Matrices

Matrix – a way of presenting information in table-like form:

$$\begin{bmatrix} 2 & 7 \\ 5 & -3 \end{bmatrix}$$

This matrix has two rows and two columns, so it is said to be a matrix of <u>order</u> 2 x 2. An m x n matrix would have m rows and n columns. Each number in a matrix is called an <u>element</u>.

Operations with matrices – When adding and subtracting, each element is added to or subtracted from its corresponding element in the other matrix. For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Addition and subtraction can only be done with matrices of the same order. This is because if matrices of different orders were used, some elements would not have another corresponding element to be added to or subtracted from.

When multiplying matrices, the top-left element in the first matrix is multiplied by the top-left element in the second matrix. The product is then added to the product of the second element in the first row of the first matrix and the first element in the second row of the second matrix. This continues in a similar fashion for the rest of the elements. For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Because of this process, matrices can only be multiplied together if the number of columns in the first matrix equals the number of rows in the second matrix.

The <u>additive identity</u> is a matrix with all elements zero. The <u>additive inverse</u> is a matrix with elements such that when added to another matrix, the resulting matrix has all zeros.

Determinant – the <u>determinant</u> of a 2 x 2 matrix is obtained as follows:

$$\det \boldsymbol{A} = |\boldsymbol{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The first three portions of the equation are simply different ways to notate "determinant of matrix **A**". To find the determinant of a 3 x 3 matrix, do the following:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} = a\begin{vmatrix} e & f \\ h & j \end{vmatrix} - b\begin{vmatrix} d & f \\ g & j \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

In essence, finding the determinant of a 3 x 3 matrix is the same as finding the determinants of three 2 x 2 matrices and combining them. **A** singular matrix is a matrix with determinant zero.

For example:

$$\begin{vmatrix} 1 & 4 & 3 \\ 5 & 8 & 4 \\ 9 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 8 & 4 \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} 5 & 4 \\ 9 & 1 \end{vmatrix} + 3\begin{vmatrix} 5 & 8 \\ 9 & 3 \end{vmatrix} = -51$$

Identity – the identity of a matrix **A** is notated $\mathbf{A}^{-1}$ and is such that $\mathbf{AA}^{-1} = \mathbf{I}$, the identity matrix. The identity matrix for a 2 x 2 matrix is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In order to find the inverse of a 2 x 2 matrix, use this form:

$$\boldsymbol{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If the determinant of a matrix is zero, the inverse cannot be determined because 1/0 is meaningless. Therefore, the inverse of any singular matrix cannot be found.

More advanced techniques

Systems of equations – solving systems of equations with matrices is exceptionally easy, especially if a calculator is used. Consider this system:

$$3x + 4y - z = 8$$
$$-7x - y + 3z = 2$$
$$x + y - z = 3$$

Ordinarily, this would be difficult and time-consuming to solve. However, we can put this into matrix form as such:

$$\begin{bmatrix} 3 & 4 & -1 \\ -7 & -1 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 3 \end{bmatrix}$$

This is possible because of the nature of multiplying matrices. Recall from page 1 that multiplying two matrices together involves adding multiple products together. If these matrices are multiplied, the original system of equations is obtained. In other words, multiplying the first two matrices together would yield 3x+4y-z=8 for the top row, and so on. The leftmost matrix is called the coefficient matrix, the middle matrix is called the variable matrix, and the rightmost matrix is called the solution matrix.

In order to solve this system, the leftmost matrix needs to be moved to the right side of the equation. Recall that any matrix multiplied by its inverse gives the identity matrix. This identity matrix multiplied by the variable matrix would result in simply the variable matrix unchanged. So, if both sides of the equation are multiplied by the inverse of the coefficient matrix, the result is the variable matrix equal to the answer matrix (which contains the values for x, y, and z). In the case of this particular system, the result is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1.4375 \\ 2.625 \\ -1.8125 \end{bmatrix}$$

So, x=-1.4375, y=2.625, and z=-1.8125.

Vectors

Vector – a quantity involving both magnitude and direction. Quantities involving only magnitude are called scalar quantities.

Vectors can be notated with either a bold letter (**a**) or an arrow on top of the letter ($\vec{a}$). Vectors can be described in either matrix form or standard form. In matrix form, **a** could be written as $\binom{3}{4}$, and in standard form, it would be written as 3**i**+4**j**, where $\boldsymbol{i} = \binom{1}{0}$ and $\boldsymbol{j} = \binom{0}{1}$.

Vectors are not pinned down to any one location; they can be moved to any location.

Magnitude – the magnitude of a vector can be found by using Pythagorean Theorem on its values. For example, the magnitude of **a** as described in the previous paragraph would be:

$$\sqrt{3^2 + 4^2} = 5$$

The magnitude of a vector is notated as $|\boldsymbol{a}|$.

Parallel – parallel vectors have the same direction. Magnitude does not matter. In order for vectors to have the same direction, the elements for their direction need to be proportional. For example, $\binom{1}{1}$ is parallel to $\binom{-2}{-2}$ because they are proportional by a factor of -2.

Equal – in order for two vectors to be equal, they must have the same magnitude and direction.

$\hat{\boldsymbol{a}}$ **–** pronounced "a-hat". It is the unit vector in the direction of **a**, meaning that it has a magnitude of one. In order to find $\hat{\boldsymbol{a}}$, simply divide each of the elements in **a** by its magnitude. For example, if $\boldsymbol{a} = \binom{3}{4}$:

$$\hat{\boldsymbol{a}} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

3D Vectors – vectors in 3D space are very similar to vectors in 2D. The main difference is that they have an added z-component. As such, 3D vectors are described as either $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or xi+yj+zk.

Operations with vectors – addition and subtraction of vectors is very similar to that with matrices. Each element is added to or subtracted from its corresponding element in the other vector. For example:

$$\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ 8 \end{pmatrix}$$

Subtraction also gives the vector from one point to another. In the above example, the <u>resultant vector</u> is the vector from the second vector to the first vector.

In addition, vectors can be multiplied by a <u>scalar</u>:

$$\lambda \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3\lambda \\ -2\lambda \\ 4\lambda \end{pmatrix}$$

Position Vector – a way to define a point by giving the distance and direction of the point from the origin. For example, the position vector for the point P might be $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This is the vector $\overrightarrow{OP}$, assuming O is the origin.

Orthogonal – means the same thing as 'perpendicular'.

Dot Product – a method of gaining a scalar product from two vectors. As such, it is also referred to as the <u>scalar product</u>. Here is the form:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf$$

Another form of the dot product is as follows:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

The equation is read "a dot b". ϴ refers to the angle between the two vectors. From this second form of the dot product, it can be assumed that two perpendicular vectors will have a dot product of zero. This is because if the two vectors were perpendicular, ϴ would be 90°, and if that were true, the entire right side of the equation would be zero.

A few properties of the dot product are as follows:

- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}||\boldsymbol{a}| = |\boldsymbol{a}|^2$
- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ (commutative)
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$ (distributive over addition)
- $\lambda(\boldsymbol{a} \cdot \boldsymbol{b}) = (\lambda\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (\lambda\boldsymbol{b}) = \lambda|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ (multiplication by a scalar λ)

Direction Cosines – only applies to 3D vectors. Consider vector $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. If this vector makes angles α, β, and γ with the x-axis, y-axis, and z-axis, respectively, then:

$$\cos\alpha = \frac{x}{|\boldsymbol{r}|}$$
$$\cos\beta = \frac{y}{|\boldsymbol{r}|}$$
$$\cos\gamma = \frac{z}{|\boldsymbol{r}|}$$

<u>More advanced techniques</u>

Find the angle between two vectors – use the two forms of the dot product to solve for ϴ. For example, consider vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$:

Find the dot product of the two vectors using the first form of the dot product: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 20$

Plug the dot product into the second form, along with the magnitudes of the two vectors: $20 = \sqrt{14} * \sqrt{29} * \cos\theta$

Solve for ϴ: $\theta = 6.98°$

Write a vector in terms of two other vectors – because vectors are not pinned down to any one location, any vector can be written in terms of any two other vectors. Consider, for example, the vector $\begin{pmatrix}6\\5\end{pmatrix}$. In order to be written in terms of component vectors $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$, the following must be done:

Write the three vectors in a form that allows the component vectors to be multiplied by a scalar:	$\begin{pmatrix}6\\5\end{pmatrix} = \lambda\begin{pmatrix}2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\1\end{pmatrix}$
Now λ and μ must be found, so that the component vectors can be multiplied by the correct scalars. In order to do this, write the equation as a system of equations:	$6 = 2\lambda + \mu$ $5 = \lambda + \mu$
Solve the system of equations (using any method):	$\lambda = 1$ $\mu = 4$

Now it is known what the values of λ and μ are. Simply multiply the component vectors by these values and then add the resulting vectors in order to obtain the original vector.

Lines

Vector equation – a straight line is uniquely defined if given the position of one point that lies on the line and the direction of the line. In terms of vectors, this means that a line is defined if the following is known:

- The position vector of any point on the line
- Any vector that is parallel to the line

Therefore, the vector equation of a line is written as follows:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$$

Where **a** (position vector) is the position vector of any point on the line, **b** (direction vector) is any vector parallel to the line, and **r** is the position vector of another point on the line. This vector equation works for both 2D and 3D lines.

Because **r** is the position vector of any point on the line, it can be written as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (in 3D space), where x, y, and z are coordinates of the point. In a similar manner, the entire vector equation can be written as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

Parametric equations – these equations of a line can be easily derived from the vector equation. The parametric equation takes each line of the second form of the vector equation separately to form a system of equations, as follows:

$$x = a + \lambda d$$
$$y = b + \lambda e$$
$$z = c + \lambda f$$

Each separate equation gives a component (x, y, or z) of the position vector of any point on the line.

Cartesian equation – this equation of a line can also be easily derived from the vector equation. The form is as follows:

$$\frac{x-a}{d}=\frac{y-b}{e}=\frac{z-c}{f}$$

Where all of the variables are taken from the second form of the vector equation. Note that it is acceptable to place zeros in the Cartesian equation if needed. For example:

$$\frac{x-1}{0}=\frac{y+5}{3}=\frac{z-0}{1}$$

More advanced techniques

State whether two lines intersect, are skew, or are parallel – in 2D space, lines can either intersect or be parallel to each other. In order to tell which is the case, simply look at each line's direction vector and see if they are parallel. If the lines' direction vectors are parallel, both lines are parallel to each other.

In 3D space, lines can either intersect, be skew, or be parallel. To be skew means to not intersect and to not be parallel. In order to decide which the case is, first look at the direction vectors of each line. As with 2D lines, if the direction vectors are parallel, the lines are parallel. For example, these two lines are not parallel:

$$\boldsymbol{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\boldsymbol{r}=\begin{pmatrix}4\\5\\6\end{pmatrix}+\mu\begin{pmatrix}3\\1\\2\end{pmatrix}$$

If the two direction vectors are not parallel, then proceed to the next step.

Convert each line to parametric form, and then set each pair corresponding equations (x, y, and z) equal to each other to form a system of three equations as such:

$$1+\lambda=4+3\mu$$
$$\lambda=5+\mu$$
$$2+\lambda=6+2\mu$$

Rearrange the system:

$$\lambda-3\mu=3$$
$$\lambda-\mu=5$$
$$\lambda-2\mu=4$$

Solve for λ and μ using any two equations (the first and third equations will be used): $\lambda = 6$ $\mu = 1$

Check for consistency with the remaining equation. If the system is not consistent, the lines are skew and do not intersect. In this case, the system is consistent, so the lines do intersect. $(6) - (1) = 5$

Find the point of intersection of two lines – the example from above will be continued here. After it is known that two lines do indeed intersect, it is a simple matter to find the point of intersection:

Plug either of the values for λ or μ into their corresponding vector equation to find the intersection point (μ will be used): $\boldsymbol{r} = \begin{pmatrix}4\\5\\6\end{pmatrix} + 1\begin{pmatrix}3\\1\\2\end{pmatrix} = \begin{pmatrix}7\\6\\8\end{pmatrix}$

Therefore, the intersection point of the two lines is $\begin{pmatrix}7\\6\\8\end{pmatrix}$.

Find the distance from a point to a line – the shortest distance from a point to a line is along the vector that begins at the line and ends at the point and is perpendicular to the line (or in the other order). Considering the point with position vector $\begin{pmatrix}1\\2\end{pmatrix}$ and the line $\boldsymbol{r} = \begin{pmatrix}2\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\1\end{pmatrix}$, follow the steps:

Use the vector equation of the line to find any point on the line (in terms of λ): $\boldsymbol{r} = \begin{pmatrix}2+\lambda\\5+\lambda\end{pmatrix}$

Find the vector between the original point and the point on the line: $\begin{pmatrix}2+\lambda\\5+\lambda\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1+\lambda\\3+\lambda\end{pmatrix}$

Make the vector between the two points perpendicular to the direction vector of the line using dot product: $\begin{pmatrix}1+\lambda\\3+\lambda\end{pmatrix} \cdot \begin{pmatrix}1\\1\end{pmatrix} = 0$

Solve for λ: $\lambda = -2$

Plug λ into the vector between the two points: $\begin{pmatrix}1-2\\3-2\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$

Find the magnitude of the resulting vector: $\left|\begin{pmatrix}-1\\1\end{pmatrix}\right| = \sqrt{2}$

Therefore, the distance between the point and the line is $\sqrt{2}$.

Find the distance from a line to another line – the process for this depends on whether the lines are parallel, intersecting, or skew. If they intersect, the distance is zero. The shortest distance between two lines is along the vector that is perpendicular to both lines. In the case of parallel lines, using a random point on one line and the full vector equation of the other line, use the method described above for finding the distance from a point to a line.

Consider these two lines:

$$\boldsymbol{r} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\2\end{pmatrix}$$
$$\boldsymbol{r} = \begin{pmatrix}2\\1\\2\end{pmatrix} + \mu\begin{pmatrix}1\\2\\2\end{pmatrix}$$

Because these lines are skew, follow this method:

As stated earlier, the shortest distance between two lines is along the vector that is perpendicular to both lines. Start by finding any point on each of the lines in terms of λ and μ:

$$\boldsymbol{r} = \begin{pmatrix}1+\lambda\\2+\lambda\\1+2\lambda\end{pmatrix}$$
$$\boldsymbol{r} = \begin{pmatrix}2+\mu\\1+2\mu\\2+2\mu\end{pmatrix}$$

Now, find the vector between these two points:

$$\begin{pmatrix}1+\lambda\\2+\lambda\\1+2\lambda\end{pmatrix} - \begin{pmatrix}2+\mu\\1+2\mu\\2+2\mu\end{pmatrix} = \begin{pmatrix}-1+\lambda-\mu\\1+\lambda-2\mu\\-1+2\lambda-2\mu\end{pmatrix}$$

Use dot product to make this vector perpendicular to both of the lines' direction vectors:

$$\begin{pmatrix}-1+\lambda-\mu\\1+\lambda-2\mu\\-1+2\lambda-2\mu\end{pmatrix} \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} = 0$$
$$\begin{pmatrix}-1+\lambda-\mu\\1+\lambda-2\mu\\-1+2\lambda-2\mu\end{pmatrix} \cdot \begin{pmatrix}1\\2\\2\end{pmatrix} = 0$$

Simplify each of the equations:

$$6\lambda - 7\mu = 2$$
$$7\lambda - 9\mu = 1$$

Solve the system of equations for λ and μ:

$$\lambda = {}^{11}/_{5}$$
$$\mu = {}^{8}/_{5}$$

Plug λ and μ back into the vector between the two points:

$$\begin{pmatrix}-1+{}^{11}/_{5}-{}^{8}/_{5}\\1+{}^{11}/_{5}-{}^{16}/_{5}\\-1+{}^{22}/_{5}-{}^{16}/_{5}\end{pmatrix} = \begin{pmatrix}{}^{-2}/_{5}\\0\\{}^{1}/_{5}\end{pmatrix}$$

Find the magnitude of this resulting vector:

$$\left|\begin{pmatrix}{}^{-2}/_{5}\\0\\{}^{1}/_{5}\end{pmatrix}\right| = \sqrt{{}^{1}/_{5}}$$

Therefore, the distance between the two lines is $\sqrt{1/5}$.

Planes

Vector equation – one of the ways to uniquely define a plane involves two vectors that are parallel to the plane and one point that is on the plane. As such, the vector equation of a plane is as follows:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$$

Where **a** is the position vector of any point on the plane and **b** and **c** are two vectors that are parallel to the plane, but not parallel to each other.

Dot product equation – the equation of a plane can be written using the dot product as well. Another way to uniquely define a plane involves any two points on the plane as well as one vector that is perpendicular to the plane. As such, the dot product equation is as follows:

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

Where **r** and **a** are any two points on the plane and **n** is any vector perpendicular to the plane. Vector **r** is usually left as a variable when writing this form of a plane. For example:

$$\boldsymbol{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

The dot product equation can be proved easily. Consider a plane with two random points A and R, along with a perpendicular vector **n**. Then, consider the vector from point A to point R ($\overrightarrow{AR}$). Because both point A and point R lie on the plane, $\overrightarrow{AR}$ is parallel to the plane and therefore perpendicular to vector **n**. Using the dot product:

$$\boldsymbol{n} \cdot \overrightarrow{AR} = 0$$

This can also be written as:

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{a}) = 0$$

After using the distributive property of the dot product, the result is:

$$(\boldsymbol{n} \cdot \boldsymbol{r}) - (\boldsymbol{n} \cdot \boldsymbol{a}) = 0$$

And finally:

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

Scalar product equation – this form is based on the dot product equation. Simply carry out the dot product on the right side of the equation. The form is:

$$\boldsymbol{r} \cdot \boldsymbol{n} = p$$

Where p is the scalar product of **a** and **n**. Considering the example plane in the previous paragraph, the scalar product equation would be:

$$\boldsymbol{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 9$$

It should be noted that planes with parallel **n** vectors are parallel.

Cartesian equation – this form of a plane can be derived very easily from the scalar product equation. Considering the plane used in the previous paragraph, the Cartesian equation for it is:

$$x + 2y + z = 9$$

Basically, the coefficients for x, y, and z are taken from the vector **n** and the right side of the equation is simply p.

More advanced techniques

Find distance from plane to origin – the shortest distance from a plane to the origin is along a vector that is perpendicular to the plane and ends at the origin. The formula is as follows:

$$d = \frac{\boldsymbol{a} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}$$

Or, in terms of the scalar product equation, the formula translates to:

$$d = \frac{p}{|\boldsymbol{n}|}$$

These formulas can easily be proved. Consider the same situation as used in the proof of the dot product equation (a plane, two points on the plane, and a perpendicular vector to the plane). Add a vector **d** from the origin to the plane, so that it is perpendicular to the plane and parallel to **n**.

Position vector **a** and vector **d** form a right triangle. Make the angle between them (at the origin) be θ. This means that:

$|\boldsymbol{d}| = |\boldsymbol{a}| * \cos\theta$

Recall that the alternate form of the dot product is:
$\boldsymbol{a} \cdot \boldsymbol{n} = |\boldsymbol{a}| * |\boldsymbol{n}| * \cos\theta$

Using this form of the dot product and our previous equation, it can be proven that:
$\boldsymbol{a} \cdot \boldsymbol{n} = |\boldsymbol{n}| * |\boldsymbol{d}|$

Finally:
$d = \frac{\boldsymbol{a} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}$

Where d symbolizes the magnitude (or length) of vector **d**.

For example, the distance from plane with scalar product formula $\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ 7 \\ 9 \end{pmatrix} = 3$ to the origin is $\frac{3}{\sqrt{134}}$.

Find distance from plane to plane – the easiest way to find this distance is to simply find the distance from the origin to each of the planes, and then subtract the distance. For example, taking the plane from the previous example and adding the plane $\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ 7 \\ 9 \end{pmatrix} = 5$, the distance between them is:

$$\frac{5}{\sqrt{134}} - \frac{3}{\sqrt{134}} = \frac{2}{\sqrt{134}}$$

Note that this method only works with parallel planes, because two non-parallel planes intersect and the distance between them is zero.

Find distance from plane to line – the easiest way to find this distance is to pretend that the line is a plane, and then use the method described in the previous paragraph. For example, considering the plane $\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ 7 \\ 9 \end{pmatrix} = 5$ and the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, the line can be changed into a parallel plane as follows:

$$\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ 7 \\ 9 \end{pmatrix} = 45$$

Now, the distance between the original plane and the new "plane" can be found easily:

$$\frac{45}{\sqrt{134}} - \frac{5}{\sqrt{134}} = \frac{40}{\sqrt{134}}$$

Find distance from plane to point – this method is almost exactly the same as the previous method. Simply change the point into a plane that is parallel to the original plane as shown in the above example, then find the distance between the two planes.

Find line intersection of two planes – take, for example, the planes 7x-4y+3z=-3 and 4x+2y+z=4:

Eliminate z from both equations (system of equations):

$5x + 10y = 15$

Or $x + 2y = 3$

Or $x = 3 - 2y$

Eliminate y from both of the original equations:

$3x + z = 1$

Or $x = \frac{1-z}{3}$

Therefore, the Cartesian equation of the intersecting line is:

$$x = 3 - 2y = \frac{1 - z}{3}$$