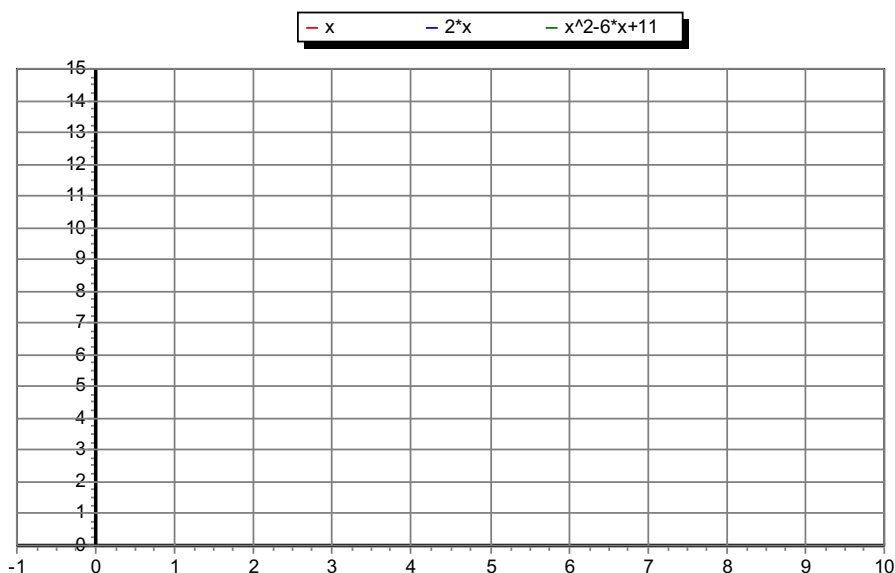


Ken Chen

Consider the parabola $y = (x - 3)^2 + 2 = x^2 - 6x + 11$, which is drawn together with the lines $y = x$ and $y = 2x$ on the same axes.

Using the program called Qax Grapher, which has functions same as a GDC, the parabola $y = (x - 3)^2 + 2$ and the two lines $y = x$ and $y = 2x$ are graphed as illustrated below.

Qax Grapher



There are 4 intersections in the axes which have the x-values labeled as x_1 , x_2 , x_3 and x_4 as they appear from left to right on the x-axis.

x_1 and x_4 – the x-values of the intersections between the line $y = 2x$ and the parabola $y = x^2 - 6x + 11$ on the left and right hand side of the graph respectively.

x_2 and x_3 – the x-values of the intersections between the line $y = x$ and the parabola $y = x^2 - 6x + 11$ on the left and right hand side of the graph respectively.

Using GDC, these x-values can be found.

$$x_1 = 1.764 \quad x_2 = 2.382 \quad x_3 = 4.618 \quad x_4 = 6.236$$

Call SL as the difference between x_1 and x_2 and SR as the difference between x_3 and x_4 .

$$SL = x_2 - x_1 = 2.382 - 1.764 = 0.618$$

and

$$SR = x_4 - x_3 = 6.236 - 4.618 = 1.618$$

D is the difference between the two differences SL and SR.

$$D = |SL - SR| = 1.618 - 0.618 = 1.$$

In order to find out the pattern of the D-value, other parabolas which have same characteristics as the parabola above (*the parabolas need to be in the form $y = ax^2 + bx + c$ ($a > 0$) with vertices in quadrant 1*) are considered.

- Consider the parabolas with **a = 1**

When the parabola $y = (x - 4)^2 + 1 = x^2 - 8x + 17$ and the lines $y = x$ and $y = 2x$ are graphed, four values of x are obtained.

$$x_1 = 2.172 \quad x_2 = 2.697 \quad x_3 = 6.303 \quad x_4 = 7.828$$

$$SL = x_2 - x_1 = 2.697 - 2.172 = 0.525$$

$$SR = x_4 - x_3 = 7.828 - 6.303 = 1.525$$

$$D = |SL - SR| = 1.525 - 0.525 = 1.$$

When the parabola $y = (x - 3)^2 + 3 = x^2 - 6x + 12$ and the lines $y = x$ and $y = 2x$ are graphed, four values of x are obtained.

$$x_1 = 2.000 \quad x_2 = 3.000 \quad x_3 = 4.000 \quad x_4 = 6.000$$

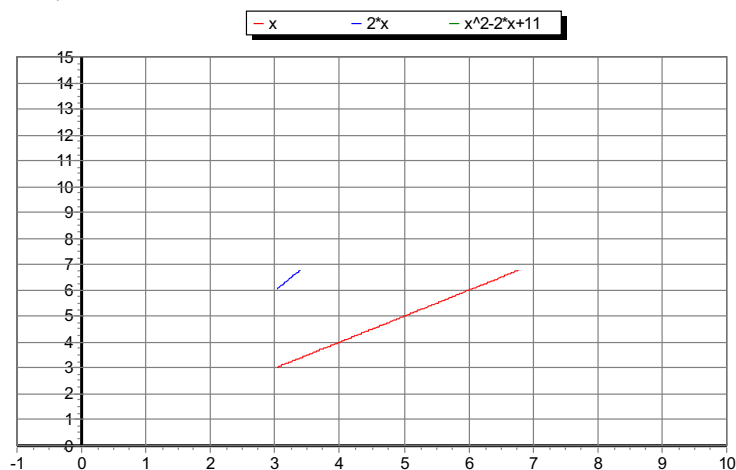
$$SL = x_2 - x_1 = 3.000 - 2.000 = 1.000$$

$$SR = x_4 - x_3 = 6.000 - 4.000 = 2.000$$

$$D = |SL - SR| = 2.000 - 1.000 = 1.$$

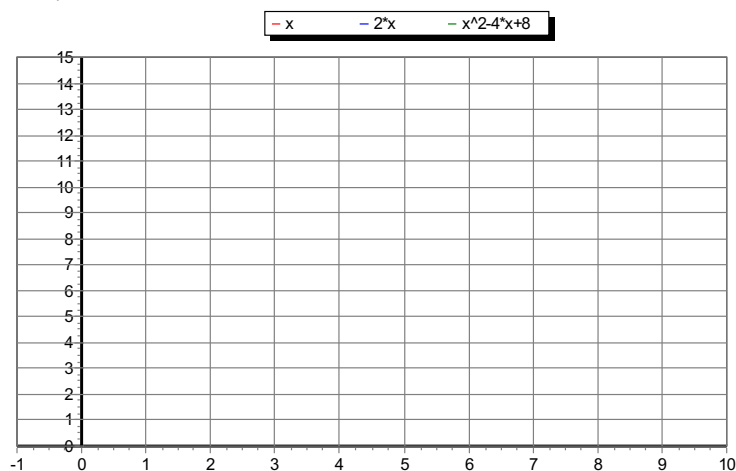
When the parabola $y = x^2 - 2x + 11$ is graph together with the two lines $y = x$ and $y = 2x$, there is no intersections between the parabola and the lines therefore there is no x-value obtained. Thus the D-value cannot be found.

Qax Grapher



When the function of the parabola is $y = x^2 - 4x + 8$, there are only intersections between the line $y = 2x$ and the parabola so only x_1 and x_3 are known. Therefore, the D-value cannot be found.

Qax Grapher

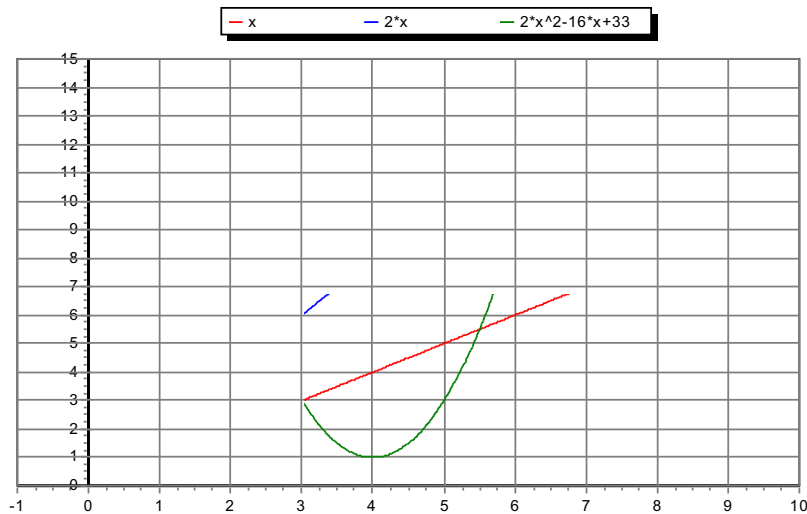


→ It can be seen that when D-value is 1 when the parabolas have $a = 1$ and there are four intersections between the parabola and the lines.

- Consider the parabolas with $a = 2$

When the parabola $y = 2(x - 4)^2 + 1$ and the lines $y = x$ and $y = 2x$ are graphed, four values of x are obtained.

Qax Grapher



$$x_1 = 2.564 \quad x_2 = 3.000 \quad x_3 = 5.500 \quad x_4 = 6.436$$

$$SL = x_2 - x_1 = 3.000 - 2.564 = 0.436$$

$$SR = x_4 - x_3 = 6.436 - 5.500 = 0.936$$

$$D = |SL - SR| = 0.936 - 0.436 = 0.5 = \text{Error! Reference source not found..}$$

When the parabola $y = 2(x - 2)^2 + 1$ and the lines $y = x$ and $y = 2x$ are graphed, four values of x are obtained.

$$x_1 = 1.177 \quad x_2 = 1.500 \quad x_3 = 3.000 \quad x_4 = 3.823$$

$$SL = x_2 - x_1 = 1.500 - 1.177 = 0.323$$

$$SR = x_4 - x_3 = 3.823 - 3.000 = 0.823$$

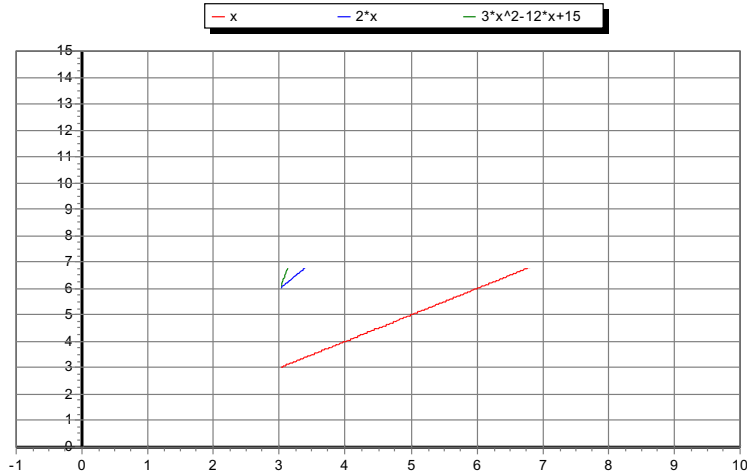
$$D = |SL - SR| = 0.823 - 0.323 = 0.5 = \text{Error! Reference source not found..}$$

→ It can be seen that when D-value is **Error! Reference source not found.** when the parabolas have $a = 2$ and there are four intersections between the parabola and the lines.

- Consider the parabolas with $a = 3$

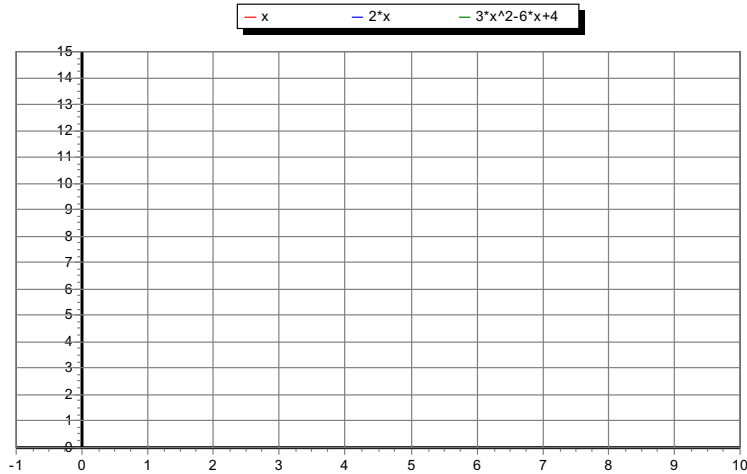
When the parabola $y = 3(x - 2)^2 + 3$ and the lines $y = x$ and $y = 2x$ are graphed, the parabola does not cut the line $y = x$ so there is only two x -values and the D-value cannot be found.

Qax Grapher



When the parabola $y = 3(x - 1)^2 + 1$ and the lines $y = x$ and $y = 2x$ are graphed, four values of x are obtained.

Qax Grapher



$$x_1 = 0.667 = \text{Error! Reference source not found.} \quad x_2 = 1.000 \quad x_3 = 1.333 = \text{Error!}$$

$$\text{Reference source not found.} \quad x_4 = 2.000$$

$$SL = x_2 - x_1 = 1.000 - \text{Error! Reference source not found.} = \text{Error! Reference source not found.}$$

$$SR = x_4 - x_3 = 2.000 - \text{Error! Reference source not found.} = \text{Error! Reference source not found.}$$

$$D = |SL - SR| = \text{Error! Reference source not found.} - \text{Error! Reference source not found.} = \text{Error! Reference source not found.}$$

When the parabola $y = 3(x - 4)^2 + 2$ and the lines $y = x$ and $y = 2x$ are graphed, four values of x are obtained.

$$x_1 = 2.880 \quad x_2 = 3.333 \quad x_3 = 5.000 \quad x_4 = 5.786$$

$$SL = x_2 - x_1 = 3.333 - 2.880 = 0.453$$

$$SR = x_4 - x_3 = 5.786 - 5.000 = 0.786$$

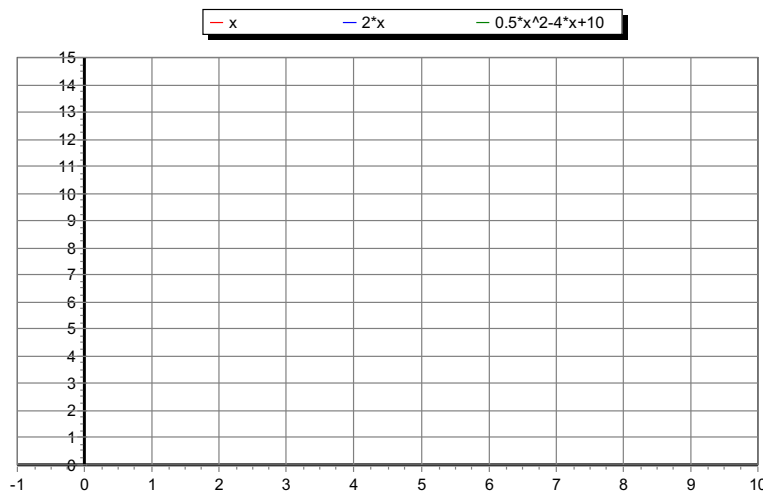
$$D = |SL - SR| = 0.786 - 0.453 = 0.333 = \text{Error! Reference source not found.}$$

→ It can be seen that when D-value is **Error! Reference source not found.** when the parabolas have $a = 3$ and there are four intersections between the parabola and the lines.

- Consider the parabolas with $a = \text{Error! Reference source not found.}$

When the parabola $y = \text{Error! Reference source not found.} (x - 4)^2 + 2$ and the lines $y = x$ and $y = 2x$ are graphed, four values of x are obtained.

Qax Grapher



$$x_1 = 2.000 \quad x_2 = 2.764 \quad x_3 = 7.236 \quad x_4 = 10.000$$

$$SL = x_2 - x_1 = 2.764 - 2.000 = 0.764$$

$$SR = x_4 - x_3 = 10.000 - 7.236 = 2.764$$

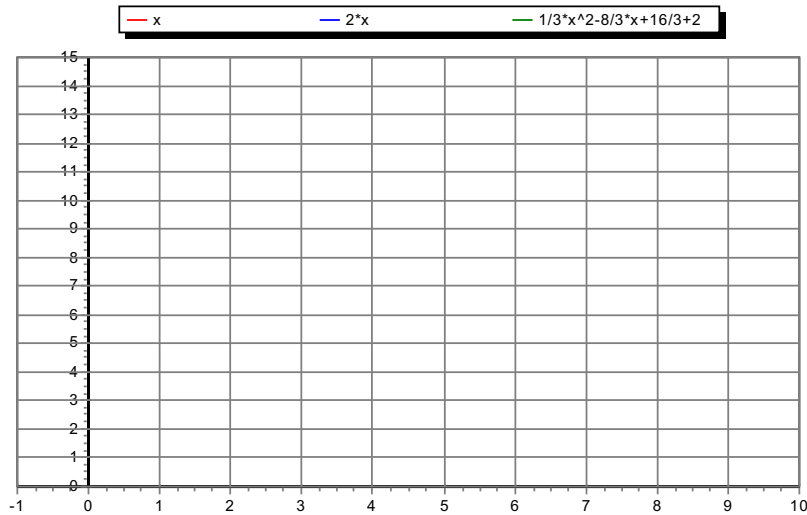
$$D = |SL - SR| = 2.764 - 0.764 = 2$$

→ It can be seen that when D-value is 2 when the parabolas have a = **Error! Reference source not found.** and there are four intersections between the parabola and the lines.

- Consider the parabolas with a = **Error! Reference source not found.**

When the parabola $y = \text{Error! Reference source not found.}$ $(x - 4)^2 + 2$ and the lines $y = x$ and $y = 2x$ are graphed, four values of x are obtained.

Qax Grapher



$$x_1 = 1.804 \quad x_2 = 2.628 \quad x_3 = 8.372 \quad x_4 = 12.196$$

$$SL = x_2 - x_1 = 2.628 - 1.804 = 0.824$$

$$SR = x_4 - x_3 = 12.196 - 8.372 = 3.824$$

$$D = |SL - SR| = 3.824 - 0.824 = 3$$

→ It can be seen that when D-value is 3 when the parabolas have a = **Error! Reference source not found.** and there are four intersections between the parabola and the lines.

From these examples, the **conjecture** can be obtained.

*“For any four intersections of the two lines $y = x$, $y = 2x$ and the parabola $ax^2 + bx + c$ ($a > 0$ and its vertex is in quadrant 1); the value of D which is calculated by $D = |(x_2 - x_1) - (x_4 - x_3)|$ equals to **Error! Reference source not found.**”*

Testing the Validity of the Statement

To test for the validity of the conjecture above, other parabolas which have **a as a real number and have vertices place in any quadrants** are investigated.

Firstly, the pattern in intersections of any parabolas that have **real value of a** and the lines $y = x$ and $y = 2x$ is investigated out to see whether the conjecture above still holds true.

- $a = -4$ (a **Error! Reference source not found.** Z)

When the parabola $y = -4x^2 + 32x - 45$ and the lines $y = x$ and $y = 2x$ are graphed, four values of x are obtained.

$$x_1 = 1.934 \quad x_2 = 2.073 \quad x_3 = 5.427 \quad x_4 = 5.816$$

$$SL = x_2 - x_1 = 2.073 - 1.934 = 0.139$$

$$SR = x_4 - x_3 = 5.816 - 5.427 = 0.389$$

$$D = |SL - SR| = 0.389 - 0.139 = 0.25 = \text{Error! Reference source not found.}$$

- $a = \text{Error! Reference source not found.}$ a **Error! Reference source not found.** Q)

When the parabola $y = -\frac{1}{2}x^2 + 3x + 5$ and the lines $y = x$ and $y = 2x$ are graphed, four values of x are obtained.

$$x_1 = -2.317 \quad x_2 = -1.742 \quad x_3 = 5.742 \quad x_4 = 4.317$$

$$SL = x_2 - x_1 = -1.742 + 2.317 = 0.575$$

$$SR = x_4 - x_3 = 4.317 - 5.742 = -1.425$$

$$D = |SL - SR| = 0.575 + 1.425 = 2.$$

[...]

Formal proof

To prove the conjecture mathematically, the parabolas with $a = k$ (k **Error! Reference source not found.** R) is considered.

x_2 and x_3 – the x -value of the intersections of the line $y = x$ and the parabola $y = kx^2 + bx + c$

$$\text{Error! Reference source not found. } kx^2 + bx + c = x$$

$$\text{Error! Reference source not found. } kx^2 + (b - 1)x + c = 0$$

$$\Delta = (b - 1)^2 - 4kc$$

$$\text{Error! Reference source not found. } x_2, x_3 = \text{Error! Reference source not found.}$$

$$\text{Error! Reference source not found. } x_2 + x_3 = \text{Error! Reference source not found.} = \text{Error!}$$

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x_1 and x_4 – the x -value of the intersections of the line $y = 2x$ and the parabola $y = kx^2 + bx + c$

$$\text{Error! Reference source not found. } kx^2 + bx + c = 2x$$

$$\text{Error! Reference source not found. } kx^2 + (b - 2)x + c = 0$$

$$\Delta = (b - 2)^2 - 4kc$$

Error! Reference source not found. $x_1, x_4 =$ **Error! Reference source not found.**

Error! Reference source not found. $x_1 + x_4 =$ **Error! Reference source not found.** = **Error!**

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$$D = |(x_2 - x_1) - (x_4 - x_3)| = |(x_2 + x_3) - (x_1 + x_4)| = | \text{Error! Reference source not found.} - \text{Error! Reference source not found.} \text{Error! Reference source not found.}$$

Therefore, the conjecture is mathematically proven.

Further investigation

[The pattern in intersections of the parabolas and two different intersecting lines]

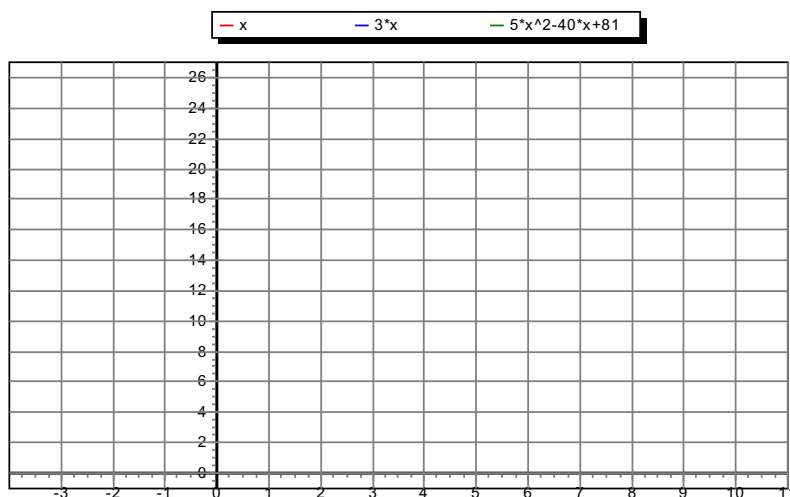
For further investigation, the intersecting lines are changed to see whether the conjecture holds true.

Many different combinations of intersecting lines other than $y = x$ and $y = 2x$ are considered.

- $y = x$ and $y = 3x$

When the parabola $y = 5(x - 4)^2 + 1$ and the lines $y = x$ and $y = 3x$ are graphed, four values of x are obtained.

Qax Grapher



x_1 and x_4 – the x -values of the intersections between the line $y = 3x$ and the parabola on the left and right hand side of the graph respectively.

x_2 and x_3 – the x-values of the intersections between the line $y = x$ and the parabola on the left and right hand side of the graph respectively.

$$x_1 = 2.787 \quad x_2 = 3.319 \quad x_3 = 4.881 \quad x_4 = 5.813$$

$$SL = x_2 - x_1 = 3.319 - 2.787 = 0.532$$

$$SR = x_4 - x_3 = 5.813 - 4.881 = 0.932$$

$$D = |SL - SR| = 0.932 - 0.532 = 0.400 = \text{Error! Reference source not found.}$$

- **$y = 2x$ and $y = 3x$**

When the parabola $y = 5(x - 4)^2 + 1$ and the lines **$y = 2x$ and $y = 3x$** are graphed, four values of x are obtained.

x_1 and x_4 – the x-values of the intersections between the line $y = 3x$ and the parabola on the left and right hand side of the graph respectively.

x_2 and x_3 – the x-values of the intersections between the line $y = 2x$ and the parabola on the left and right hand side of the graph respectively.

$$x_1 = 2.787 \quad x_2 = 3.000 \quad x_3 = 5.400 \quad x_4 = 5.813$$

$$SL = x_2 - x_1 = 3.000 - 2.787 = 0.213$$

$$SR = x_4 - x_3 = 5.813 - 5.400 = 0.413$$

$$D = |SL - SR| = 0.413 - 0.213 = 0.2 = \text{Error! Reference source not found.}$$

- **$y = x$ and $y = 5x$**

When the parabola $y = 5(x - 4)^2 + 1$ and the lines **$y = x$ and $y = 5x$** are graphed, four values of x are obtained.

x_1 and x_4 – the x-values of the intersections between the line $y = 5x$ and the parabola on the left and right hand side of the graph respectively.

x_2 and x_3 – the x-values of the intersections between the line $y = x$ and the parabola on the left and right hand side of the graph respectively.

$$x_1 = 2.488 \quad x_2 = 3.319 \quad x_3 = 4.881 \quad x_4 = 6.513$$

$$SL = x_2 - x_1 = 3.319 - 2.488 = 0.832$$

$$SR = x_4 - x_3 = 6.513 - 4.881 = 1.632$$

$$D = |SL - SR| = 1.632 - 0.832 = 0.8 = \text{Error! Reference source not found.}$$

Justification

The conjecture above can be proved as below.

x_2 and x_3 – the x-value of the intersections of the line $y = mx$ and the parabola $y = kx^2 + bx + c$

Error! Reference source not found. $kx^2 + bx + c = mx$

Error! Reference source not found. $kx^2 + (b - m)x + c = 0$

$$\Delta = (b - m)^2 - 4kc$$

Error! Reference source not found. $x_2, x_3 = \text{Error! Reference source not found.}$

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x_1 and x_4 – the x-value of the intersections of the line $y = nx$ and the parabola $y = kx^2 + bx + c$

Error! Reference source not found. $kx^2 + bx + c = nx$

Error! Reference source not found. $kx^2 + (b - n)x + c = 0$

$$\Delta = (b - n)^2 - 4kc$$

Error! Reference source not found. $x_1, x_4 = \text{Error! Reference source not found.}$

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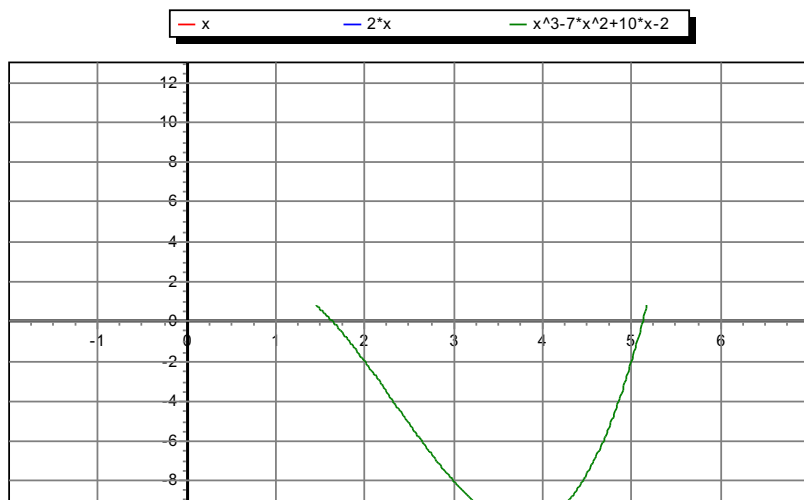
$D = |(x_2 - x_1) - (x_4 - x_3)| = |(x_2 + x_3) - (x_1 + x_4)| = | \text{Error! Reference source not found.} - \text{Error! Reference source not found.} \text{Error! Reference source not found.}$

[The pattern in intersections of the graph of cubic polynomial and the two lines $y = x$ and $y = 2x$]

In this section, the parabola is replaced by the cubic polynomial. The cubic polynomial will be drawn together with the two lines $y = x$ and $y = 2x$ to see whether there is any pattern in intersections of these lines.

Consider the cubic $y = x^3 - 7x^2 + 10x - 2$ and the lines $y = x$ and $y = 2x$. There are 6 intersections formed.

Qax Grapher



x_1, x_2 and x_3 – the x-values of the intersections between the line $y = x$ and the cubic from left to right respectively.

x_4, x_5 and x_6 – the x-values of the intersections between the line $y = 2x$ and the cubic from left to right respectively.

$$x_1 = 0.281 \quad x_2 = 1.316 \quad x_3 = 5.403 \quad x_4 = 0.354 \quad x_5 = 1.000 \quad x_6 = 5.646$$

SL is the difference of the two left hand roots x_1 and x_4 .

$$SL = x_4 - x_1 = 0.354 - 0.281 = 0.073$$

SM is the difference of the two middle roots x_2 and x_5 .

$$SM = x_2 - x_5 = 1.316 - 1.000 = 0.316$$

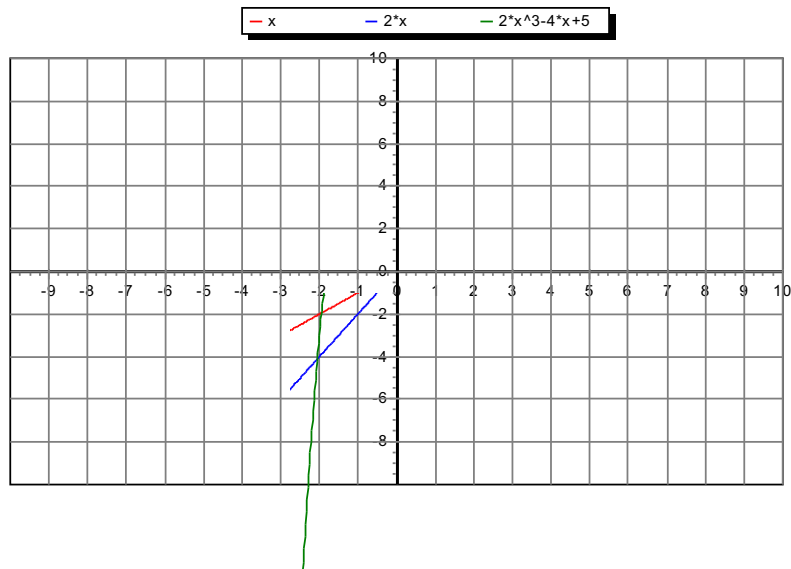
SR is the difference of the two right hand roots x_3 and x_6 .

$$SR = x_6 - x_3 = 5.646 - 5.403 = 0.243$$

→ It can be seen that $SM = SL + SR$ ($0.243 + 0.073 = 0.316$) or $x_1 + x_2 + x_3 = x_4 + x_5 + x_6$ ($= 7$).

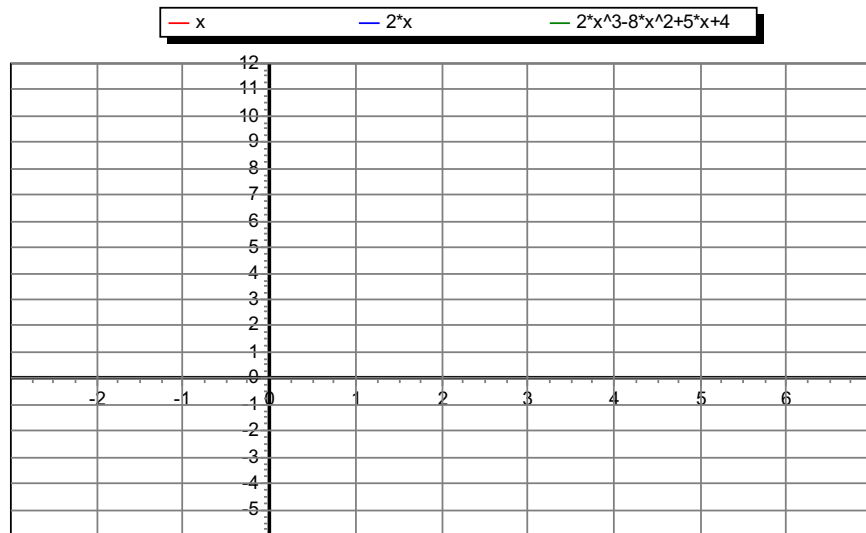
Consider the cubic $y = 2x^3 - 4x + 5$. There are only two intersections between the line $y = 2x$ and the cubic therefore there is no relationship deduced from it.

Qax Grapher



Consider the cubic $y = 2x^3 - 8x^2 + 5x + 4$. There are 6 intersections.

Qax Grapher



$$x_1 = -0.481 \quad x_2 = 1.311 \quad x_3 = 3.170 \quad x_4 = -0.520 \quad x_5 = 1.138 \quad x_6 = 3.382$$

$$SL = 0.520 - 0.481 = 0.039$$

$$SM = 1.311 - 1.138 = 0.173$$

$$SR = 3.382 - 3.170 = 0.212$$

$$\rightarrow SR = SM + SL \quad (0.039 + 0.173 = 0.212)$$

From these examples, the conjecture can be deduced:

“For any 6 intersections of any cubic polynomial $y = ax^3 + bx^2 + cx + d$ (**Error! Reference source not found.**) and the lines $y = x$ and $y = 2x$, there are three differences between the x -values of six roots SL , SM , SR . The sum of the two smaller differences equals to the bigger difference, i.e. $|x_1 - x_4| + |x_2 - x_5| = |x_3 - x_6|$.”

Justification

For the general cubic equation $ax^3 + bx^2 + cx + d = 0$, the sum of the roots is **Error! Reference source not found.** [Viète's theorem]

x_1 , x_2 and x_3 are the roots of the equation

$$ax^3 + bx^2 + cx + d = x$$

$$ax^3 + bx^2 + (c - 1)x + d = 0$$

$$\rightarrow x_1 + x_2 + x_3 = \text{Error! Reference source not found.}$$

x_4 , x_5 and x_6 are the roots of the equation

$$ax^3 + bx^2 + cx + d = 2x$$

$$ax^3 + bx^2 + (c - 2)x + d = 0$$

$$\rightarrow x_1 + x_2 + x_3 = \text{Error! Reference source not found.}$$

\rightarrow It can be seen that the sum of the roots of the cubic equations does not depend on the coefficient of x .

$$\rightarrow x_1 + x_2 + x_3 = x_4 + x_5 + x_6$$

$$\rightarrow |x_1 - x_4| + |x_2 - x_5| = |x_3 - x_6|$$

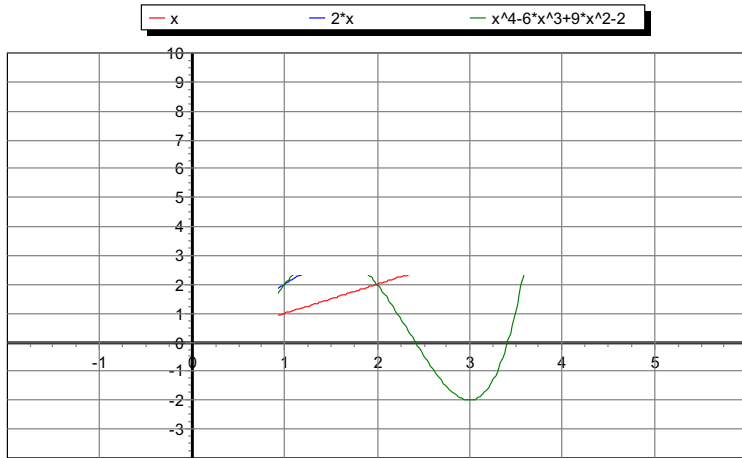
[The pattern in intersections of the graph of higher order polynomial and the two lines $y = x$ and $y = 2x$]

In this section, the parabola is replaced by the higher order polynomial (start from 4). The cubic polynomial will be drawn together with the two lines $y = x$ and $y = 2x$ to see whether there is any pattern in intersections of these lines.

- **Quadric polynomial**

Consider $y = x^4 - 6x^3 + 9x^2 - 2$. There are 8 intersections.

QaxGrapher



$$x_1 = -0.377 \quad x_2 = 0.726 \quad x_3 = 2.000 \quad x_4 = 3.651 \quad x_5 = -0.343 \quad x_6 = 1.000$$

$$x_7 = 1.529 \quad x_8 = 3.814$$

x_1, x_2, x_3, x_4 are the x-values of the intersections of the quadric polynomial and the $y = x$

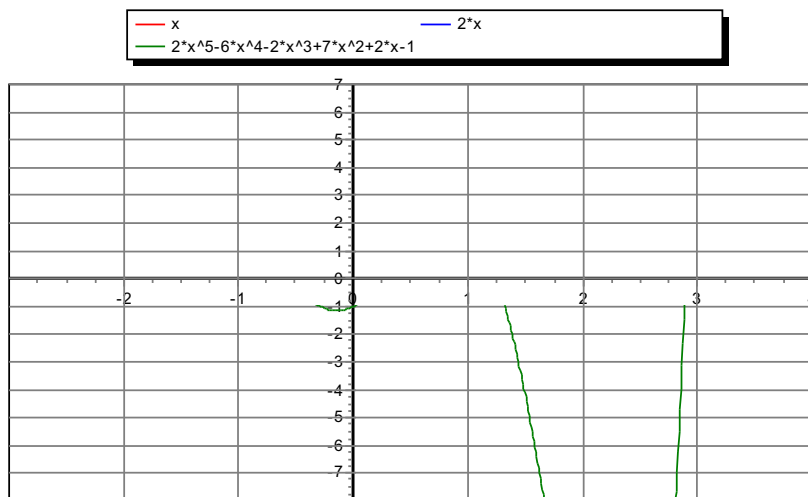
x_5, x_6, x_7, x_8 are the x-values of the intersections of the quadric polynomial and the $y = 2x$

$$D = (x_1 + x_2 + x_3 + x_4) - (x_5 + x_6 + x_7 + x_8) = 6 - 6 = 0.$$

- Polynomial of order 5**

Consider $y = 2x^5 - 6x^4 - 2x^3 + 7x^2 + 2x - 1$. There are 10 intersections.

Qax Grapher



$$x_1 = -0.914 \quad x_2 = -0.484 \quad x_3 = 0.340 \quad x_4 = 1.139 \quad x_5 = 2.919 \quad x_6 = -1.000$$

$$x_7 = -0.384 \quad x_8 = 0.442 \quad x_9 = 1.000 \quad x_{10} = 2.942$$

x_1, x_2, x_3, x_4, x_5 are the x-values of the intersections of the quadric polynomial and the $y = x$

$x_6, x_7, x_8, x_9, x_{10}$ are the x-values of the intersections of the quadric polynomial and the $y = 2x$

$$D = (x_1 + x_2 + x_3 + x_4 + x_5) - (x_6 + x_7 + x_8 + x_9 + x_{10}) = 3 - 3 = 0$$

The conjecture can be made: "The difference between the sum of all the x-values of the intersections of the polynomial and the line $y = x$ and the sum of all those of the intersections of the polynomial the lines $y = 2x$ equals to 0"

Justification

The justification is same as the justification for cubic polynomial. According to the Viète's theorem, it can be seen that the sum of all the roots is independent of the coefficient of x . So the sum always equals to **Error! Reference source not found.** which is a constant for any intersections of the polynomial (with order higher than 3) and any changing lines.