Maths Portfolio

SL Type 1

Infinite Surds

In this mathematics portfolio we are instructed to investigate different expression of infinite surds in square root form and then find the exact value and statement for these surds.

INFINITE SURDS

The following expression is an example of an infinite surd.

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}$$

The first ten terms of the surd can be expressed in the sequence:

$$a_1 = \sqrt{1+\sqrt{1}} = 1.414213562$$

$$a_2 = \sqrt{1+\sqrt{1+\sqrt{1}}} = 1.553773974$$

$$a_3 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}} = 1.598053182$$

$$a_4 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}} = 1.611847754$$

$$a_5 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}} = 1.616121207$$

$$a_6 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}} = 1.617442799$$

$$a_7 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}} = 1.617851291$$

$$a_8 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}} = 1.617977531$$

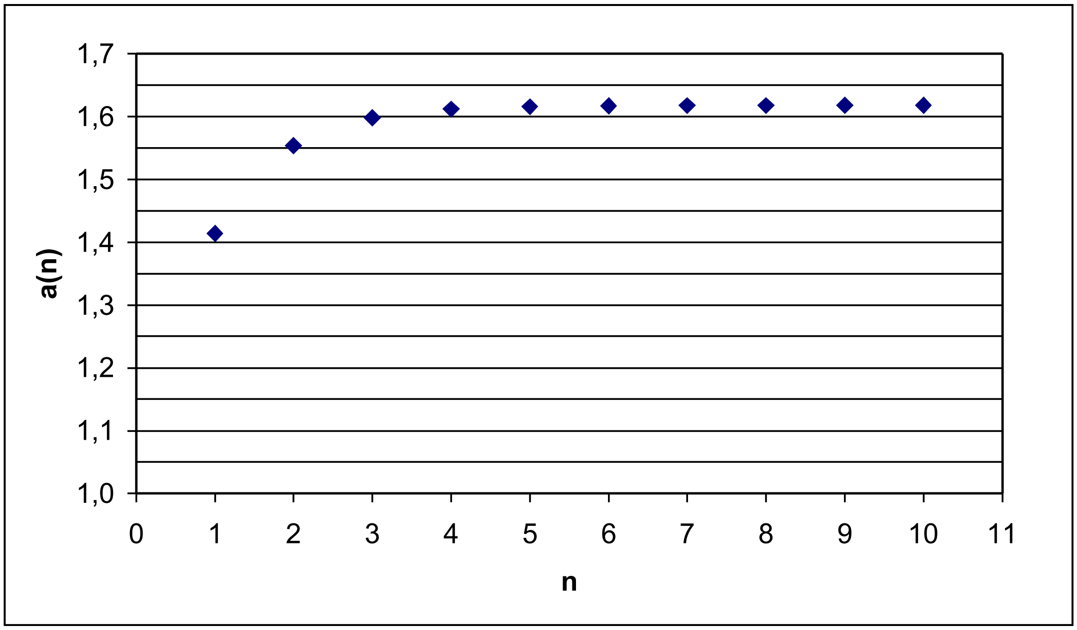
$$a_9 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}}} = 1.618016542$$

$$a_{10} = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}}}}}}}} = 1.618028597$$

From the tem terms of the sequence we can observe that the formu la for the sequence is displayed as:

$$a_{n+1} = \sqrt{1 + a_n}$$

The results had to be plotted in a graph as shown below:



The graph above shows us the relationship between *n* and a_n. We can observe that as *n* increases, a_n also increases but each time less than before, suggesting that at a large certain point of *n* it stops increasing and just follows in a straight line.

To find the exact infinite value for this sequence we would use the equation and rearrange in the correct way to find the value.

Finding the exact value for the surd

$$a_{n+1} = \sqrt{1 + a_n}$$

$$a_n = \sqrt{1 + a_n}$$

In this equation we can observe that the +1 from the a_{n+1} is eliminated as, we can see in the graph, that *n* and *n*+1 are very close values with a difference of almost zero. Like this we can assume a_{n+1} as a_n for the solving of the equation.

$$\frac{1+\sqrt{5}}{2} = 1.618033989$$

$$\frac{1-\sqrt{5}}{2} = -0.6180339885$$

The exact value for the surd is 1.618033989 as the second answer does not fit in the problem.

Another example of an infinite surd is:

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+...}}}}$$

The first ten terms of this surd can be expressed in the sequence:

$$a_1 = \sqrt{2+\sqrt{2}} = 1.847759065$$

$$a_2 = \sqrt{2+\sqrt{2+\sqrt{2}}} = 1.961570561$$

$$a_3 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} = 1.990369453$$

$$a_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} = 1.997590912$$

$$a_5 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}} = 1.999397637$$

$$a_6 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}} = 1.999849404$$

$$a_7 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}} = 1.999962351$$

$$a_8 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}} = 1.999990588$$

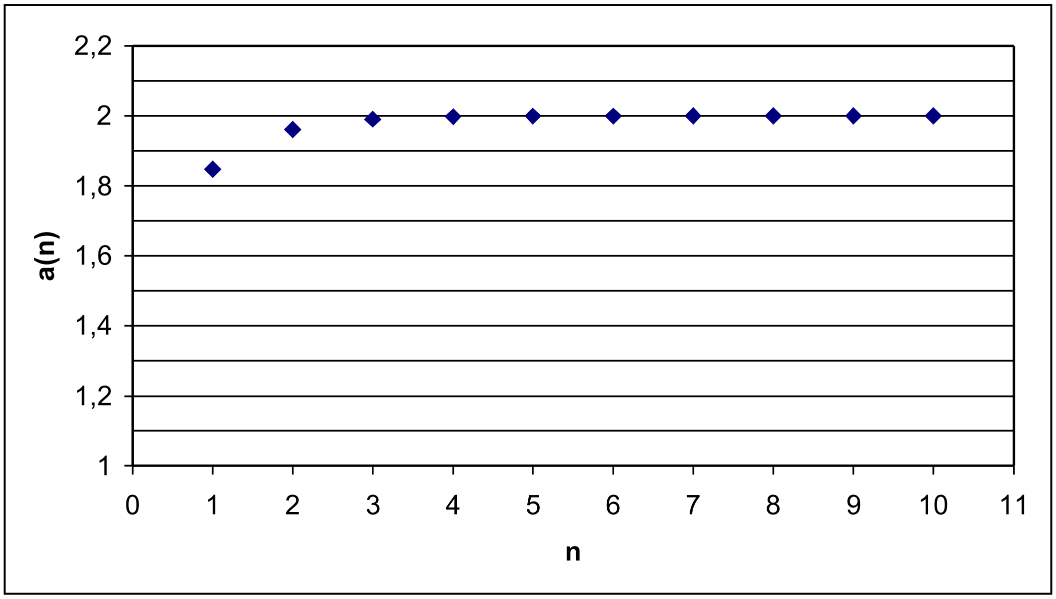
$$a_9 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}} = 1.999997647$$

$$a_{10} = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}}}}} = 1.999999412$$

From the tem terms of the sequence we can observe that the formula for the sequence is displayed as:

$$a_{n+1} = \sqrt{2 + a_n}$$

The results had to be plotted in a graph as shown below:



The graph above also shows us the relationship between *n* and a_n. We can observe that as *n* increases, a_n also increases but each time less than befo re, suggesting that at a large certain point of *n* it stops increasing and just follows in a straight line.

To find the exact infinite value for this sequence we would use the equation and rearrange in the correct way to find the value.

Finding the exact value for the surd

$$a_{n+1} = \sqrt{2 + a_n}$$

$$a_n = \sqrt{2 + a_n}$$

$$a_n^2 = a_n + 2$$

$$a_n^2 - a_n - 2 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{1 \pm \sqrt{-1^2 - 4x1x - 2}}{2x1}$$

$$\frac{1 \pm \sqrt{1 + 8}}{2}$$

$$\frac{1 \pm \sqrt{9}}{2}$$

$$\frac{1 + \sqrt{9}}{2} = 2$$

$$\frac{1 - \sqrt{9}}{2} = -1$$

In this equation we can observe that the +1 from the a_{n+1} is eliminated as, we can see in the graph, that *n* and *n*+1 are very close values with a difference of almost zero. Like this we can assume a_{n+1} as a_n for the solving of the equation.

The exact value for the surd is 2 as the second answer does not fit in the problem

The general infinite surd in terms of *k*

$$\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\dots}}}}}$$

Finding the expression for the exact value of the surd in terms of *k*:

$$a_{n+1} = \sqrt{k+a_n}$$

$$a_n = \sqrt{k+a_n}$$

$$a_n^2 = a_n + k$$

$$a_n^2 - a_n - k = 0$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\frac{1 \pm \sqrt{-1^2 - 4x1x-k}}{2x1}$$

$$\frac{1 \pm \sqrt{1+4k}}{2}$$

In this equation we can observe that the +1 from the a_{n+1} is eliminated as, we can see in the graph, that *n* and *n*+1 are very close values with a difference of almost zero. Like this we can assume a_{n+1} as a_n for the solving of the equation.

The value of a surd is not always an integer. An integer is an entire number, not a decimal or a fraction.

Finding values of k that make the expression an integer.

$$k = \frac{1+\sqrt{1+4k}}{2}$$

$$k(1) = \frac{1+\sqrt{1+4x1}}{2} = 1.618033989$$

$$k(2) = \frac{1+\sqrt{1+4x2}}{2} = 2$$

$$k(3) = \frac{1+\sqrt{1+4x3}}{2} = 2.302775638$$

$$k(4) = \frac{1+\sqrt{1+4x4}}{2} = 2.561552813$$

$$k(5) = \frac{1+\sqrt{1+4x5}}{2} = 2.791287847$$

$$k(6) = \frac{1+\sqrt{1+4x6}}{2} = 3$$

$$k(12) = \frac{1+\sqrt{1+4x12}}{2} = 4$$

$$k(20) = \frac{1+\sqrt{1+4x20}}{2} = 5$$

a_1 a_2 a_3 a_4 a_5

Sequence: 2, 6, 12, 20, 30

$k - n =$ 1, 4, 9, 16, 25

$$k - n = n^2$$

Therefore the general statement that represents all the values of k for which the expression is an integer is:

$$k = n^2 + n$$

To prove my equation is valid I choose a term for n and found the value of *k*. This value gives an integer in the expression for the exact value of the general infinite surd.

$$k = n^2 + n$$
$$k = 88^2 + 88$$
$$k = 7832$$

$$k = \frac{1+\sqrt{1+4k}}{2}$$
$$k = \frac{1+\sqrt{1+4x7832}}{2}$$
$$k = \frac{1+\sqrt{31329}}{2}$$
$$k = \frac{1+177}{2} = \frac{178}{2}$$
$$k = 89$$

$$k = n^2 + n$$
$$k = 231^2 + 231$$
$$k = 53592$$

$$k = \frac{1+\sqrt{1+4k}}{2}$$
$$k = \frac{1+\sqrt{1+4x53592}}{2}$$
$$k = \frac{1+\sqrt{214369}}{2}$$
$$k = \frac{1+463}{2} = \frac{464}{2}$$
$$k = 232$$

The results show us that the limitations found in the general statement are those that didn't give an integer as a final answer. Therefore *k* has to be an integer, however not a decimal or a fraction as results wouldn't be a whole number. It also isn't possible to obtain an integer if *k* is a negative number because there is no square root for a negative number. For a pleasing result, 1+4*k* , needs to give a number that possesses a perfect square root, an integer.

I arrived at this general statement by subtracting the *n* term from the *k*, and like that finding out that the product of this subtraction is the *n* term to the power of 2. Like that I found the equation $k - n = n^2$. By rearranging the formula to make *k* the subject I came to the conclusion that $k = n^2 + n$, the general statement.