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Introduction

This parabola investigation is to investigate the patterns in the intersections of parabolas the lines y = x and y = 2x. The investigation includes graphs for visual aid and a table showing relationships. Each answer includes examples of how the answer was found and proof.

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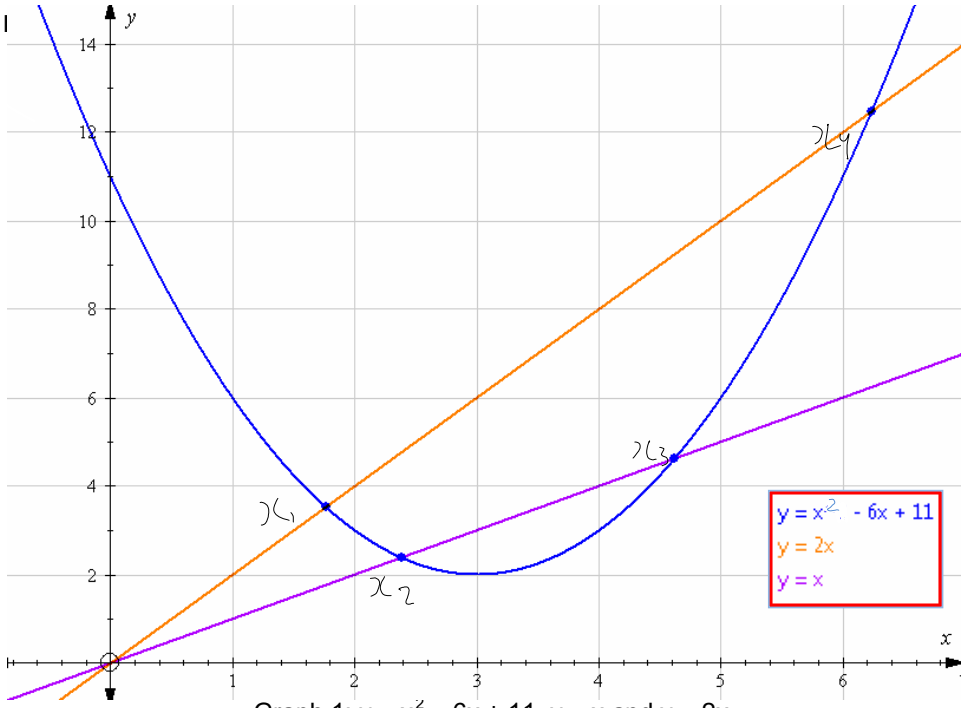
Method

1. **Consider the parabola $y = (x-3)^2 = x^2 - 6x + 11$ and the lines $y = x$ and $y = 2x$.**

- **Using technology find the four intersections illustrated on the right**
  Using technology the three graphs were plotted on the same graph and the intersections were found. Graph one shows the graph and the points of intersection. The points of intersection are



Graph 1: $y = x^2 - 6x + 11$, $y = x$ and $y = 2x$

- **Label the x-values of these intersections as they appear from left to right on the x-axis as x_1, x_2, x_3, and x_4.**
  All four points are labelled on graph one above. The x-values for the above points x_1, x_2, x_3, and x_4 are 1.764, 2.382, 4.618 and 6.236 respectively.

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- **Find the values of $x_2 - x_1$ and $x_4 - x_3$ and name them S_L and S_R respectively.**
  $S_L = x_2 - x_1$
  $S_R = x_4 - x_4$
  $S_L = 2.382 - 1.764 \quad = 0.618$
  $S_R = 6.236 - 4.618 \quad = 1.618$

- **Finally, calculate $D = | S_L - S_R |$.**
  D is equal to the absolute value of $S_L - S_R$, this means D can never be a negative number.
  $D = | S_L - S_R |$
  $D = | 0.618 - 1.618 |$
  $D = | -1 |$
  $D = 1$
  Therefore the difference between the S_L and S_R is one.

2. **Find values of D for other parabolas of the form $y = ax^2 + bx + c$, $a>0$, with vertices in quadrant 1, intersected by the lines $y=x$ and $y = 2x$. Consider various values of a, beginning with a = 1. Make a conjecture about the value of D for these parabolas.**
   The graphs below demonstrate the other parabolas with different values of A and at the bottom is a spreadsheet containing the values for x_1, x_2, x_3, x_4, S_L and S_R and the subsequent values for D. Blue dots represent the intersections between the lines y = x and y = 2x with the parabola.

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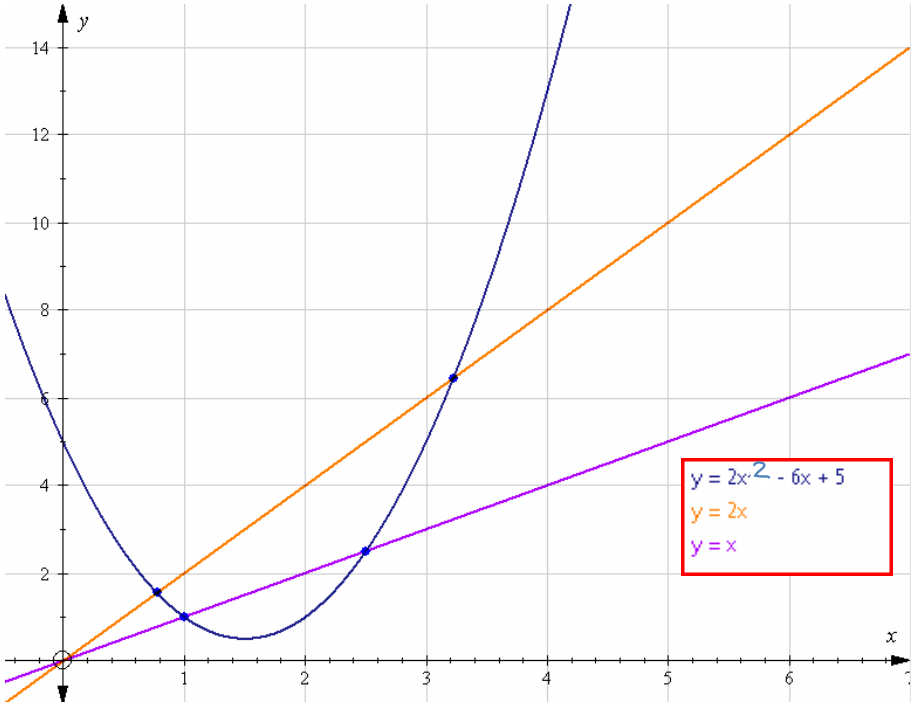
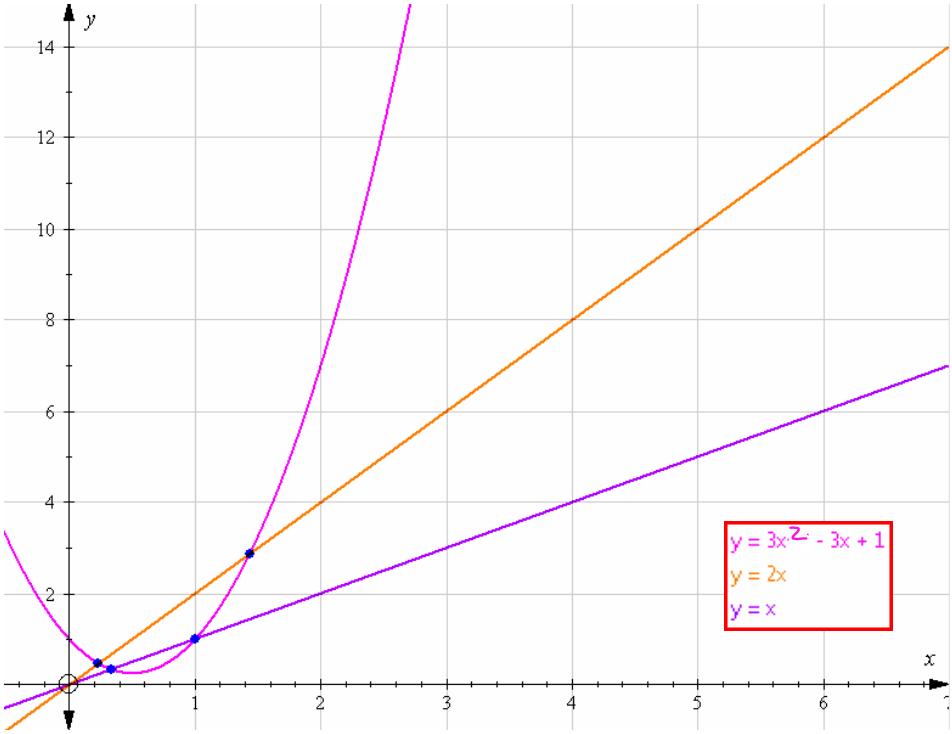


Graph 2: $y = 2x^2 – 6x + 5$



Graph 3: $y = 3x^2 – 3x + 1$

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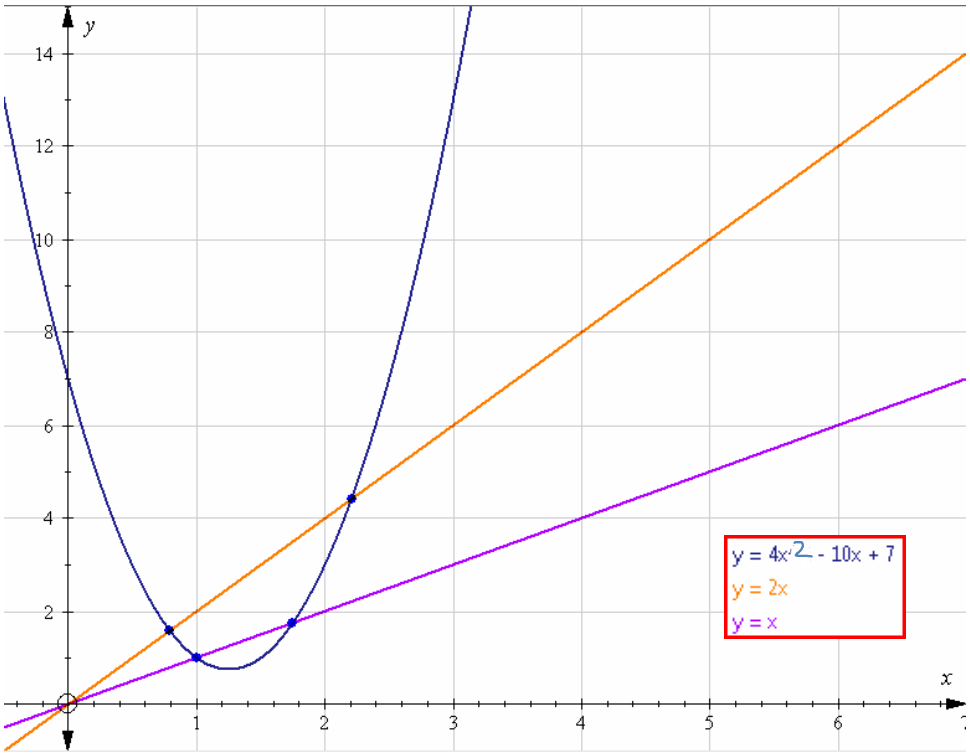
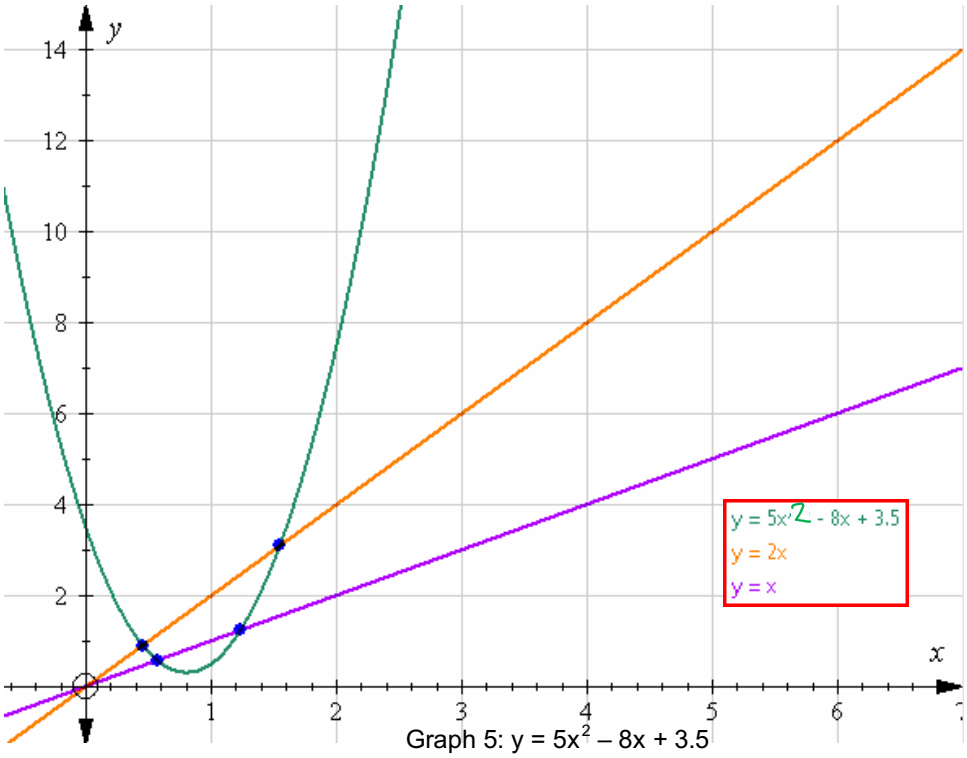


Graph 4: $y = 4x^2 – 10x +7$



Graph 5: $y = 5x^2 – 8x + 3.5$

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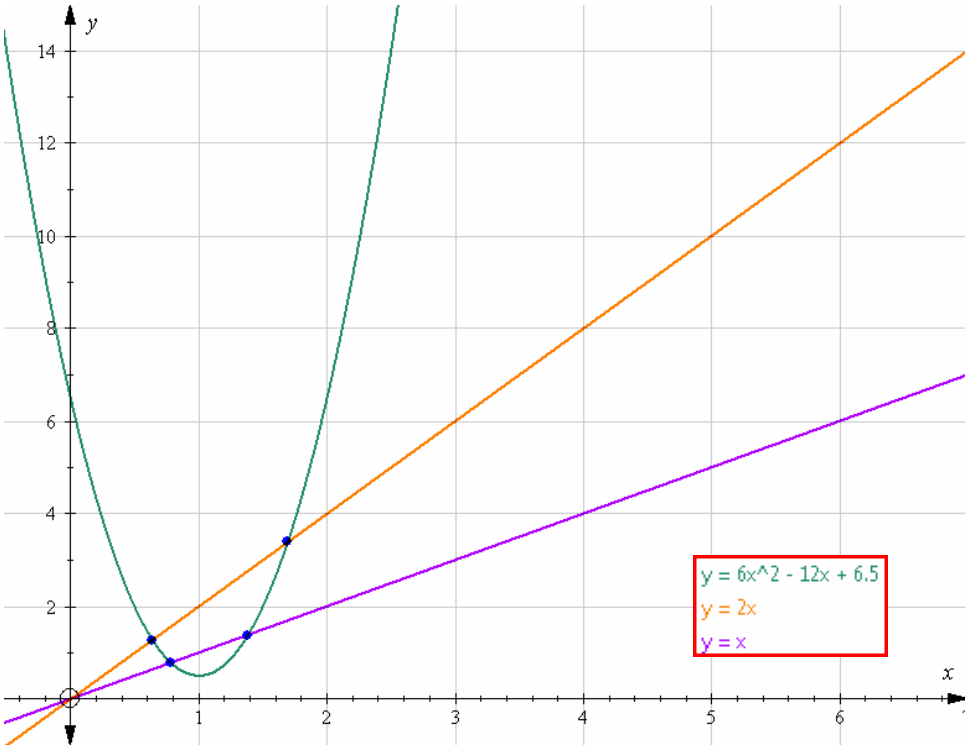

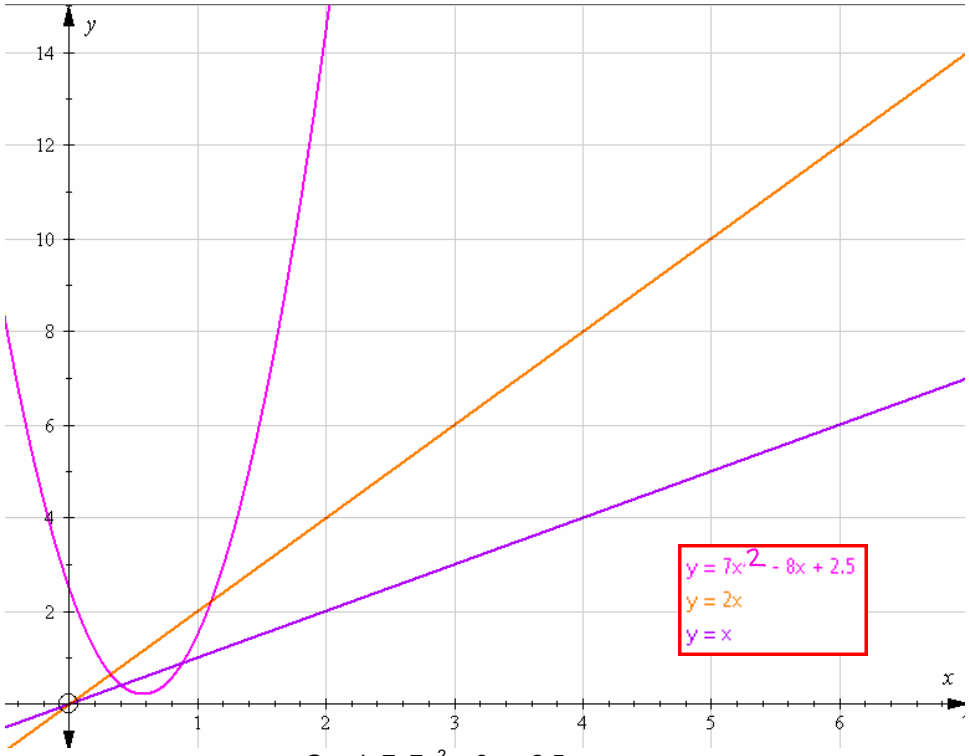

Graph 7: $7x^2 - 8x + 2.5$

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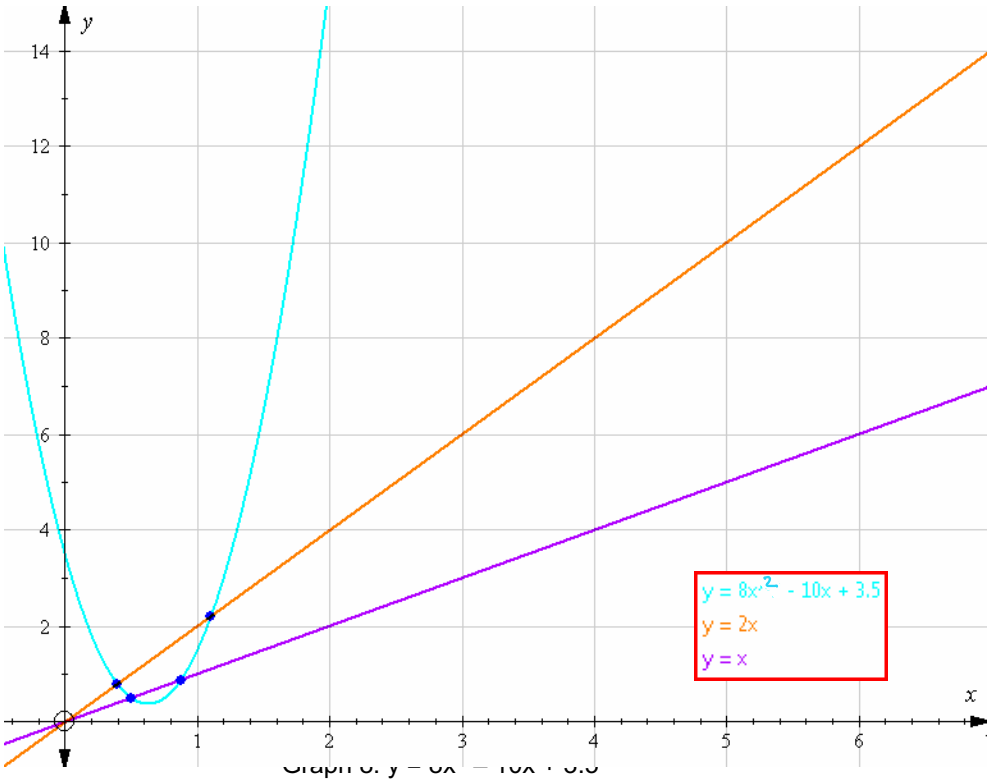

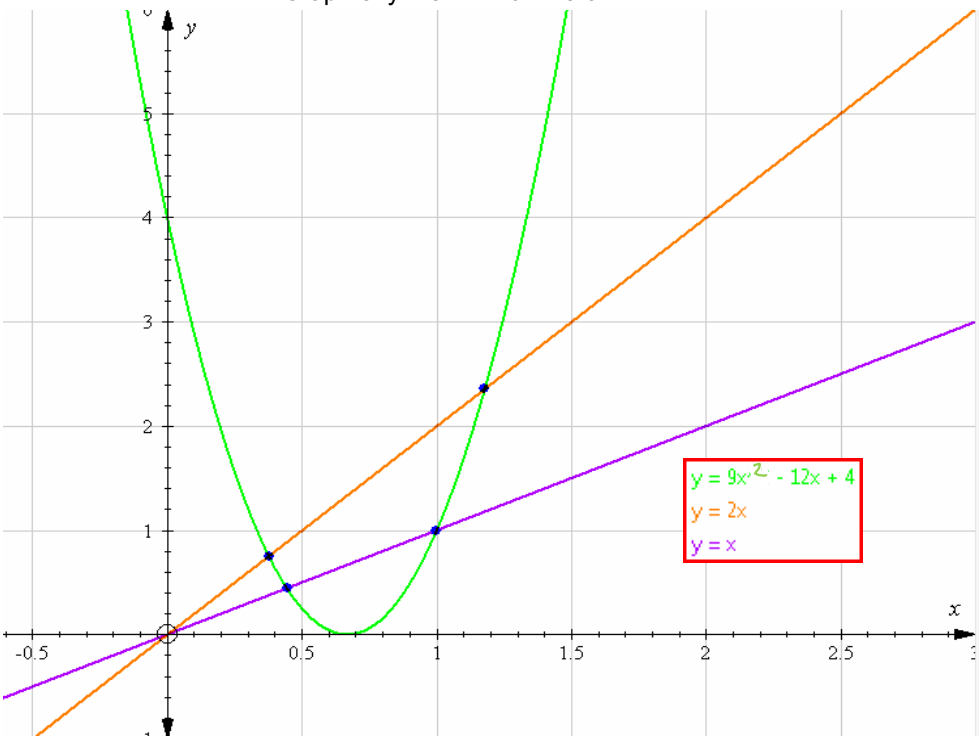

Graph 9: $y = 9x^2 - 12x + 4$

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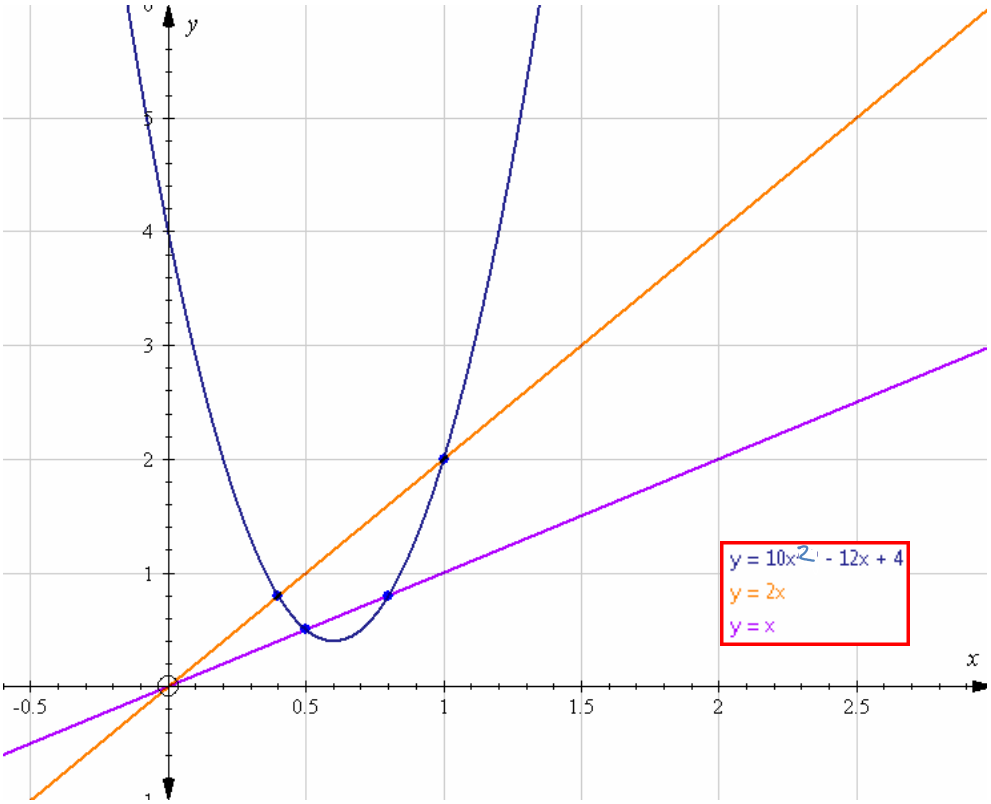
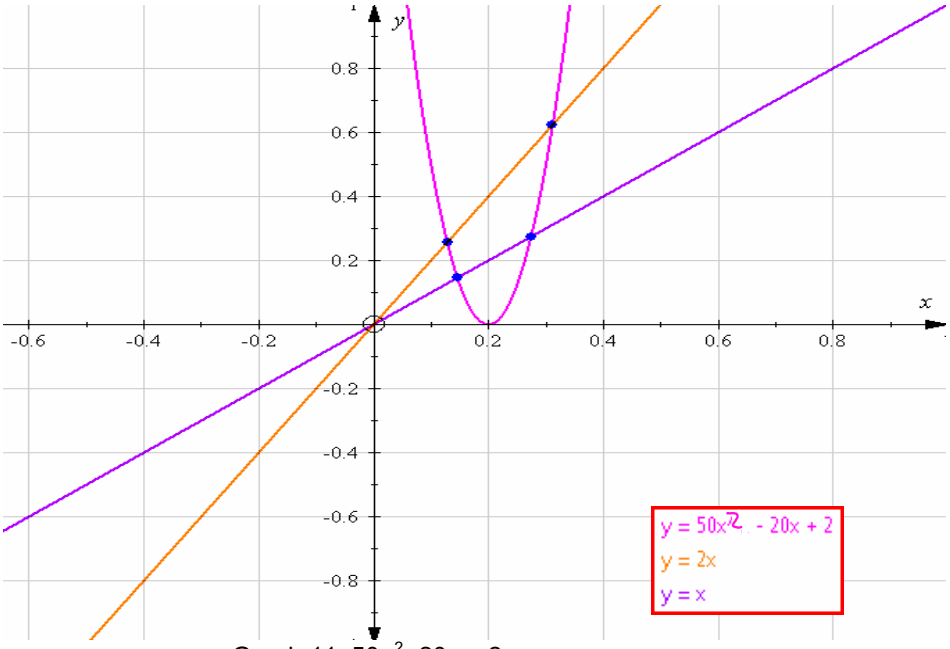


Graph 10: $10x^2 – 12x +4$



Graph 11: $50x^2 -20x + 2$

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	Graph 2	Graph 3	Graph 4	Graph 5	Graph 6
A - Value	2	3	4	5	6
X_1	0.7752	0.2341	0.7929	0.4523	0.6396
X_2	1	0.3333	1	0.5683	0.7829
X_3	2.5	1	1.75	1.2317	1.3838
X_4	3.2247	1.4325	2.207	1.5477	1.6937
S_L	0.2248	0.0992	0.2071	0.116	0.1433
S_R	0.7247	0.4325	0.457	0.316	0.3099
Difference	0.4999	0.3333	0.2499	0.2	0.1666
D	0.5	0.33	0.25	0.2	0.166
	Graph 7	**Graph 8**	**Graph 9**	**Graph 10**	**Graph 12**
A - Value	7	8	9	10	50
X_1	0.3236	0.3964	0.3772	0.4	0.1283
X_2	0.4061	0.5	0.4444	0.5	0.146
X_3	0.8797	0.875	1	0.8	0.274
X_4	1.105	1.1036	1.1783	1	0.3117
S_L	0.0825	0.1036	0.0672	0.1	0.0177
S_R	0.2253	0.2286	0.1783	0.2	0.0377
Difference	0.1428	0.125	0.1111	0.1	0.02
D	0.143	0.125	0.111	0.1	0.02

Table 1: Data from graphs 2 – 10 including value for D

From table 1 it is apparent there is a direct relationship between the value for ‘a’ and the value for d. It is evident that they are inversely proportional i.e. if ‘a’ goes up d goes down. From the data a conjecture can be made. The conjecture is $D = | a^{-1} |$. Therefore if ‘a’ is 5 then D is 0.2, this is supported in table 1.

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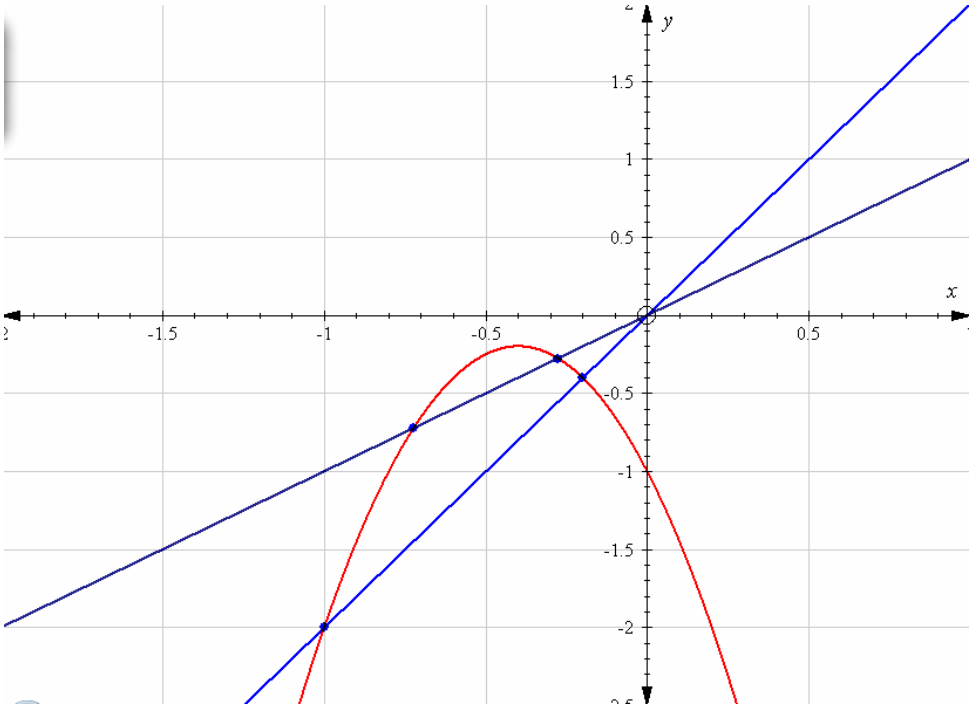
3. Investigate your conjecture for any real value of a and any placement of the vertex. Refine your conjecture as necessary, and prove it. Maintain the labelling convention in parts 1 and 2 by having the intersections of the first line to be x_2 and x_3 and the intersections with the second line to be x_1 and x_4.

Graph 12 below of $y = -5x^2 - 4x + 1$ has a vertex in the 3rd quadrant, according to the conjecture the value for D should be 0.2 and indeed this is true for Graph 12.



Graph 12: $-5x^2 – 4x + 1$

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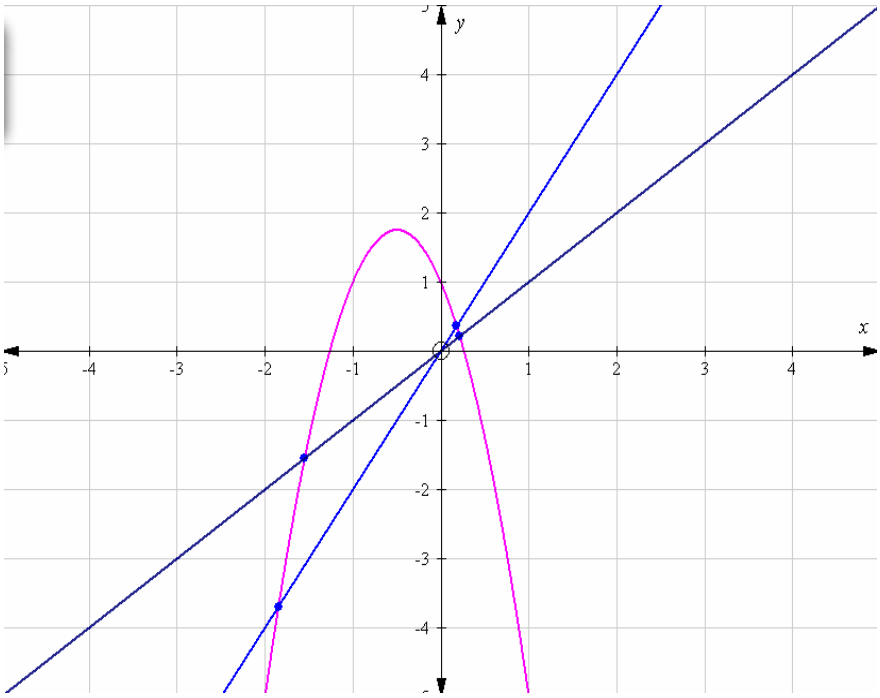
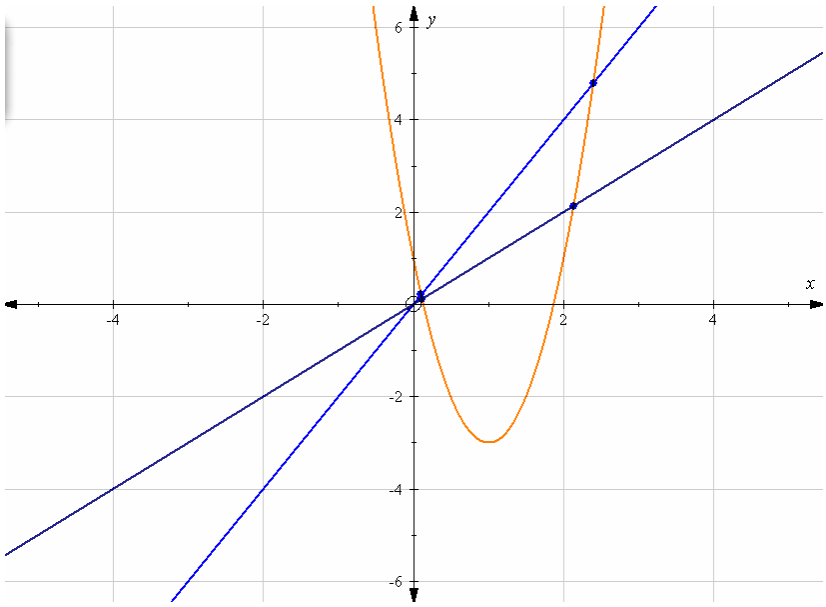


Graph 13: $-3x^2 – 3x + 1$

Graph 13 also proves the conjecture for a vertex in quadrant 2. The D value in graph 13 should be 3^{-1} and it is. Finally Graph 14 represents a graph in quadrant four.



Graph 14: $4x^2 – 8x + 1$

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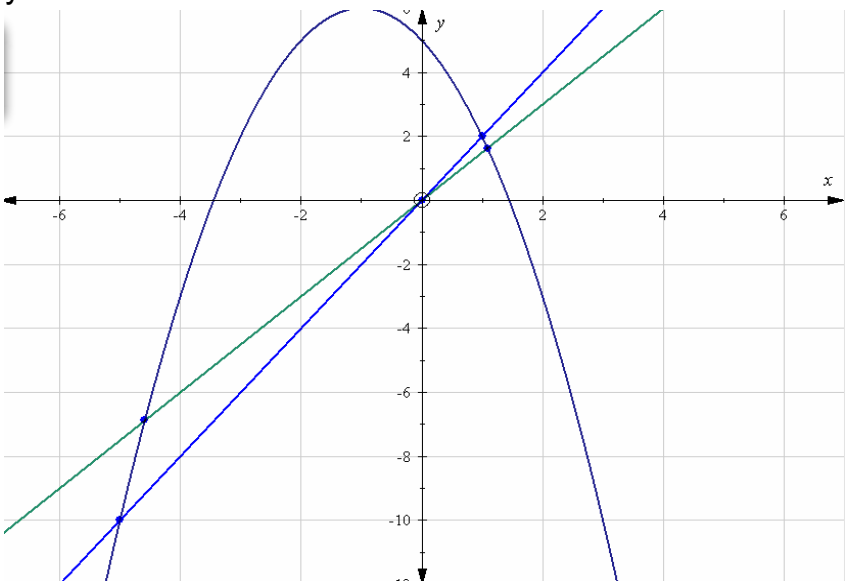
The above three graphs all prove that $D = | a^{-1} |$ is correct as long as the intersections for y = x are x_2 and x_3 and the intersections with y = 2x are x_1 and x_4.

4. Does your conjecture hold if the intersecting lines are changed? Modify your conjecture if necessary then prove it.

The conjecture does not work for different intersecting lines therefore another conjecture must be made to accommodate different lines. Graph 15 below has intersecting lines of y = 2x and y = 1.5x.



Graph 15: $y = -x^2 -2x +5$

The D value for graph 15 is 0.5 not one as expected, therefore a new conjecture must be made. Through making different graphs with different lines there is an obvious correlation between the difference in the values for the intersecting lines and the D value. In Graph 15 the difference between the two values for the intersecting lines is 0.5, not one as in all the other examples. If the difference is multiplied by the inverse of a the new value for D can be found, this was not needed in the other questions as the difference in the intersecting lines was one and multiplying by one

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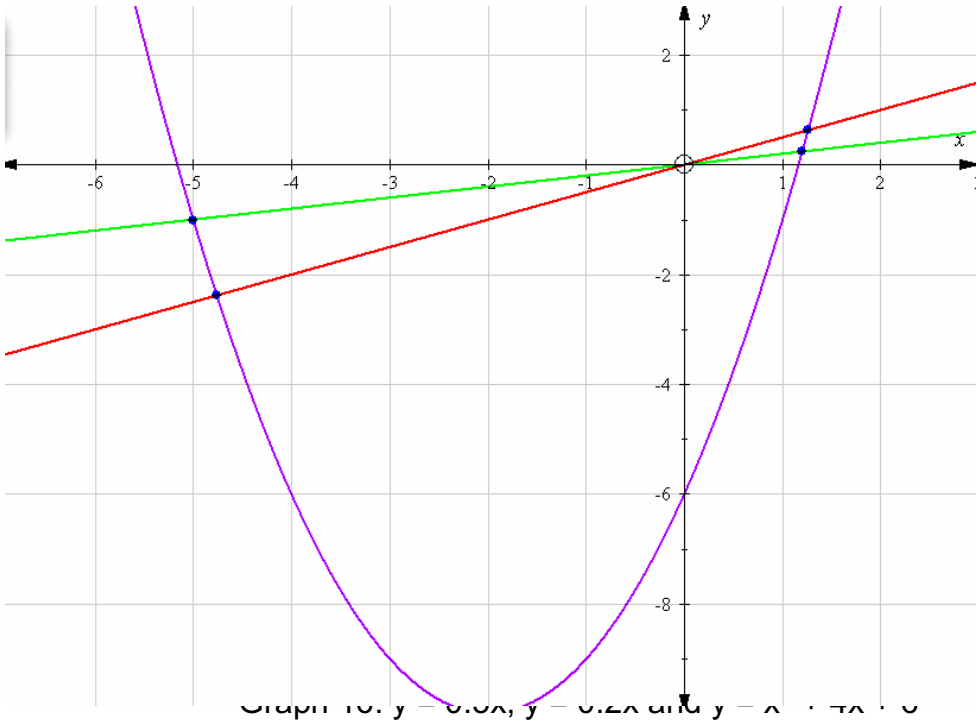
is not needed. The new conjecture for D is:

$D = (y_2 - y_1)\,(|\,a^{-1}\,|)$. The graph below proves this conjecture.



Graph 16: y = 0.5x, y = 0.2x and y = x² + 4x + 6

Using the above conjecture it would be expected that the value for D for Graph 16 is $(0.5 - 0.2)\,(1^{-1}) = 0.3$ which is correct. Therefore the conjecture for graphs with different intersecting lines is

$D = (y_2 - y_1)\,(|\,a^{-1}\,|)$ or $D = \dfrac{|y_2 - y_1|}{a}$

This can be proven algebraically

If $y = ax^2 + bx + c$, $y = y_1x$ and $y = y_2x$

$D = |\,S_L - S_R\,|$

$D = |\,(x_2 - x_1) - (x_4 - x_3)\,|$

$D = |\,(x_2 + x_3) - (x_4 + x_1)\,|$

x_2 and x_3 are the intersections between $y = y_2x$ and $y = ax^2 + bx + c$

x_1 and x_4 are the intersections between $y = y_1x$ and $y = ax^2 + bx +$

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c

By substituting into the above equations it can be found that

$y = ax^2 + (b - y_2)x + c$ and $y = ax^2 + (b - y_1)x + c$

x_1 and x_4 are the roots of the equation which means $x_1 + x_4$ is equal to the sum of the roots which is defined by the equation $\frac{-b}{a}$

By substituting into the formula for D we can see that

$$D = \left| (x_2 + x_3) - (x_4 + x_1) \right|$$

$$D = \left| \frac{(b - y_1)}{a} - \frac{(b - y_2)}{a} \right|$$

$$D = \left| \frac{(b - y_1 - b + y_2}{a} \right|$$

$$D = \left| \frac{y_2 - y_1}{a} \right|$$

This is proof for the above conjecture.

5. Determine whether a conjecture can be made for cubic polynomials.

To determine whether a conjecture can be made for cubic polynomials in the form $ax^3 + bx^2 + cx + d$ there must be a system made for naming the intersections and how D is calculated with these values. Graph 17 gives an example of what a cubic polynomial would look like intersecting each line three times.

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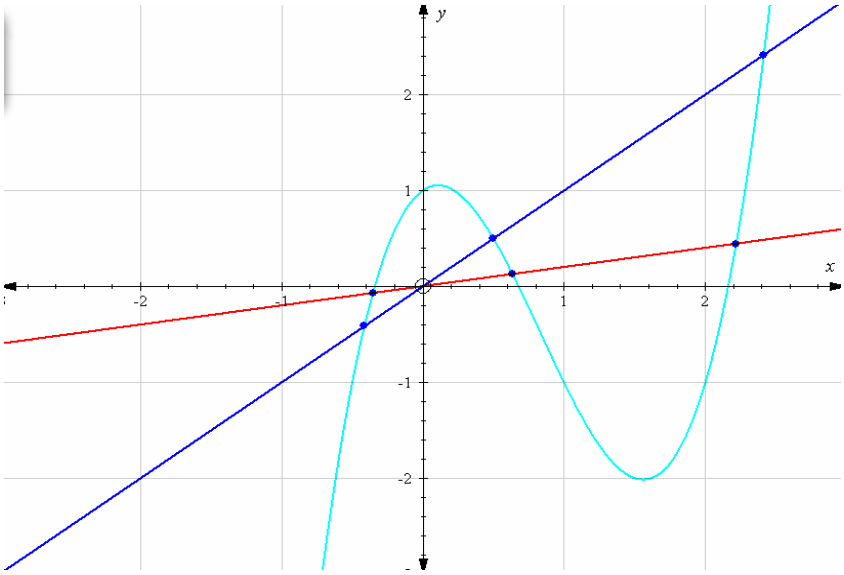

Graph 17: $y = 0.2x$, $y = x$, $y = 2x^3 - 5x^2 + x + 1$

If the conjecture for quadratics was correct the value for D for the above graph would be 0.4. This is not true and the conjecture for quadratics is untrue. The intersections must be named differently to get a value for D. The new formula to find D is:

$$D = \left| (x_2 - x_1) - (x_4 - x_3) - (x_6 - x_5) \right|$$

Where the points of intersection for the line with a higher gradient are labelled x_1, x_3 and x_6 from left to right and for the other line is x_2, x_4 and x_5. The values S_L is still $x_2 - x_1$, but the two others are different. S_M is equal to $x_4 - x_3$ and S_R is $x_6 - x_5$. With this definition the value for D is equal to:

Graph 17	
X_1	-0.41
X_2	-0.35
X_3	0.5
X_4	0.636
X_5	2.2181
X_6	2.4141
S_L	0.06

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S_M	0.136
S_R	0.196
D	0

$D = | S_L - S_M - S_R |$

Using graph 17 the following table was made calculating the value for D.

Through various trials with other cubic polynomials the same value for D being 0 was found. Therefore a new conjecture can be made that D = 0 a≠ 0.

6. Consider whether the conjecture might be modified to include higher order polynomials.

The general equation for a polynomial is:

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$

It can also be written as:

$f(x) = ax^n + bx^{n-1} + \ldots + qx + k$

Using the equation the roots will be $q_1, q_2, \ldots. q_n$

The sum of the roots as stated above is $\frac{-b}{a}$

When a polynomial intersects with another polynomial of order x^{n-1} i.e.

$y_1 = c_1x^{n-1} + d_1xn^{-2} + \ldots + m_1x^1 + k_1$ and $y_2 = c_2x^{n-1} + d_2xn^{-2} + \ldots + m_2x^1 + k_2$ the sum of the roots will always be $\frac{-b - y_1}{a}$ and $\frac{-b - y_2}{a}$

The two roots will be the

Therefore D will be equal to:

$D = |(\text{Sum of roots in } y_1) - (\text{sum of roots in } y_2)|$

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$$D = \left| \left(\frac{-b - y_1}{a}\right) - \left(\frac{-b - y_2}{a}\right) \right|$$

$$D = \left| \frac{-b - y_1 + b + y_2}{a} \right|$$

$$D = \frac{y_2 - y_1}{a}$$

That is the conjecture for intersecting lines one degree lower than the polynomial, this is the same conjecture as question two, because question two fit the criteria. If the intersecting lines are more than one order apart a different conjecture must be used which is stated in question five. The conjecture is:

$D = 0$

Absolute value symbols are not needed as zero is the same whether they are there or not.

Conclusion

There is a definite relationship between the intersecting lines and the parabola. This was found through using quadratic equations and visual aids of graphs to show how a value for D was found throughout the investigation. Through the use of proof to justify each answer it could be used as a way to check each answer and each answer could be used to check the proof, using this method increased the likelihood of the conjectures being correct.