Matrix Binomials

Question

A.) Let $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *and* $Y = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ *Calculate* $X^2, X^3, X^4, Y^2, Y^3, Y^4$

Solution

$$X^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

$$X^4 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} * \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$$

$$Y^2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$Y^3 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$$

$$Y^4 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} * \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}$$

Question

B.) By considering integer powers of X and Y, find expressions for $X^n, Y^n, (X+Y)^n$

Following the pattern in the first question and noticing from it, I deduce that the formula for:

$$\boxed{X^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}}$$

$$\boxed{Y^n = \begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}}$$

Also to find the n^{th} power of (X+Y) I will use some examples to find the pattern in it:

$$X+Y=\begin{pmatrix}1&1\\1&1\end{pmatrix}+\begin{pmatrix}1&-1\\-1&1\end{pmatrix}=\begin{pmatrix}2&0\\0&2\end{pmatrix}$$

$$(X+Y)^2=\begin{pmatrix}2&0\\0&2\end{pmatrix}*\begin{pmatrix}2&0\\0&2\end{pmatrix}=\begin{pmatrix}4&0\\0&4\end{pmatrix}$$

$$(X+Y)^3=\begin{pmatrix}4&0\\0&4\end{pmatrix}*\begin{pmatrix}2&0\\0&2\end{pmatrix}=\begin{pmatrix}8&0\\0&8\end{pmatrix}$$

From the examples above, I deduced the equation of $(X+Y)^n$ by following the pattern, and it is:

$$\boxed{(X+Y)^n=\begin{pmatrix}2^n&0\\0&2^n\end{pmatrix}}$$

Question

Let **A**=a**X** and **B**=b**Y**, where a and b are constants, use different values of a and b to calculate A^2,A^3,A^4,B^2,B^3,B^4

Solution

First we need to find **A**, **B**:

$$A=a*\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}a&a\\a&a\end{pmatrix}$$

$$B=b*\begin{pmatrix}1&-1\\-1&1\end{pmatrix}=\begin{pmatrix}b&-b\\-b&b\end{pmatrix}$$

Find A^2,A^3,A^4,B^2,B^3,B^4 using different values of a and b

Let's assume that a= 2

$$A=a*\begin{pmatrix}1&1\\1&1\end{pmatrix}=2*\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}2&2\\2&2\end{pmatrix}$$

$$\therefore\ A^2=\begin{pmatrix}2&2\\2&2\end{pmatrix}*\begin{pmatrix}2&2\\2&2\end{pmatrix}=\begin{pmatrix}8&8\\8&8\end{pmatrix}$$

$$\therefore\ A^3=\begin{pmatrix}2&2\\2&2\end{pmatrix}*\begin{pmatrix}8&8\\8&8\end{pmatrix}=\begin{pmatrix}32&32\\32&32\end{pmatrix}$$

Also, let's assume that b=3

$$B = b * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 3 * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$$

$$\therefore\ B^2 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} * \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} 18 & -18 \\ -18 & 18 \end{pmatrix}$$

$$\therefore\ B^3 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} * \begin{pmatrix} 18 & -18 \\ -18 & 18 \end{pmatrix} = \begin{pmatrix} 108 & -108 \\ -108 & 108 \end{pmatrix}$$

To obtain the general formula for $A^n, B^n, (A+B)^n$ we need to find a formula for $A^2, A^3, A^4, B^2, B^3, B^4$ <u>without using different values</u> of a and b, ie:

$$A^2 = \begin{pmatrix} a & a \\ a & a \end{pmatrix} * \begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} a & a \\ a & a \end{pmatrix} * \begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix} = \begin{pmatrix} 4a^3 & 4a^3 \\ 4a^3 & 4a^3 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix} * \begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix} = \begin{pmatrix} 8a^4 & 8a^4 \\ 8a^4 & 8a^4 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} b & -b \\ -b & b \end{pmatrix} * \begin{pmatrix} b & -b \\ -b & b \end{pmatrix} = \begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix}$$

$$B^3 = \begin{pmatrix} b & -b \\ -b & b \end{pmatrix} * \begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix} = \begin{pmatrix} 4b^3 & -4b^3 \\ -4b^3 & 4b^3 \end{pmatrix}$$

$$B^4 = \begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix} * \begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix} = \begin{pmatrix} 8b^4 & -8b^4 \\ -8b^4 & 8b^4 \end{pmatrix}$$

Question

By considering integer powers of **A** and **B**, find expressions for $A^n, B^n, (A+B)^n$

Solution

By following the pattern in the previous question, one can deduce that the formulas are:

$$\boxed{A^n = \begin{pmatrix} 2^{n-1}a^n & 2^{n-1}a^n \\ 2^{n-1}a^n & 2^{n-1}a^n \end{pmatrix}}$$

$$\boxed{B^n = \begin{pmatrix} 2^{n-1}b^n & -2^{n-1}b^n \\ -2^{n-1}b^n & 2^{n-1}b^n \end{pmatrix}}$$

And to find the formula for $(A+B)^n$, one should give some examples and work done on it, so the pattern can be traced and noticed:

$$A+B=\begin{pmatrix} a & a \\ a & a \end{pmatrix}+\begin{pmatrix} b & -b \\ -b & b \end{pmatrix}=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$$

$$(A+B)^2=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}*\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}=\begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}$$

$$(A+B)^3=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}*\begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}=\begin{pmatrix} 4a^3+4b^3 & 4a^3-4b^3 \\ 4a^3-4b^3 & 4a^3+4b^3 \end{pmatrix}$$

$$(A+B)^4=\begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}*\begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}$$
$$=\begin{pmatrix} 8a^4+8b^4 & 8a^4-8b^4 \\ 8a^4-8b^4 & 8a^4+8b^4 \end{pmatrix}$$

So, by noticing the pattern in the examples provided above, I conclude that the formula will be:

$$\boxed{(A+B)^n=\begin{pmatrix} 2^{n-1}a^n+2^{n-1}b^n & 2^{n-1}a^n-2^{n-1}b^n \\ 2^{n-1}a^n-2^{n-1}b^n & 2^{n-1}a^n+2^{n-1}b^n \end{pmatrix}}$$

Question

Now consider $\boldsymbol{M}=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$, show that **M= A+B** and that $\boldsymbol{M^2}=\boldsymbol{A^2}+\boldsymbol{B^2}$

Solution

This is solved by keeping a and b as constants, but without using number, so it is kind of a general formula.

$$\boxed{A+B=\begin{pmatrix} a & a \\ a & a \end{pmatrix}+\begin{pmatrix} b & -b \\ -b & b \end{pmatrix}=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}=M}$$

$$M^2=\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}*\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}=\begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}$$

$$A^2=\begin{pmatrix} a & a \\ a & a \end{pmatrix}*\begin{pmatrix} a & a \\ a & a \end{pmatrix}=\begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix}$$

$$B^2=\begin{pmatrix} b & -b \\ -b & b \end{pmatrix}*\begin{pmatrix} b & -b \\ -b & b \end{pmatrix}=\begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix}$$

$$\boxed{A^2+B^2=\begin{pmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{pmatrix}+\begin{pmatrix} 2b^2 & -2b^2 \\ -2b^2 & 2b^2 \end{pmatrix}=\begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}=M^2}$$

To make sure of it, an example is giving, where we replace a=2, and b=3

$$A+B=\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}+\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}=\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix} = \begin{pmatrix} 2+3 & 2-3 \\ 2-3 & 2+3 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} = A + B$$

Question

Hence, find **the** general statement that expresses $\boldsymbol{M^n}$ in terms of α**X** and β**Y**.

Solution

After following the pattern in the previous question, I concluded the formula of $\boldsymbol{M^n}$ to be:

$$M^n = \begin{pmatrix} 2^{n-1}a^n + 2^{n-1}b^n & 2^{n-1}a^n - 2^{n-1}b^n \\ 2^{n-1}a^n - 2^{n-1}b^n & 2^{n-1}a^n + 2^{n-1}b^n \end{pmatrix}$$

To obtain the general formula of $\boldsymbol{M^n}$ in terms of α**X** and β**Y:**

$$\boxed{M^n = 2^{n-1}a^n \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 2^{n-1}b^n \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}$$

Question

Test the validity of your general statement by using different values of α, β and n.

Solution

Let's assume that a=2, b=3 and n=3

$$\boxed{M^3 = 2^2 2^3 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 2^2 3^3 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 32 & 32 \\ 32 & 32 \end{pmatrix} + \begin{pmatrix} 108 & -108 \\ -108 & 108 \end{pmatrix} = \begin{pmatrix} 140 & -76 \\ -76 & 140 \end{pmatrix}}$$

And by using the formulas we got from before $M^2 * M$

Where $M = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}, M^2 = \begin{pmatrix} 2a^2+2b^2 & 2a^2-2b^2 \\ 2a^2-2b^2 & 2a^2+2b^2 \end{pmatrix}$

So, also by assuming the same variables as before a=2, b=3 and n=3, and then replacing them in the equation $M^2 * M$, we get the exact result as before:

$$\boxed{M^3 = \begin{pmatrix} 2+3 & 2-3 \\ 2-3 & 2+3 \end{pmatrix} * \begin{pmatrix} 8+18 & 8-18 \\ 8-18 & 8+18 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} * \begin{pmatrix} 26 & -10 \\ -10 & 26 \end{pmatrix} = \begin{pmatrix} 140 & -76 \\ -76 & 140 \end{pmatrix}}$$