A **matrix** is a rectangular array of numbers (or letters) arranged in rows and columns. These numbers (or letters) are known as **entries**. Entries can be added and multiplied, but also squared. The aim of this portfolio is to investigate squaring matrices.

When we square the matrix $\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ what we receive is a) $\boldsymbol{M}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.

Calculating the matrices for $n = 3, 4, 5, 10, 20$ and $50$:

$$\boldsymbol{M}^3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

$$\boldsymbol{M}^4 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^4 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$$

$$\boldsymbol{M}^5 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^5 = \begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix}$$

$$\boldsymbol{M}^{10} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{10} = \begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix}$$

$$\boldsymbol{M}^{20} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{20} = \begin{pmatrix} 1048576 & 0 \\ 0 & 1048576 \end{pmatrix}$$

$$\boldsymbol{M}^{50} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{50} = \begin{pmatrix} 1.125899907 \times 10^{15} & 0 \\ 0 & 1.125899907 \times 10^{15} \end{pmatrix}$$

Examples shown above clearly indicate that while zero entries remain the same, non-zero entries change. Each entry is raised to a given power separately. Raising them to any power does not change the zero-entries.

Hence, we can observe that the general expression for the matrix $\boldsymbol{M}^n = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}^n = \begin{pmatrix} k^n & 0 \\ 0 & k^n \end{pmatrix}$

When considering the matrices $\boldsymbol{P} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $\boldsymbol{S} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$, more conclusions can be drawn.

Raising each of them to the second power gives:

$$\boldsymbol{P}^2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^2 = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} = 2\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

$$\boldsymbol{S}^2 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^2 = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix} = 2\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$$

These matrices are simplified to $2\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$ and $2\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$ to make it easier to notice any patterns. Calculating $\mathbf{P}^n$ and $\mathbf{S}^n$ for other values of $n$ we obtain:

$$P^3 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^3 = \begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix} = 2\begin{pmatrix} 18 & 14 \\ 14 & 18 \end{pmatrix} = 4\begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix}$$

$$S^3 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^3 = \begin{pmatrix} 112 & 104 \\ 104 & 112 \end{pmatrix} = 2\begin{pmatrix} 56 & 52 \\ 52 & 56 \end{pmatrix} = 4\begin{pmatrix} 28 & 26 \\ 26 & 28 \end{pmatrix}$$

$$P^5 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^5 = \begin{pmatrix} 528 & 496 \\ 496 & 528 \end{pmatrix} = 2\begin{pmatrix} 264 & 248 \\ 248 & 264 \end{pmatrix} = 16\begin{pmatrix} 33 & 31 \\ 31 & 33 \end{pmatrix}$$

$$S^5 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^5 = \begin{pmatrix} 3904 & 3872 \\ 3872 & 3904 \end{pmatrix} = 2\begin{pmatrix} 1952 & 1936 \\ 1936 & 1952 \end{pmatrix} = 16\begin{pmatrix} 244 & 242 \\ 242 & 244 \end{pmatrix}$$

$$P^{10} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{10} = \begin{pmatrix} 524800 & 523776 \\ 523776 & 524800 \end{pmatrix} = 512\begin{pmatrix} 1025 & 1023 \\ 1023 & 1025 \end{pmatrix}$$

$$S^{10} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^{10} = \begin{pmatrix} 30233600 & 30232576 \\ 30232576 & 30233600 \end{pmatrix} = 512\begin{pmatrix} 59050 & 59048 \\ 59048 & 59050 \end{pmatrix}$$

Analysis of above pattern shows that:

a) dividing the first result by 2 raised to the power 1 less than the number to which the matrix is raised (e.g. $P^{10}$; number 2 will be raised to 10 – 1 = 9 power, which gives 512), results in a new matrix with two pairs of similar entries situated diagonally;

b) the difference between two entries in one row or one column is equal 2, therefore we can either add or subtract 1 from their mean to get all entries of one matrix.

Matrix $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$ is the general expression for the pattern described above. Matrices ***M, P*** and ***S*** in this portfolio can be used as examples, where $n$ = 1, 2 and 3.

Summarising:

a) $k$ is the arithmetic mean of two entries found in one row or column

b) $k$ can be both odd or even

c) $k$ does not have to be an integer number

$$A^4 = \begin{pmatrix} 5.5 & 3.5 \\ 3.5 & 5.5 \end{pmatrix}^4 = \begin{pmatrix} 3288.5 & 3772.5 \\ 3772.5 & 3288.5 \end{pmatrix} = 8\begin{pmatrix} 411.06 & 409.06 \\ 409.06 & 411.06 \end{pmatrix}$$

d) $k$ does not have to be a positive number

$$B^5 = \begin{pmatrix} -12 & -10 \\ -10 & -12 \end{pmatrix}^5 = \begin{pmatrix} -2576832 & -2576800 \\ -2576800 & -2576832 \end{pmatrix} = 16\begin{pmatrix} -161052 & -161050 \\ -161050 & -161052 \end{pmatrix}$$

e) $n$ is the power to which the matrix is raised

f) to obtain the entries where the difference equals 2 (in a row or a column) we have to divide each of them by $2^{n-1}$

$$A^n = \begin{pmatrix} x & y \\ y & x \end{pmatrix}^n = 2^{n-1}\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$$

When technology was used to investigate what happened with further values of $k$ and $n$ it occurred, that for greater numbers the pattern was not true. In the case of, e.g. $k$ = 1500 and $n$ = 2, the pattern worked. Increasing $n$ to 3, however, caused all the entries to be the same.
This was also checked for the matrices ***P*** and ***S***.

$$P^{17} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{17} = \begin{pmatrix} 8590000128 & 8589869056 \\ 8589869056 & 8590000128 \end{pmatrix} = 65536\begin{pmatrix} 131073 & 131071 \\ 131071 & 131073 \end{pmatrix}$$

$$P^{18} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{18} = \begin{pmatrix} 3.44 \times 10^{10} & 3.44 \times 10^{10} \\ 3.44 \times 10^{10} & 3.44 \times 10^{10} \end{pmatrix}$$

$$S^{13} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^{13} = \begin{pmatrix} 6530351104 & 6530342912 \\ 6530342912 & 6530351104 \end{pmatrix} = 4096\begin{pmatrix} 1594324 & 1594322 \\ 1594322 & 1594324 \end{pmatrix}$$

$$S^{14} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^{14} = \begin{pmatrix} 3.92 \times 10^{10} & 3.92 \times 10^{10} \\ 3.92 \times 10^{10} & 3.92 \times 10^{10} \end{pmatrix}$$

The pattern works as long as the results are less than 10 billions. If they exceed this number, all the entries will be exactly the same.

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^n > 10\,000\,000\,000$$

Therefore, this does not seem to be true for every number.
The results hold true in general because in real life situations so large numbers are not frequently used. For smaller numbers the pattern fits thoroughly.