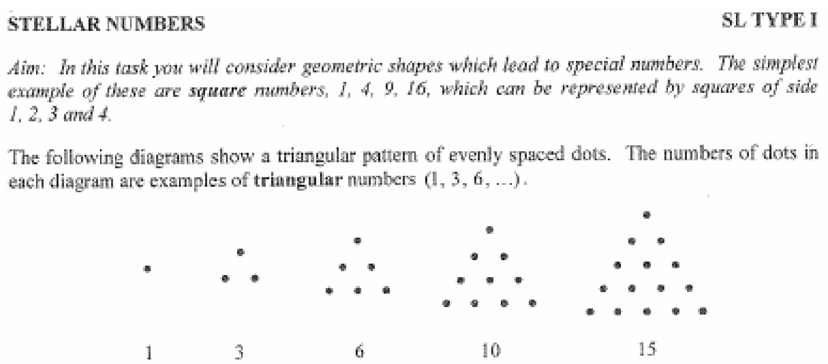
Maths Internal Assessment

Term 2 2011



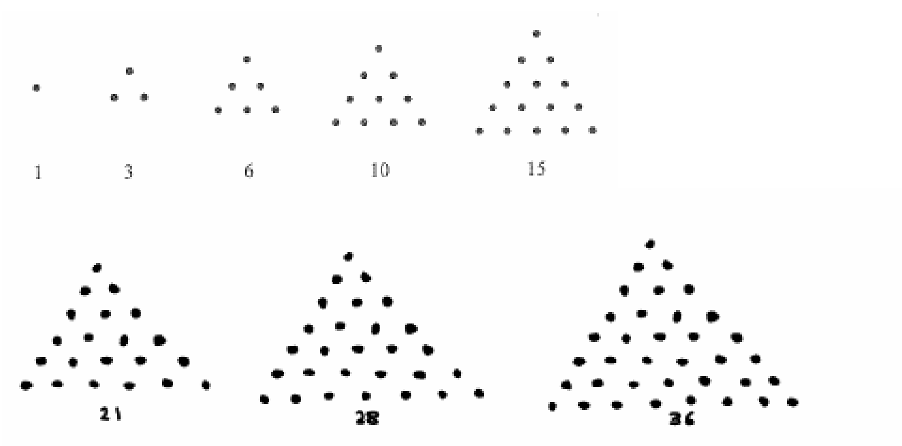
STELLAR NUMBERS　　SL TYPE I

Aim: In this task you will consider geometric shapes which lead to special numbers. The simplest example of these are ***square*** *numbers, 1, 4, 9, 16, which can be represented by squares of side 1, 2, 3 and 4.*

The following diagrams show a triangular pattern of evenly spaced dots. The numbers of dots in each diagram are examples of **triangular** numbers (1, 3, 6, ...).

1　3　6　10　15

Question 1:
Complete the triangular numbers sequence with three more terms.



The table below summarises the data (above) into a table so it is easier to analyse.

Term (Tn)	1	2	3	4	5	6	7	8
No. of Dots	1	3	6	10	15	21	28	36

Question 2:

Find a general statement that represents the nth triangular number in terms of n.

The table below shows the connection and calculation between t he term and the number of dots for the given triangular numbe rs.

Term	**Row**	**Calculation**	**Answer**
N_1 (1st)	1st row = 1	1	1
N_2 (2nd)	1st row = 1 2nd row = 2	1 + 2	3
N_3 (3rd)	1st row = 1 2nd row = 2 3rd row = 3	1 + 2 + 3	6
N_4 (4th)	1st row = 1 2nd row = 2 3rd row = 3 4th row = 4	1 + 2 + 3 + 4	10

Square numbers are also geometric shapes which lead to special nu mbers. Therefore, we must take into account these square numbers in order to find out the general statement.

Term	**Row**	**Calculation**	**Answer**
N_1 (1st)	1st row = 1	1	1
N_2 (2nd)	1st row = 2 2nd row = 2	2 + 2	4
N_3 (3rd)	1st row = 3 2nd row = 3 3rd row = 3	3 + 3 + 3	9
N_4 (4th)	1st row = 4 2nd row = 4 3rd row = 4 4th row = 4	4 + 4 + 4 + 4	16

When we compare the square numbers to the triangular numb ers, a connection is discovered. We can see that:

Qualitative	Quantitative
N1 (triangular numbers) + N2 (triangular numbers) = N2 (square numbers)	1 + 3 = 4
N2 (triangular numbers) + N3 (triangular numbers) = N3 (square numbers)	3 + 6 = 9
N3 (triangular numbers) + N4 (triangular numbers) = N4 (square numbers)	6 + 10 = 16

When this is written as a formula...:

$$T_{n-1} + T_n = n^2$$
$$T_{n-1} + T_{n-1} + n = n^2$$
$$2(T_{n-1}) + n = n^2$$
$$2(T_{n-1}) = n^2 - n$$
$$T_{n-1} = \frac{n^2 - n}{2}$$
$$T_{n-1} = \frac{n(n-1)}{2}$$
$$\downarrow$$
$$T_n = \frac{n(n+1)}{2}$$

Another method in finding the general statement is by using multiplication. By this I mean:

Quantitative	Qualitative
(N1 x N2)/2 = T1	(1 x 2)/2 = 1
(N2 x N3)/2 = T2	(2 x 3)/2 = 3
(N3 x N4)/2 = T3	(3 x 4)/2 = 6
(N4 x N5)/2 = T4	(4 x 5)/2 = 10

$$\rightarrow T_n = \frac{n(n+1)}{2}$$

Therefore, by multiplying one term and the term next to it, then divid ing the multiplication by 2, we can determine the number of dots in each triangle.

Yet another method in determining this general statement is by thin king logically. Technically, a triangle is half a square. Therefore, as the number of dots in a squ are number is n2, the equation should be n2/2. Except we must take into account the dots that we c ut in half and include them in the equation. The dots that we cut in half wi ll be represented as n/2. Therefore the statement is:

$$T_n = \frac{n^2}{2} + \frac{n}{2}$$
$$= \frac{n^2+n}{2}$$
$$= \frac{n(n+1)}{2} \longrightarrow T_n = \frac{n(n+1)}{2}$$

Examples:

$$T_1 = \frac{1(1+1)}{2} = \frac{1\times 2}{2} = \frac{2}{2}, \quad T_1 = 1$$

$$T_2 = \frac{2(2+1)}{2} = \frac{2\times 3}{2} = \frac{6}{2}, \quad T_2 = 3$$

$$T_3 = \frac{3(3+1)}{2} = \frac{3\times 4}{2} = \frac{12}{2}, \quad T_3 = 6$$

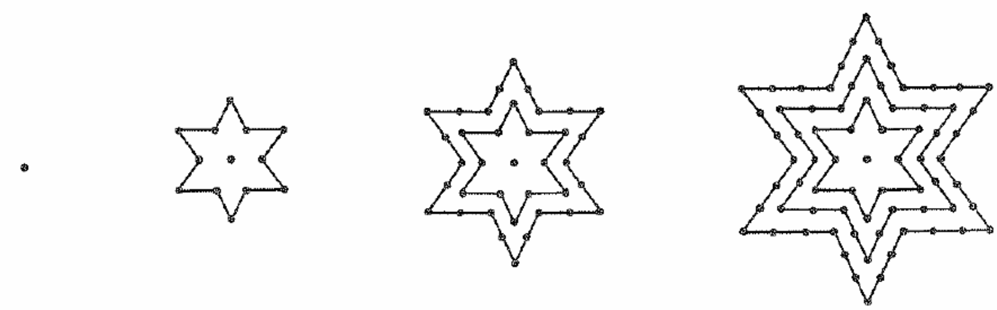
$$T_4 = \frac{4(4+1)}{2} = \frac{4\times 5}{2} = \frac{20}{2}, \quad T_4 = 10$$

$$T_5 = \frac{5(5+1)}{2} = \frac{5\times 6}{2} = \frac{30}{2}, \quad T_5 = 15$$

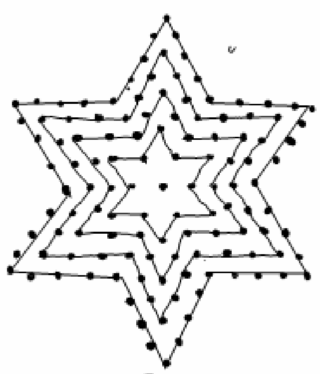
$$T_6 = \frac{6(6+1)}{2} = \frac{6\times 7}{2} = \frac{42}{2}, \quad T_6 = 21$$

$$T_7 = \frac{7(7+1)}{2} = \frac{7\times 8}{2} = \frac{56}{2}, \quad T_7 = 28$$

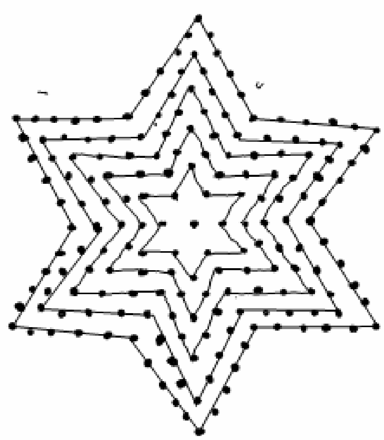
Consider **stellar** (star) shapes with p vertices, leading to p-stellar numbers. The first four representations for a star with six vertices are shown in the four stages S_1–S_4 below. The 6-stellar number at each stage is the total number of dots in the diagram.



S_1 S_2 S_3 S_4



S_5



S_6

Question 3:

Find the number of dots (i.e. the stellar numbers) in each stage up to S6. Organize the data so that you can recognize and describe any patterns.

The table below organiz es the data so that we can recognise and describe the patterns.

	S1	S2	S3	S4	S5	S6
Layer 1	1	1	1	1	1	1
Layer 2	0	12	12	12	12	12
Layer 3	0	0	24	24	24	24
Layer 4	0	0	0	36	36	36
Layer 5	0	0	0	0	48	48
Layer 6	0	0	0	0	0	60
Calculation	1	1+12	1+12+24	1+12+24+36	1+12+24+36+48	1+12+24+36+48+60
Total no. of dots	1	13	37	73	121	181

Excluding the first layer (layer 1), as the stage number (Sn) and layer increases by 1, the number of dots correlatively increase by 12.

Question 4:
Find an expression for t he 6-stellar number at S7.

As we discovered the connection between the number of layers and the number of dot s, an expression for S7 can be calculated.

$$S_7 = \overset{1}{1} \underset{11}{\curvearrowright} + \overset{2}{12} \underset{12}{\curvearrowright} + \overset{3}{24} \underset{12}{\curvearrowright} + \overset{4}{36} \underset{12}{\curvearrowright} + \overset{5}{48} \underset{12}{\curvearrowright} + \overset{6}{60} \underset{12}{\curvearrowright} + \overset{7}{72}$$

$$= 12(1+2+3+4+5+6)+1$$

$$= (12 \times 21) + 1$$

$$= 252 + 1$$

$$= 253$$

Therefore, the expression for the 6 -stellar number at stage S7 is (12 x 21) +1.

Question 5:
Find a general statement for the 6 -stellar number at stage Sn in terms of n.

Similar to using square numbers to find out triangular numbers, finding the general statement for the 6-stellar number will require the manipulation of triangular numbers.

Firstly we shall find more expressions for the 6-stellar numbers:

$$S_2 = \overset{1}{1} + \overset{2}{12} = 12(1)+1 = (12\times 1)+1 = 12+1 = 13$$

$$S_4 = \overset{1}{1} + \overset{2}{12} + \overset{3}{24} + \overset{4}{36} = 12(1+2+3)+1 = (12\times 6)+1 = 72+1 = 73$$

$$S_6 = \overset{1}{1} + \overset{2}{12} + \overset{3}{24} + \overset{4}{36} + \overset{5}{48} + \overset{6}{60} = 12(1+2+3+4+5)+1 = (12\times 15)+1 = 180+1 = 181$$

When the expressions are written this way, we can relate the 6-stellar numbers with triangular numbers. The focus is on the circled number in the expression as they correspond to the n-1th of the triangular term. In other words the general statement for the 6-stellar numbers is (12 x (n - 1)) + 1.

However, this expression is not complete as it does not apply to the 6-stellar numbers. Therefore, as previously stated, for the triangular numbers, n-1 = n(n-1)/2. So if this is substituted into the equation, the general statement should be discovered.

Examples:

$$S_1 = 12\times\frac{1(1-1)}{2}+1 = (12\times 0)+1 = 1$$

$$S_3 = 12\times\frac{3(3-1)}{2}+1 = 12\times\frac{6}{2}+1 = (12\times 3)+1 = 36+1 = 37$$

$$S_5 = 12\times\frac{5(5-1)}{2}+1 = 12\times\frac{20}{2}+1 = (12\times 10)+1 = 120+1 = 121$$

Therefore, the general statement for the 6-stellar numbers is 12 x ((n(n-1)/2)+1

Question 6:
Now repeat the steps above for other values of p.
The chosen value for p is 7. This means that we are exploring the 7-stellar number therefore, the star has 7 points.

Question 6.1:
Find the number of dots (i.e. the stellar numbers) in each stage up to S6. Organize the data so that you can recognize and describe any patterns.

	S1	S2	S3	S4	S5	S6
Layer 1	1	1	1	1	1	1
Layer 2	0	14	14	14	14	14
Layer 3	0	0	28	28	28	28
Layer 4	0	0	0	42	42	42
Layer 5	0	0	0	0	56	56
Layer 6	0	0	0	0	0	70
Calculation	1	1+14	1+14+28	1+14+28+42	1+14+28+42+56	1+14+28+42+56+70
Total no. of dots	1	15	43	85	141	211

Excluding the first layer (layer 1), as the stage number (Sn) and layer increases by 1, the number of dots correlatively increase by 14.

Question 6.2:
Find an expression for the 6 -stellar number at S7.

$$S_n = \overset{1}{1} \underbrace{+}_{13} \overset{2}{14} \underbrace{+}_{14} \overset{3}{28} \underbrace{+}_{14} \overset{4}{42} \underbrace{+}_{14} \overset{5}{56} \underbrace{+}_{14} \overset{6}{70} \underbrace{+}_{14} \overset{7}{84}$$

$$\begin{aligned} S_n &= 14(1+2+3+4+5+6)+1 \\ &= (14 \times 21)+1 \\ &= 295 \end{aligned}$$

Therefore, the expression for the 7 -stellar number at stage S7 is (14 x 21)+1.

Question 6.3:
Find a general statement for the 7 -stellar number at stage Sn in terms of n.
We must verify if the same theory, behind the general statement for the 6-stellar number, can be used to determine the general statement for
the 7-stellar number. This means that the manipulation of triangular numbers is essential.

$$\begin{aligned} S_2 &= \overset{1}{1} + \overset{2}{14} \\ &= 14(1)+1 \\ &= (14 \times 1)+1 \\ &= 14+1 \\ &= 15 \end{aligned}$$

$$\begin{aligned} S_4 &= \overset{1}{1} + \overset{2}{14} + \overset{3}{28} + \overset{4}{42} \\ &= 14(1+2+3)+1 \\ &= (14 \times 6)+1 \\ &= 84+1 \\ &= 85 \end{aligned}$$

$$\begin{aligned} S_6 &= \overset{1}{1} + \overset{2}{14} + \overset{3}{28} + \overset{4}{42} + \overset{5}{56} + \overset{6}{70} \\ &= 14(1+2+3+4+5)+1 \\ &= (14 \times 15)+1 \\ &= 210+1 \\ &= 211 \end{aligned}$$

Therefore, from the examples above, we can see that the same logic from the previous statement (6-stellar number) can be used. For example, the expression for the 7 -stellar number at S4 is S4 = (14 x 6) + 1

It should be recognised that the '6' (circled in the calculations above) is the 3rd triangular number. Thus, there is the same logic behind this general statement as the previous general statement (6-stellar number). The general statement of the 6 - stellar number was 12 x ((n(n-1)/2)+1. There was 12 more dots added onto each sequential layer and the equation commenced with 12 x… However for the 7 - stellar number, 14 more dots are added onto each sequential layer.

By taking this into account, the general statement for the 7 - stellar number is 14 x ((n(n-1)/2) + 1.

Examples:

$$S_1 = 14 \times \frac{1(1-1)}{2} + 1 = (14 \times 0) + 1 = 1$$

$$S_3 = 14 \times \frac{3(3-1)}{2} + 1 = 14 \times \frac{6}{2} + 1 = (14 \times 3) + 1 = 42 + 1 = 43$$

$$S_5 = 14 \times \frac{5(5-1)}{2} + 1 = 14 \times \frac{20}{2} + 1 = (14 \times 10) + 1 = 140 + 1 = 141$$

Therefore, the general statement for the 7 - stellar number is 14 x ((n(n-1)/2)+ 1.

Question 7:
Hence, produce the general statement in terms of *p* and *n*, that generates the sequence of p-stellar numbers for any value of *p* at stage Sn

Sn for the 6-stellar numbers is 12 x ((n(n-1)/2)+ 1.
Sn for the 7-stellar numbers is 14 x ((n(n-1)/2)+ 1.

However, the question is asking for a general statement that generates the sequence of p - stellar numbers for any value of p. Thus, when we compare these two statements; we can see that the only integer which changes is the first, and hence, finding its mathematical relation between the first integer and its corresponding 'p' value is necessary to form this general statement.

The 'p' value is the number of points on a star or the p -stellar number.
Therefore, in this case, the 'p' values are 6 and 7.
The first integer for the 6 - stellar number (p) is 12.
The first integer for the 7 - stellar number (p) is 14.

We can see that the first integer is double its 'p' value.
6 x 2 = 12
7 x 2 = 14

Therefore the general statement can be expressed like so:

$$S_n = 2p \times \left(\frac{n(n-1)}{2}\right) + 1$$

But the simplified general statement is:

$$S_n = 2p \times \left(\frac{n(n-1)}{2}\right) + 1$$

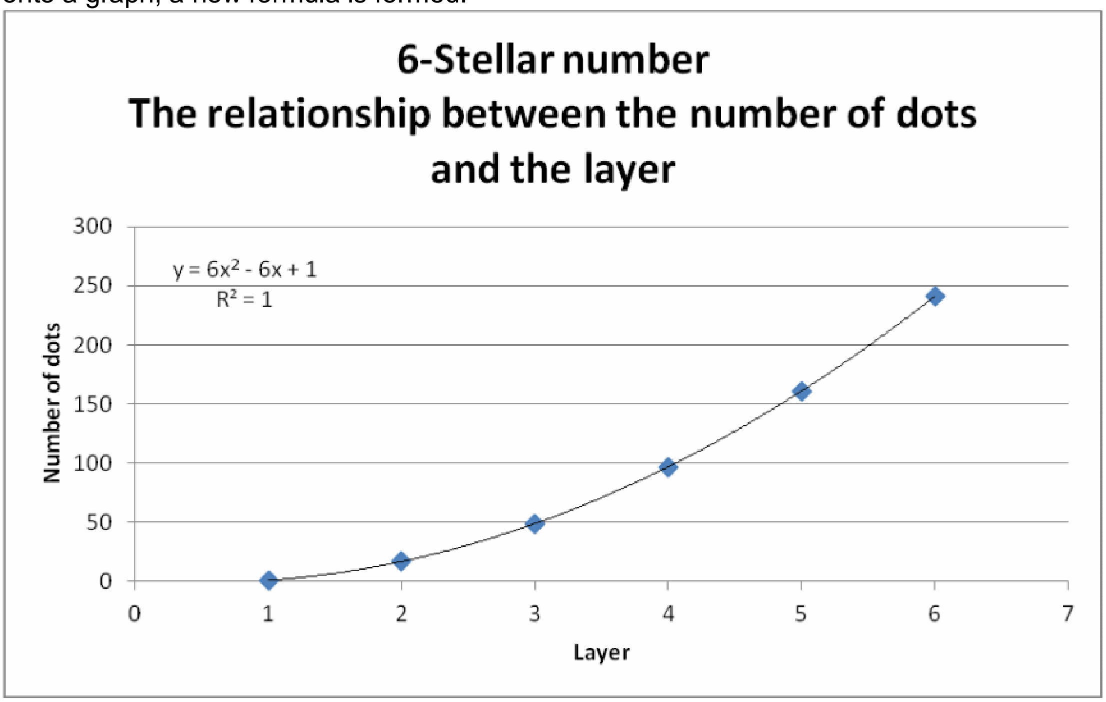
$$= \cancel{2}p \times \left(\frac{n(n-1)}{\cancel{2}}\right) + 1$$

$$= p \times (n(n-1)) + 1$$

$$S_n = pn(n-1) + 1$$

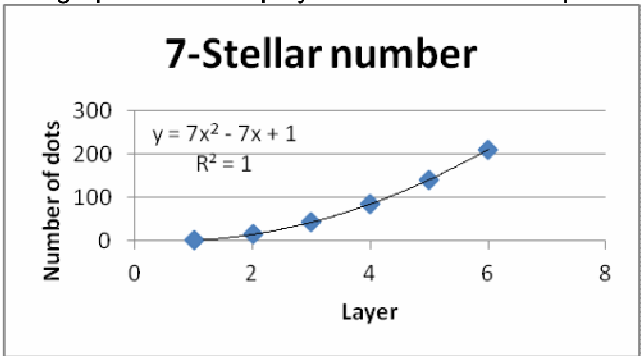
Therefore, the general statement i n terms of *p* and *n* is:
S_n = pn (n-1) + 1

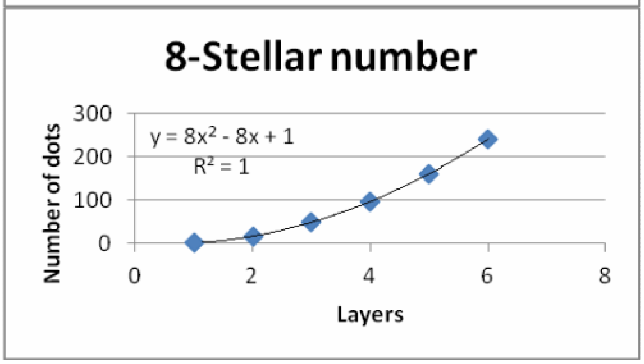
However, when the relationship between the total number of dots and the layers are plotted onto a graph, a new formula is formed.



A polynomial trend-line is used to determine the link between each layer to dete rmine the formula for the 6-stellar number. This formula is $y=6x^2 - 6x + 1$. The 6 in '$6x^2$' and '6x' from the n-stellar number. Therefore, if it was a 7-stellar number, the equation should be $y = 7x^2 - 7x + 1$; which it is shown in the graph below. If this equation is expressed in terms of 'p' and 'n', it will be $pn^2 - pn + 1$. This simply justifies the general statement previously found which was: S_n = pn (n-1) + 1 as the statement from this graph is just the expanded version of the found general statement. T he 'R' value is the accuracy of the plotted data when compared to the polynomial trend-line. This value state 1 which means that it is 100% accuracy which supports the equation by stating that it is 100% accurate.

The graphs below display the other valu es of ‘p’ to test the validity of this equation.





The graphs on the left have an equation of $y=7x^2 - 7x + 1$ (7-stellar number) and $y=8x^2 - 8x + 1$ (8-stellar number). Which furthermore proves that the even if the ‘p’ value is different, the general statement will still be $S_n = pn^2 - pn + 1$.

We can see that only the co-efficient changes therefore for this graph, the ‘p’ is the stellar point’s number and the ‘x’ is the ‘n’ value which is the layer number.

Question 8:
Test the validity of the general statement

3- Stellar, 8th layer
$p = 3 \quad n = 8$
$S_n = pn(n-1)+1$
$S_8 = (3\times8)(8-1)+1$
$S_8 = (24\times7)+1$
$S_8 = 168+1$
$S_8 = 169$

4- Stellar, 7th layer
$p = 4 \quad n = 7$
$S_n = pn(n-1)+1$
$S_7 = (4\times7)(7-1)+1$
$S_7 = (28\times6)+1$
$S_7 = 168+1$
$S_7 = 169$

5- Stellar, 6th layer
$p = 5 \quad n = 6$
$S_n = pn(n-1)+1$
$S_6 = (5\times6)(6-1)+1$
$S_6 = (30\times5)+1$
$S_6 = 150+1$
$S_6 = 151$

6- Stellar, 5th layer
$p = 6 \quad n = 5$
$S_n = pn(n-1)+1$
$S_5 = (6\times5)(5-1)+1$
$S_5 = (30\times4)+1$
$S_5 = 120+1$
$S_5 = 121$

7- Stellar, 4th layer
$p = 7 \quad n = 4$
$S_n = pn(n-1)+1$
$S_4 = (7\times4)(4-1)+1$
$S_4 = (28\times3)+1$
$S_4 = 84+1$
$S_4 = 85$

8- Stellar, 3rd layer
$p = 8 \quad n = 3$
$S_n = pn(n-1)+1$
$S_3 = (8\times3)(3-1)+1$
$S_3 = (24\times2)+1$
$S_3 = 48+1$
$S_3 = 49$

9- Stellar, 2nd layer
$p = 9 \quad n = 2$
$S_n = pn(n-1)+1$
$S_2 = (9\times2)(2-1)+1$
$S_2 = (18\times1)+1$
$S_2 = 18+1$
$S_2 = 19$

10- Stellar, 1st layer
$p = 10 \quad n = 1$
$S_n = pn(n-1)+1$
$S_1 = (10\times1)(1-1)+1$
$S_1 = (10\times0)+1$
$S_1 = 0+1$
$S_1 = 0$

Question 9
Discuss the scope and limitations of this general statement

A few limitations come with the use of this general statement.

There are three positive numbers that cannot be used in this general formula for the values of 'p'. They are: 0, 1 and 2. The simple reason is because stellar -numbers are presented in the shape of a star. Therefore as the 'p' value represents the number of poin ts on a star, - e.g. for a '2-stellar', this would take the form of a straight line which does not fulfil the definition of 'stellar number'.

However, as stellar numbers must take the form of a star; p = ∞ (infinity) cannot be applied to the general statement as the shape would be a circle.

As stated in the previous limitation, that p= ∞ (infinity) will be a circle; this indicates that the higher the 'p' value is; the harder it will be to create that 'n - stellar number' and thus hard er to prove.

With logical reasoning, it is obvious that no decimal numbers can fill the 'p' value when creating the diagram for it. There is no such thing as 'half -a-dot' (p+0.5) nor 'a-quarter' (p+0.25). Although these numbers can be substituted into 'p' when working out how many dots are in that stellar-number, there is no way to calculate if the number of dots is correct. Therefore, a 'decimal numbered stellar' cannot be created.

Similar to the reason behind why decimal numbers cannot be a ‘p’ value of the stellar number, negative numbers cannot take the value of ‘p’ because there is no such thing as a negative dot. It would simply mean that there is no dot.

E.g. -3 Stellar, layer 3
S_n = pn (n-1) + 1
S_n = ((-3)x(3))(3-1) + 1
S_n = (-9) x (2) + 1
S_n = -18 + 1
S_n = -17
= drawing -17 dots is impossible.

Question 10:
Explain how you arrived at the general statement.
The results from a few different techniques were required to arrive and form the general statement. Firstly, it started with the trian gular number formula which was supported by the geometric reasoning behind the formation of square numbers. It was essential to understand and explore these triangular numbers as without this, there would be the need to write out the sequential sum for the further layers.

Secondly, the knowledge of the triangular shapes was also applied to the creation of the 6 - stellar numbers to discover the connection between the number of dots and the layers. By finding this link, it was possible to create an expression for the 6-stellar number which linked with the triangular numbers. All these previous steps led to the formation of the general formula which was 12 x ((n(n-1)/2)+1.

Then, by drawing the diagram of the 6 -stellar number a particular pattern emerged. It wa s about the consistent addition in the number of dots as the layers increased. In other words, the first 'star' consisted of one dot, the second layer (star) contained 12 dots, the third had 24 dots, the fourth layer had 36 dots. As one can see, the number of dots is increasing by 12 excluding the first -> second layer. Without exploring the triangular numbers, it would have been difficult to express this abnormality.

Fourthly, the relation between the 'p' value and the first integer of the general formula must be determined. This was found by comparing the 6 -stellar and 7-stellar numbers, and was found that the first integer was always equal to 2 x p.

As all of the steps above was completed, what one needed to do now was just combine the results from the steps above to determine the general statement. Therefore, by doing so the general statement was found to be: Sn = pn (n -1) + 1.

The final step was to test if the general statement could be applied to the stellar numbers in terms of 'p' and 'n'; which was successful; however with some few limitations such as the 0, 1, 2 and decimal numbers substituting the 'p' value.