LOGARITHM BASES

This internal assessment focuses on the logarithms. There are a few rules which govern all the concepts of logarithms:

$\log_a b$=c, a^c=b   where a>0, a≠1, b>0

$\log_a b = \frac{\log_c b}{\log_c a}$

- Consider the following sequences. Write down the next two terms of each sequence.

log□8, log□8, log□8, log□□8, log□□8, log□□8, log□□□8

log□81, log□81, log□□81, log□□81, log□□□81, log□□□81

log□25, log□□25, log□□□25, log□□□25, log□□□□25, log□□□□□25

:
$\log_m m^k$, $\log_{m^2} m^k$, $\log_{m^3} m^k$,$\log_{m^4} m^k$, $\log_{m^5} m^k$, $\log_{m^6} m^k$

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- Find an expression for the n^{th} term of each sequence. Write down your expression in the form $\frac{p}{q}$, where p, q $\in \mathbb{Z}$. Justify your answers using technology.

1. $\log_{2^n} 8 = \log_{2^n} 2^3$

Use the $\log_a b = \frac{\log_c b}{\log_c a}$ rule.

$$\frac{\log_2 2^3}{\log_2 2^n}$$

Then we apply the rule: $\log_c b^a$ = a $\log_c b$.

$$\frac{\log_2 2^3}{\log_2 2^n} = \frac{3\log_2 2}{n\log_2 2}$$

We can cross $\log_2 2$ away on both sides.

What remains is: $\frac{3}{n}$

We can do this for all the rows.

2. $\log_{3^n} 81 = \log_{3^n} 3^4$

Use the $\log_a b = \frac{\log_c b}{\log_c a}$ rule.

$$\frac{\log_3 3^4}{\log_3 3^n}$$

Then we apply the rule: $\log_c b^a$ = a $\log_c b$

$$\frac{\log_3 3^4}{\log_3 3^n} = \frac{4\log_3 3}{n\log_3 3}$$

We can cross $\log_3 3$ away on both sides.

What remains is: $\frac{4}{n}$

3. $\log_{5^n} 25 = \log_{5^n} 5^2$

Use the $\log_a b = \frac{\log_c b}{\log_c a}$ rule.

$$\frac{\log_5 5^2}{\log_5 5^n}$$

Then we apply the rule: $\log_c b^a$ = a $\log_c b$

$$\frac{\log_5 5^2}{\log_5 5^n} = \frac{2\log_5 5}{n\log_5 5}$$

We can cross $\log_5 5$ away on both sides.

What remains is: $\frac{2}{n}$

4. Expressed in m, n and k.

$\log_{m^n} m^k$

Use the $\log_a b = \frac{\log_c b}{\log_c a}$ rule and then we can change the base. We change the base to 10.

$$\frac{\log_m m^k}{\log_m m^n}$$

Then we apply the rule: $\log_c b^a$ = a $\log_c b$

$$\frac{\log_m m^k}{\log_m m^n} = \frac{\text{k}\log_m m}{n\log_m m}$$

We can cross $\log_m m$ away on both sides.

What remains is: $\frac{k}{n}$. Derived from this we can conclude that the general expression for the nth term of each sequence in the form $\frac{p}{q}$ thus is $\frac{k}{n}$.

Examples to justify this statement using technology:

$\log_4 8 = \log_{2^2} 2^3$ $(\log_{m^n} m^k)$

$\frac{\log 8}{\log 4} = \frac{3}{2} = 1{,}5$

$\log_{27} 81 = \log_{3^3} 3^4$ $(\log_{m^n} m^k)$

$\frac{\log 81}{\log 27} = \frac{4}{3} \approx 1{,}33$

$\log_{\frac{1}{5}} 125 = \log_{5^{-1}} 5^3 \quad (\log_{m^n} m^k)$

$\frac{\log 125}{\log \frac{1}{5}} = \frac{3}{-1} \approx -3$

- Now calculate the following, giving your answer in the form $\frac{p}{q}$, where p, q $\in \mathbb{Z}$.

The answer was $\frac{k}{n}$. This form will be used.

1. $\log_4 64 = \log_{2^2} 2^6 \quad (\log_{m^n} m^k)$

$\frac{\log 64}{\log 4} = \frac{6}{2} = 3$

$\log_8 64 = \log_{2^3} 2^6 \quad (\log_{m^n} m^k)$

$\frac{\log 64}{\log 8} = \frac{6}{3} = 2$

$\log_{32} 64 = \log_{2^5} 2^6 \quad (\log_{m^n} m^k)$

$\frac{\log 64}{\log 32} = \frac{6}{5} = 1{,}2$

2. $\log_7 49 = \log_{7^1} 7^2 \quad (\log_{m^n} m^k)$

$\frac{\log 49}{\log 7} = \frac{2}{1} = 2$

$\log_{49} 49 = \log_{7^2} 7^2 \quad (\log_{m^n} m^k)$

$\frac{\log 49}{\log 49} = \frac{2}{2} = 1$

$\log_{343} 49 = \log_{7^3} 7^2 \quad (\log_{m^n} m^k)$

$\frac{\log 49}{\log 343} = \frac{2}{3} \approx 0{,}67$

3. $\log_{\frac{1}{5}} 125 = \log_{5^{-1}} 5^3 \quad (\log_{m^n} m^k)$

$\frac{\log 125}{\log \frac{1}{5}} = \frac{3}{-1} = -3$

$\log_{\frac{1}{125}} 125 = \log_{5^{-3}} 5^3 \quad (\log_{m^n} m^k)$

$\frac{\log 125}{\log\frac{1}{125}} = \frac{3}{-3} = -1$

$\log_{\frac{1}{625}} 125 = \log_{5^{-4}} 5^3$ $(\log_{m^n} m^k)$

$\frac{\log 125}{\log\frac{1}{625}} = \frac{3}{-4} = -0{,}75$

4. $\log_8 512 = \log_{2^3} 2^9$ $(\log_{m^n} m^k)$

$\frac{\log 512}{\log 8} = \frac{9}{3} = 3$

$\log_2 512 = \log_{2^1} 2^9$ $(\log_{m^n} m^k)$

$\frac{\log 512}{\log 2} = \frac{9}{1} = 9$

$\log_{16} 512 = \log_{2^4} 2^9$ $(\log_{m^n} m^k)$

$\frac{\log 512}{\log 16} = \frac{9}{4} = 2{,}25$

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- Describe how to obtain the third answer in each row from the first two answers. Create two more examples that fit the pattern above.

1. $\log_4 64, \log_8 64, \log_{32} 64 = \log_{2^2} 2^6, \log_{2^3} 2^6, \log_{2^5} 2^6$

As we can see, n=2 in the first logarithm and in the second logarithm n=3. If we add these together, we get n=2+3=5. That means that in the first row, the third answer is obtained by adding the first two n up together. The pattern is therefore that you add up the two n in front of the next logarithm.

The next two examples which would fit in the pattern would therefore be:

$\log_{2^8} 2^6, \log_{2^{13}} 2^6 = \log_{256} 64, \log_{8192} 64$

2. $\log_7 49, \log_{49} 49, \log_{343} 49 = \log_{7^1} 7^2, \log_{7^2} 7^2, \log_{7^3} 7^2$

As we can see, n=1 in the first logarithm and in the second logarithm n=2. There is an arithmetic increase, with the fixed number of 1. The next number in the second row will therefore be n=3. The pattern thus is that there is an arithmetic increase with the fixed number of 1.

The next two examples which would fit the pattern would therefore be:

$\log_{7^4} 7^2, \log_{7^5} 7^5 = \log_{2401} 49, \log_{16087} 49$

3. $\log_{\frac{1}{5}} 125, \log_{\frac{1}{125}} 125, \log_{\frac{1}{625}} 125 = \log_{5^{-1}} 5^3, \log_{5^{-3}} 5^3, \log_{5^{-4}} 5^3$

As we can see, n=-1 in the first logarithm and in the second logarithm n= -3. If we add these up together, we get n= -1 + -3 = -4. That means that the third row, the third answer is obtained by adding the first two n up together. The pattern is therefore that you add up the two n in front of the next logarithm.

The next two examples which would fit the pattern would therefore be:

$\log_{5^{-7}} 5^3, \log_{5^{-11}} 5^3 = \log_{\frac{1}{78125}} 125,\ \log_{\frac{1}{48828125}} 125$

4. $\log_8 512, \log_2 512, \log_{16} 512 = \log_{2^3} 2^9, \log_{2^1} 2^9, \log_{2^4} 2^9$

As we can see, n=3 in the first logarithm and in the second logarithm n=1. Something has to have been before the n=3, which means in front of the fir st logarithm. It should have started from n=0, as we can derive, from the second to the third logarithm, wherein there is an increase in n of 3. The pattern is therefore that you add 3 and you subtract 2 from the next logarithm and so forth.

The next two examples which would fit the pattern would therefore be:

$\log_{2^2} 2^9, \log_{2^5} 2^9 = \log_4 512,\ \log_{32} 512$

- Let $\log_a x$=c and $\log_b x$=d. Find **the** general statement that expresses $\log_{ab} x$ in terms of c and d.

$\log_a x$=c and $\log_b x$=d then find $\log_{ab} x$

One law of logarithms state that:

$$\log_a x + \log_b x = \log_{ab} x$$

We use the change of base rule:

$\log_a x$=c then a^c=x

$\therefore \log_a c$=$\log x$

$\log_b x$=d then b^d=x

$\therefore \log_b d$=$\log x$

We are taking logarithms in base x:

$\therefore c \log_x a$=$\log_x x$

$\therefore c = \frac{\log_x x}{\log_x b} = \frac{1}{\log_x a}$

$\therefore d \log_x b = \log_x x$

$\therefore$ d = $\frac{\log_x x}{\log_x b} = \frac{1}{\log_x b}$

Derived from $\log_a x + \log_b x = \log_{ab} x$ we can state that:

$$\frac{1}{\log_x a} + \frac{1}{\log_x b} = \frac{1}{(\log_x a + \log_x b)}$$

If we change the base again we get the following equation:

$$\frac{1}{\left(\frac{1}{\log_a x} + \frac{1}{\log_b x}\right)}$$

We substitute $\log_a x = c\ and\ \log_b x = d$:

$$\frac{1}{(\frac{1}{c}+\frac{1}{d})}$$

The following step is to multiply both sides by cd:

$$\frac{cd}{(c+d)}$$

The general statement that expresses $\log_{ab} x$ in terms of c and d thus is:

$$\frac{cd}{(c+d)}$$

- Test the validity of your general statement using other values of a, b, and x.

$\log_a x$=c and $\log_b x$=d

1. Example: a=2, b=4, x=8

$\log_2 8 = c \; and \; \log_4 8 = d$

$\log_2 8 + \log_4 8 = \log_8 8 = 1$

Check with the general statement:

c = $\log_2 8 = \log_{2^1} 2^3 = \frac{3}{1} = 3$

d = $\log_4 8 = \log_{2^2} 2^3 = \frac{3}{2} = 1{,}5$

$\frac{cd}{(c+d)} = \frac{(3x1{,}5)}{(3+1{,}5)} = 1$

General statement justified.

2. Example: a=5, b=125, x=25

$\log_5 25 = c \; and \; \log_{125} 25 = d$

$\log_5 25 + \log_{125} 25 = \log_{625} 25 = 0{,}5$

Check with the general statement:

c = $\log_5 25 = \log_{5^1} 5^2 = \frac{2}{1} = 2$

d = $\log_{125} 25 = \log_{5^3} 5^2 = \frac{2}{3} \approx 0{,}67$

$\frac{cd}{(c+d)} = \frac{(2x\frac{2}{3})}{(2+\frac{2}{3})} = 0{,}5$

General statement justified.

3. Example: a=10000, b=$\frac{1}{10}$, x=10

$\log_{10000} 10 = c \text{ and } \log_{\frac{1}{10}} 10 = d$

$\log_{10000} 10 + \log_{\frac{1}{10}} 10 = \log_{1000} 10 = \frac{1}{3} \approx 0{,}33$

Check with the general statement:

c = $\log_{10000} 10 = \log_{10^4} 10^1 = \frac{1}{4} = 0{,}25$

d = $\log_{\frac{1}{10}} 10 = \log_{10^{-1}} 10^1 = \frac{1}{-1} \approx -1$

$\frac{cd}{(c+d)} = \frac{(\frac{1}{4}x(-1))}{(\frac{1}{4}-1)} = \frac{1}{3} \approx 0{,}33$

General statement justified.

- Discuss the scope and/or limitations of a, b, and x.

The limitations of logarithms are usually, as stated in the second sentence of this internal assessment:

a>0, a≠1, b>0;

which would mean for this question that the limitations are:

a>0, b>0, a≠1, b≠1, x>0

We can do a check for this:

Example: a=-2, b=2, x=4

$\log_{-2} 4 = c \text{ and } \log_{2} 4 = d$

$\log_{-2} 4 + \log_{2} 4 = \log_{-4} 4 = \text{Not possible}$

It is impossible to power a function which results in a negative number, in this case a = -2. With these numbers: $\log_{2^n} 2^2$, n>0.

The same applies for b, that a>0.

Example 2: a=10, b=1, x=100

$\log_{10} 100 = c \; and \; \log_1 100 = d$

$\log_{10} 100 + \log_1 100 = Not \; possible$

N=0 as $\log_{10^0} 10^1$, which makes it impossible, as you have to divide that number and $\frac{1}{0}$ = error/not possible.

The sample applies for a, that b≠1.

Example 3: a=4, b=8, c=-8

$\log_4 -8 = c \; and \; \log_8 -8 = d$

$\log_4 -8 + \log_8 -8 = Not \; possible$

Same reason as in example 1: it is impossible to power a function which results in a negative number, in this case x (=-8). With these numbers: $\log_{2^3} 2^k$, k>0.

As a>0 and b>0, the product x sho uld always be greater than 0, therefore x>0.

To sum up again:

a>0, b>0, a≠1, b≠1, x>0

- Explain how you arrived at your general statement.

One law of logarithms state that:

$\log_a x + \log_b x = \log_{ab} x$

We use the change of base rule

$\log_a x$=c then a^c=x

$\log_b x$=d then b^d=x

∴ $\log_a c$=$\log x$

∴ $\log_b d$=$\log x$

Take logarithms in base x:

∴ $c \log_x a$=$\log_x x$

∴ $d \log_x b = \log_x x$

∴ $c = \frac{\log_x x}{\log_x a} = \frac{1}{\log_x a}$

∴ d = $\frac{\log_x x}{\log_x b} = \frac{1}{\log_x b}$

Derived from $\log_a x + \log_b x = \log_{ab} x$ we can state that:

$$\frac{1}{\log_x a}+\frac{1}{\log_x b}=\frac{1}{(\log_x a+\log_x b\ )}$$

If we change the base again we get the following equation:

$$\frac{1}{(\frac{1}{\log_a x}+\frac{1}{\log_b x})}$$

We substitute $\log_a x = c\ and\ \log_b x = d$:

$$\frac{1}{(\frac{1}{c}+\frac{1}{d})}$$

The last and following step is to multiply both sides by cd:

$$\frac{cd}{(c+d)}$$