qquad \mathbf{B}^2 = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}^2 = \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}$$

$$\mathbf{A}^3 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}^3 = \begin{pmatrix} 32 & 32 \\ 32 & 32 \end{pmatrix} \qquad \mathbf{B}^3 = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}^3 = \begin{pmatrix} -32 & 32 \\ 32 & -32 \end{pmatrix}$$

$$\mathbf{A}^4 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}^4 = \begin{pmatrix} 128 & 128 \\ 128 & 128 \end{pmatrix} \qquad \mathbf{B}^4 = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}^4 = \begin{pmatrix} 128 & -128 \\ -128 & 128 \end{pmatrix}$$

<u>Note</u>: A GCD calculator (TI 83) has been used throughout this portfolio to calculate the matrices and other calculations.

Observing the above calculations, we can detect a certain pattern in determining the values, which gives us the general expression of A and B in terms of n as;

A^n = $\begin{pmatrix} 2^{n-1} \times a^n & 2^{n-1} \times a^n \\ 2^{n-1} \times a^n & 2^{n-1} \times a^n \end{pmatrix}$ and **B**^{n} = $\begin{pmatrix} 2^{n-1} \times a^n & -2^{n-1} \times a^n \\ -2^{n-1} \times a^n & 2^{n-1} \times a^n \end{pmatrix}$

Proof:

Taking n to be 4, we substitute the values in the expression of **A**^{n} –

A^4 = $\begin{pmatrix} 2^{4-1} \times 2^4 & 2^{4-1} \times 2^4 \\ 2^{4-1} \times 2^4 & 2^{4-1} \times 2^4 \end{pmatrix}$

A^4 = $\begin{pmatrix} 2^3 \times 2^4 & 2^3 \times 2^4 \\ 2^3 \times 2^4 & 2^3 \times 2^4 \end{pmatrix}$

A^4 = $\begin{pmatrix} 8 \times 16 & 8 \times 16 \\ 8 \times 16 & 8 \times 16 \end{pmatrix}$

A^4 = $\begin{pmatrix} 128 & 128 \\ 128 & 128 \end{pmatrix}$

And since **A**^{4} = $\begin{pmatrix} 128 & 128 \\ 128 & 128 \end{pmatrix}$ is the correct value as calculated previously, this expression is proven true and consistent.

Likewise, to find the general expression of **(A + B)** ^{n}, the values of **(A + B)** raised to the powers 2,3 and 4 are calculated;

First, we find the value of **(A + B)** –

(A + B) = $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$

(A + B) ^{2} = $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}^2 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$

(A + B) ^{3} = $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 64 \\ 64 & 0 \end{pmatrix}$

(A + B) ^{4} = $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}^4 = \begin{pmatrix} 256 & 0 \\ 0 & 256 \end{pmatrix}$

Observing the repeating patterns in the calculations above, we can deduce the general expression of **(A+B)** in terms of n to be;

$$\textbf{(A + B)}^{n} = 2^{n-1}\begin{pmatrix} a^n + b^n & a^n - b^n \\ a^n - b^n & a^n + b^n \end{pmatrix}$$

Proof:

Taking n as 3, and substituting it in the above expression –

(A + B) $^3 = 2^{3-1}\begin{pmatrix} 2^3 + (-2)^3 & 2^3 - (-2)^3 \\ 2^3 - (-2)^3 & 2^3 + (-2)^3 \end{pmatrix}$

(A + B) $^3 = 2^2\begin{pmatrix} 8-8 & 8+8 \\ 8+8 & 8-8 \end{pmatrix}$

(A + B) $^3 = 4\begin{pmatrix} 0 & 16 \\ 16 & 0 \end{pmatrix}$

(A + B) $^3 = \begin{pmatrix} 0 & 64 \\ 64 & 0 \end{pmatrix}$

So since **(A + B)** ^{3} is equal to $\begin{pmatrix} 0 & 64 \\ 64 & 0 \end{pmatrix}$ as calculated previously, it is proven that this general expression for **(A + B)** ^{n} is true and consistent with all values of n.

3. Consider **M** = $\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$, to prove that **M** = **A**+**B** and **M**^{2} = **A**^{2}+**B**^{2} , we must first calculate the value of M –

As a = 2, and b = -2,

M = $\begin{pmatrix} 2+(-2) & 2-(-2) \\ 2-(-2) & 2+(-2) \end{pmatrix}$

M = $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$

To prove that M = A+B, we already know from previous workings that **A** = $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and **B** = $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$-

So,

M = A+B

M= $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$

M = $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$

We know that the value of **M** is $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$ from the previous calculation, therefore **M=A+B** is true.

And likewise, to prove that **M**^{2} = **A**^{2}+**B**^{2} ,

M^2 = $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}^2 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$ and,

A^2+**B**^{2} = **(A+B)** ^{2} = **A**^{2}+**B**^{2} + 2**AB** -------- (a/c to Algebraic Expression)

So since; **A**^{2} = $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}^2 = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$

B^2 = $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}^2 = \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}$

2**AB** = 2 $\left[\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \times \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\right]$

= 2 $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

= $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Thus, **M**^{2} = **A**^{2}+**B**^{2} = **A**^{2}+**B**^{2} + 2**AB** would be –

M^2 **=** $\begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} + \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$$

And since **M**^{2} = $\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$ is the right value, it is proved that **M**^{2} = **A**^{2}+**B**^{2} is true.

4. Now, in order to find the general statement expressing **M**^{n} in terms of a and b, we first calculate the value of **M**^{n} where n = 1, 2, 3 and 4.

M = $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$

M^2 = $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}^2 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$

M^3 = $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 64 \\ 64 & 0 \end{pmatrix}$

M^4 = $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}^4 = \begin{pmatrix} 256 & 0 \\ 0 & 256 \end{pmatrix}$

Using the expression: **(A + B)** $^n = 2^{n-1} \begin{pmatrix} a^n + b^n & a^n - b^n \\ a^n - b^n & a^n + b^n \end{pmatrix}$ whereby a=2, b=-2, and taking n=3, we calculate **(A+B)** raised to the third power –

(A+B)^3 = $2^{3-1} \begin{pmatrix} 2^3 + (-2)^3 & 2^3 - (-2)^3 \\ 2^3 - (-2)^3 & 2^3 + (-2)^3 \end{pmatrix}$

$$= 2^2 \begin{pmatrix} 0 & 16 \\ 16 & 0 \end{pmatrix}$$

$$= 4 \begin{pmatrix} 0 & 16 \\ 16 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 64 \\ 64 & 0 \end{pmatrix}$$

So since we know from previous calculations that **M**^{3} = $\begin{pmatrix} 0 & 64 \\ 64 & 0 \end{pmatrix}$, we can say that , **M**^{3} = (**A**+**B**)^3 .

Therefore, the general statement of **M**^{n} in terms of a and b is;

$$\mathbf{M}^n = (a\mathbf{X}+b\mathbf{Y})^n$$

<u>Proof:</u>

To check the validity of this general statement, we shall take different values for a, b and n. Suppose $a = 3$, $b = 4$, and $n = 2$ –

$\mathbf{M}^2 = (3\mathbf{X}+4\mathbf{Y})^2$

$$\mathbf{M}^2 = \left[3\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + 4\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right]^2$$

$$\mathbf{M}^2 = \left[\begin{pmatrix}3 & 3\\ 3 & 3\end{pmatrix} + \begin{pmatrix}4 & -4\\ -4 & 4\end{pmatrix}\right]^2$$

$$\mathbf{M}^2 = \begin{pmatrix}7 & -1\\ -1 & 7\end{pmatrix}^2$$

$$\mathbf{M}^2 = \begin{pmatrix}50 & -14\\ -14 & 50\end{pmatrix}$$

And since $\mathbf{M} = (\mathbf{A}+\mathbf{B}) = (a\mathbf{X}+b\mathbf{Y})$,

$$\mathbf{M} = \begin{pmatrix}3 & 3\\ 3 & 3\end{pmatrix} + \begin{pmatrix}4 & -4\\ -4 & 4\end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix}7 & -1\\ -1 & 7\end{pmatrix}$$

So then,

$$\mathbf{M}^2 = \begin{pmatrix}7 & -1\\ -1 & 7\end{pmatrix}^2 = \begin{pmatrix}50 & -14\\ -14 & 50\end{pmatrix}$$

Therefore, the statement is proven true and consistent with all values of a, b and n.

5. Using the Algebraic method, the general statement is to be verified and explained again.

Taking the expression, $\mathbf{M}^n = 2^{n-1}\begin{pmatrix}a^n + b^n & a^n - b^n\\ a^n - b^n & a^n + b^n\end{pmatrix}$, we find:

M = $\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$

M^2 = $\left[\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}\right]$ = $2\begin{pmatrix} a^2+b^2 & a^2-b^2 \\ a^2-b^2 & a^2+b^2 \end{pmatrix}$

M^3 = $\left[2\begin{pmatrix} a^2+b^2 & a^2-b^2 \\ a^2-b^2 & a^2+b^2 \end{pmatrix}\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}\right]$

$$= 2\left[\begin{pmatrix} (a^2+b^2)(a+b)+(a^2-b^2)(a-b) & (a^2+b^2)(a-b)+(a^2-b^2)(a+b) \\ (a^2-b^2)(a+b)+(a^2+b^2)(a-b) & (a^2-b^2)(a-b)+(a^2+b^2)(a+b) \end{pmatrix}\right]$$

$$= 2\begin{bmatrix} (a^3+a^2b+ab^2+b^3+a^3-a^2b-ab^2+b^3) & (a^3-a^2b+ab^2-b^3+a^3+a^2b-ab^2-b^3) \\ (a^3+a^2b-ab^2-b^3+a^3-a^2b+ab^2-b^3) & (a^3-a^2b-ab^2+b^3+a^3+a^2b+ab^2+b^3) \end{bmatrix}$$

$$= 2\begin{pmatrix} 2a^3+2b^3 & 2a^3-2b^3 \\ 2a^3-2b^3 & 2a^3+2b^3 \end{pmatrix}$$

$$=4\begin{pmatrix} a^3+b^3 & a^3-b^3 \\ a^3-b^3 & a^3+b^3 \end{pmatrix}$$

<u>Proof</u>:

Substituting the above with the initial values of a and b we find **M**^{3} -

M^3= $4\begin{pmatrix} a^3+b^3 & a^3-b^3 \\ a^3-b^3 & a^3+b^3 \end{pmatrix}$

M^3 = $4\begin{pmatrix} 2^3+(-2)^3 & 2^3-(-2)^3 \\ 2^3-(-2)^3 & 2^3+(-2)^3 \end{pmatrix}$

M^3 = $4\begin{pmatrix} 8-8 & 8+8 \\ 8+8 & 8-8 \end{pmatrix}$

M^3 = $4\begin{pmatrix} 8-8 & 8+8 \\ 8+8 & 8-8 \end{pmatrix}$

M^3 = $4\begin{pmatrix} 0 & 16 \\ 16 & 0 \end{pmatrix}$

$\mathbf{M}^3 = \begin{pmatrix} 0 & 64 \\ 64 & 0 \end{pmatrix}$

Therefore, since it has been shown earlier in our work that the value $\mathbf{M}^3 = \begin{pmatrix} 0 & 64 \\ 64 & 0 \end{pmatrix}$ is true and correct, it shows that the general statement of $\mathbf{M}^n$ in terms of a$\mathbf{X}$ and b$\mathbf{Y}$ is true.