

Newton College

# Math's Portfolio SL Type 1 “Matrix Powers”

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## Introduction

Matrices are tables of numbers or any algebraic quantities that can be added or multiplied in a specific arrangement. A matrix is a block of numbers that consist of columns and rows used to present raw data, store information or to perform certain mathematical operations

The aim of this portfolio is to find a general trend in different sets of matrices, therefore find a general formula that applies for all of the matrices. The pattern will then be explained and tested to see if applies correctly to all the sets of matrices.

## Method

To calculate  $M^n$  for  $n = 2, 3, 4, 5, 10, 20, 50$  I used a GDC.

Determinants are defined as  $ad-bc$  in a  $2 \times 2$  matrix,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and is denoted by  $\det A = |A|$ . In

Other words,  $\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$ .

$$M^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \det(M^2) = 16 = 4^2$$

$$M^3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}, \det(M^3) = 64 = 4^3$$

$$M^4 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}, \det(M^4) = 256 = 4^4$$

$$M^5 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix}, \det(M^5) = 1024 = 4^5$$

$$M^{10} = \begin{bmatrix} 1024 & 0 \\ 0 & 1024 \end{bmatrix}, \det(M^{10}) = 1048576 = 4^{10}$$

$$M^{20} = \begin{bmatrix} 1056 & 0 \\ 0 & 1056 \end{bmatrix}, \det(M^2) = 1.099511628 \times 10^{12} = 4^{20}$$

$$M^{50} = \begin{bmatrix} 1.28907 \times 10^5 & 0 \\ 0 & 1.28907 \times 10^5 \end{bmatrix}, \det(M^2) = 1.2676506 \times 10^{30} = 4^{50}$$

By squaring each number in the matrix by the power of  $M^n$  you get the answer for each of the matrices. So if  $M^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ , you then square the matrix, so you multiply it by itself. .

Multiplying the matrix by itself as this:  $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$  gives you the new matrix which is, in this case  $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ .

Here, in the case of matrix  $P^n$ , if we divide each number in the resultant matrix after replacing  $n$ , by  $2^{n-1}$  we notice a pattern in the numbers inside these new matrices. To calculate this we use a GDC.

$$P^2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^2 = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} = 2 \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}, \det(P^2) = 64 = 8^2 \text{ (here } k \text{ is } 8-4)$$

$$P^3 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^3 = \begin{bmatrix} 36 & 28 \\ 28 & 36 \end{bmatrix} = 2^2 \begin{bmatrix} 9 & 7 \\ 7 & 9 \end{bmatrix}, \det(P^3) = 512 = 8^3 \text{ (here } k \text{ is } 8)$$

$$P^4 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^4 = \begin{bmatrix} 116 & 110 \\ 110 & 116 \end{bmatrix} = 2^3 \begin{bmatrix} 17 & 15 \\ 15 & 17 \end{bmatrix}, \det(P^4) = 4096 = 8^4 \text{ (here } k \text{ is } 8 \times 2)$$

$$P^5 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^5 = \begin{bmatrix} 528 & 466 \\ 466 & 528 \end{bmatrix} = 2^4 \begin{bmatrix} 33 & 31 \\ 31 & 33 \end{bmatrix}, \det(P^5) = 32768 = 8^5 \text{ (here } k \text{ is } 8 \times 4)$$

Given these matrices, we can notice that when the matrix is simplified by  $2^{n-1}$  you get a new matrix which the numbers can be found by using the number multiplying it as  $k$  in the general

formula  $\begin{bmatrix} k+1 & k-1 \\ k-1 & k+1 \end{bmatrix}$ . A pattern between the matrices' determinants is noticeable. The

determinant of each matrix is 8 powered by the same number you powered the matrix.

If  $P^3 = \begin{bmatrix} 36 & 28 \\ 28 & 36 \end{bmatrix}$ , then it's certain that its determinant is going to be  $8^3$  which is 512.

To calculate the following matrices you use a GDC. You do the same thing to the resultant matrix of  $S^n$ , you divide each number inside the matrix by  $2^{n-1}$ .

$$S^2 = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}^2 = \begin{bmatrix} 20 & 16 \\ 16 & 20 \end{bmatrix} = 2 \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}, \det(S^2) = 144 = 12^2 \text{ (here } k \text{ is } 12-3)$$

$$S^3 = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}^3 = \begin{bmatrix} 112 & 104 \\ 104 & 112 \end{bmatrix} = 2^2 \begin{bmatrix} 28 & 26 \\ 26 & 28 \end{bmatrix}, \det(S^3) = 1728 = 12^3 \text{ (here } k \text{ is } (12 \times 2) + 3)$$

$$S^4 = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}^4 = \begin{bmatrix} 666 & 640 \\ 640 & 666 \end{bmatrix} = 2^3 \begin{bmatrix} 82 & 80 \\ 80 & 82 \end{bmatrix}, \det(S^4) = 20736 = 12^4 \text{ (here } k \text{ is } (12 \times 7) - 3)$$

$$S^5 = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}^5 = \begin{bmatrix} 3904 & 3872 \\ 3872 & 3904 \end{bmatrix} = 2^4 \begin{bmatrix} 244 & 242 \\ 242 & 244 \end{bmatrix}, \det(S^5) = 248832 = 12^5 \text{ (here } k \text{ is } (12 \times 20) + 3)$$

Given these matrices, we notice that when simplifying the matrices we also use  $2^{n-1}$  as the factor. We can use the general formula of  $k$  to find the numbers inside the new matrix and still a pattern between the matrices' determinants can be noticed. The determinant of each matrix is 12 powered by the same number you powered the matrix.

If  $S^3 = \begin{bmatrix} 112 & 104 \\ 104 & 112 \end{bmatrix}$ , this matrix's determinant is  $12^3$ , which is 1728, this confirms the trend I

stated before. The trend in matrix  $P$  is the same in matrix  $S$  but 4 numbers more. The formula in matrix  $P$  is  $8^n$ , and  $8 + 4 = 12$  therefore the formula in matrix  $S$  is  $12^n$ .

I will make an example for  $k = 4$ , so the matrix will be  $V = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ .

We can predict according to the determinant's pattern that the determinants will now be  $16^n$ .

$$V^2 = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}^2 = \begin{bmatrix} 34 & 30 \\ 30 & 34 \end{bmatrix} = 2 \begin{bmatrix} 17 & 15 \\ 15 & 17 \end{bmatrix}, \det(V^2) = 256 = 16^2 \text{ (here } k \text{ is } 16)$$

$$V^3 = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}^3 = \begin{bmatrix} 206 & 222 \\ 222 & 206 \end{bmatrix} = 2^2 \begin{bmatrix} 65 & 63 \\ 63 & 65 \end{bmatrix}, \det(V^3) = 4096 = 16^3 \text{ (here } k \text{ is } 16 \times 4)$$

$$V^4 = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}^4 = \begin{bmatrix} 1266 & 1340 \\ 1340 & 1266 \end{bmatrix} = 2^3 \begin{bmatrix} 257 & 255 \\ 255 & 257 \end{bmatrix}, \det(V^4) = 65536 = 16^4 \text{ (here } k \text{ is } 16 \times 16)$$

$$V^5 = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}^5 = \begin{bmatrix} 7940 & 8168 \\ 8168 & 7940 \end{bmatrix} = 2^4 \begin{bmatrix} 1025 & 1023 \\ 1023 & 1025 \end{bmatrix}, \det(V^5) = 1048576 = 16^5 \text{ (here } k \text{ is } 16 \times 64)$$

These matrices follow the trend, because  $12 + 4 = 16$  so the determinants of these matrices must be  $16^n$ , where  $n$  is the power of the matrix. To prove this, select a random matrix such as:

$$V^5 = \begin{bmatrix} 160 & 168 \\ 168 & 160 \end{bmatrix}, \text{ according to the pattern the determinant of this matrix should be } 16 \text{ to}$$

the same power in which the matrix is powered to, in this case (5), which then  $16^5$  gives 1048576 as your answer.

On the other hand, the pattern to find the numbers inside the new matrix doesn't exist here when the factor of simplification is  $2^{n+1}$ . This a limitation for this pattern and we cannot continue it.

The pattern for matrices  $M$ ,  $P$ ,  $S$  and  $V$  is given by the general formula  $\begin{bmatrix} k+1 & k-1 \\ k-1 & k+1 \end{bmatrix}$ .

As the formula for the matrix  $M^n$  is  $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^n$ , to generalize the results in terms of  $k$  and  $n$  we can join both formulae to get one general formula for the pattern in matrices  $M$ ,  $P$ ,  $S$  and  $V$ . The new formula will be  $\begin{bmatrix} k+1 & k-1 \\ k-1 & k+1 \end{bmatrix}^n$ . We can apply this to  $P$ ,  $S$  and  $V$  to prove that the formula is true.

Matrix  $P$  occurs when  $k$  is 2 so the formula will give us  $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$  which happens to be the case.

If we then square the matrix, so  $n = 2$  we get:

$$P^2 = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

Matrix  $S$  occurs when  $k$  is 3 so the formula will give us  $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$  which happens to be true.

If we then square the matrix, so  $n = 2$  we get:

$$S^2 = \begin{bmatrix} 20 & 16 \\ 16 & 20 \end{bmatrix}$$

Matrix  $V$  occurs when  $k$  is 4 so the formula will give us  $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$  which happens to follow the trend.

If we then square the matrix, so  $n = 2$  we get:

$$V^2 = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

Furthermore, the determinants in each set of matrices follow a common pattern between them. In the set of matrices  $M^n$ ,  $P^n$ ,  $S^n$ ,  $V^n$  the determinants have a pattern in each of them. In the set of matrices  $M^n$ , the determinants are always equal to  $4^n$ . In  $P^n$ , the determinants are given by the formula  $8^n$ . In the set of matrices  $S^n$ , the determinants can be found by replacing 'n' in  $12^n$ . Finally in  $V^n$ , the determinants are given by the formula  $16^n$ .

If you then consider this individual pattern of each set of matrices at a bigger scale you can notice another trend that applies to these set of matrices as a whole. This pattern is easy to see if we display the information in a sequence.

$$\begin{array}{ccccccccc}
 \det(M^2) = 4^2, & \det(M^3) = 4^3, & \det(M^4) = 4^4, & \det(M^5) = 4^5 & \dots & \text{so } \det(M^n) = 4^n \\
 \downarrow +4 & \downarrow +4 & \downarrow +4 & \downarrow +4 & & \downarrow +4 \\
 \det(P^2) = 8^2, & \det(P^3) = 8^3, & \det(P^4) = 8^4, & \det(P^5) = 8^5 & \dots & \text{so } \det(P^n) = 8^n \\
 \downarrow +4 & \downarrow +4 & \downarrow +4 & \downarrow +4 & & \downarrow +4 \\
 \det(S^2) = 12^2, & \det(S^3) = 12^3, & \det(S^4) = 12^4, & \det(S^5) = 12^5 & \dots & \text{so } \det(S^n) = 12^n \\
 \downarrow +4 & \downarrow +4 & \downarrow +4 & \downarrow +4 & & \downarrow +4 \\
 \det(V^2) = 16^2, & \det(V^3) = 16^3, & \det(V^4) = 16^4, & \det(V^5) = 16^5 & \dots & \text{so } \det(V^n) = 16^n
 \end{array}$$

Every time  $k$  increases by one, the matrix's determinants formula increases by 4.

I also noticed that the letters that represent each matrix increase by 4 as well. This is why I select the letter 'V' to represent the matrix formed when  $k = 4$ .

To prove the validity of my statements I will consider a further value of  $k$  after making a hypothesis about the results of that same value.

For  $k = 5$ , the letter should be 'Y' according to the letter's pattern, and according to the formula the matrix should be  $\begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}$  so you end up with  $Y = \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}$ . Now if we power this matrix so as  $n = 2, 3, 4, 5$  we should see the same pattern as the former sets of matrices. The matrices determinants for this set should follow the trend and should be equal to  $20^n$ .

$$Y^2 = \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}^2 = \begin{bmatrix} 20 & 48 \\ 48 & 20 \end{bmatrix} = 2^4 \begin{bmatrix} 3.25 & 3 \\ 3 & 3.25 \end{bmatrix}, \det(Y^2) = 400 = 20^2$$

$$Y^3 = \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}^3 = \begin{bmatrix} 54 & 46 \\ 46 & 54 \end{bmatrix} = 2^5 \begin{bmatrix} 15.75 & 15.5 \\ 15.5 & 15.75 \end{bmatrix}, \det(Y^3) = 8000 = 20^3$$

$$Y^4 = \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}^4 = \begin{bmatrix} 508 & 492 \\ 492 & 508 \end{bmatrix} = 2^6 \begin{bmatrix} 78.25 & 78 \\ 78 & 78.25 \end{bmatrix}, \det(Y^4) = 160000 = 20^4$$

$$Y^5 = \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}^5 = \begin{bmatrix} 5006 & 4984 \\ 4984 & 5006 \end{bmatrix} = 2^7 \begin{bmatrix} 390.75 & 390.5 \\ 390.5 & 390.75 \end{bmatrix}, \det(Y^5) = 3200000 = 20^5$$

This further value of  $k$ , confirms my previous statements about the generalized pattern of these sets of matrices, it also proves the formula and trend of the determinants in each set of matrices.

## **Conclusions**

A general trend was found in the different sets of matrices provided by the assignment sheet. A formula was created, so that it could be applied to generate more matrices that follow the pattern found. The results were satisfactory due to the few limitations that the trend had. It can be considered a limitation that a matrix when powered to 0 gives you always the identity matrix, so  $n \neq 0$ . Another limitation found is that is useless to power a matrix to 1, because it will remain unchanged. Another fact I found was that when  $k =$  negative numbers, the answers are exactly the same as the answers for the positive numbers.