QUEENSLAND ACADEMY FOR SCIENCE, MATHS AND TECHNOLOGY

Maths IA Type 2

Modelling a Functional Building

Francis Nguyen

Mr Mathews

Introduction:

The structure of a roof for a building is parabolic. The design of this building has a fixed rectangular base which is 150m long and 72m wide. The maximum height of this building can vary between 50% - 75% of its width for stability and aesthetic purposes. The independent variable in this investigation is the height of the building. The maximum volume of a cuboid under the roof depends on the height of the roof, which is the dependant variable.

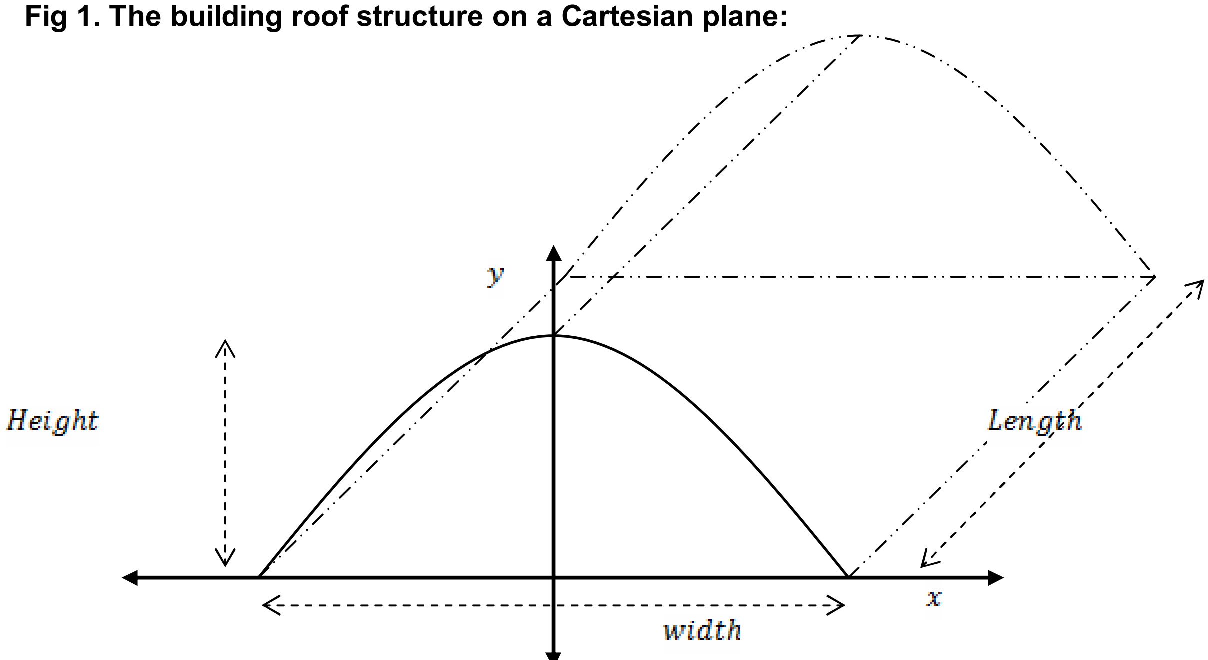
All calculations will be made through Ti-*n*Spire calculator (GDC; Graphical Display Calculator) and all figures will be rounded to 3 decimal places as architects work with millimetres where as this report works in meters.

The Function:

The model of the roof structure will be designed on a Cartesian plane using the graphing package, *Graph*. The axis of symmetry (also the maximum turning point in this case) will be modelled at $x = 0$ within this report (refer to Fig 1 below). The roots of the quadratic will be at a fixed co-ordinate of (-36,0) and (36,0) as the distance between these 2 points is 72m long for when the façade is designed at the width (refer to Fig 2).

From these requirements, we manipulate the general form of the quadratic function to fit the purposes of this report.

Fig 1. The building roof structure on a Cartesian plane:



Method 1. Finding the general formula:

The general form of a quadratic function is:

$$y = ax^2 + bx + c$$

There is no horizontal shift as the line of symmetry will be modelled at the y axis

□ $bx = 0$

The y intercept will be modelled at the maximum height of the parabola

Let y intercept $= h$

□ $c = h$

The roof is at a maximum turning point

□ $ax^2 = -ax^2$

□ The form of the model quadratic is:

$$y = -ax^2 + h$$

To solve for a, substitute in the known fixed values (the root) for x and y (36,0):

$$(0) = -a(36)^2 + h$$

$$1296a = h$$

$$a = \frac{h}{1296}$$

Substitute all these values into the general form of the quadratic to obtain the general formula of the roof:

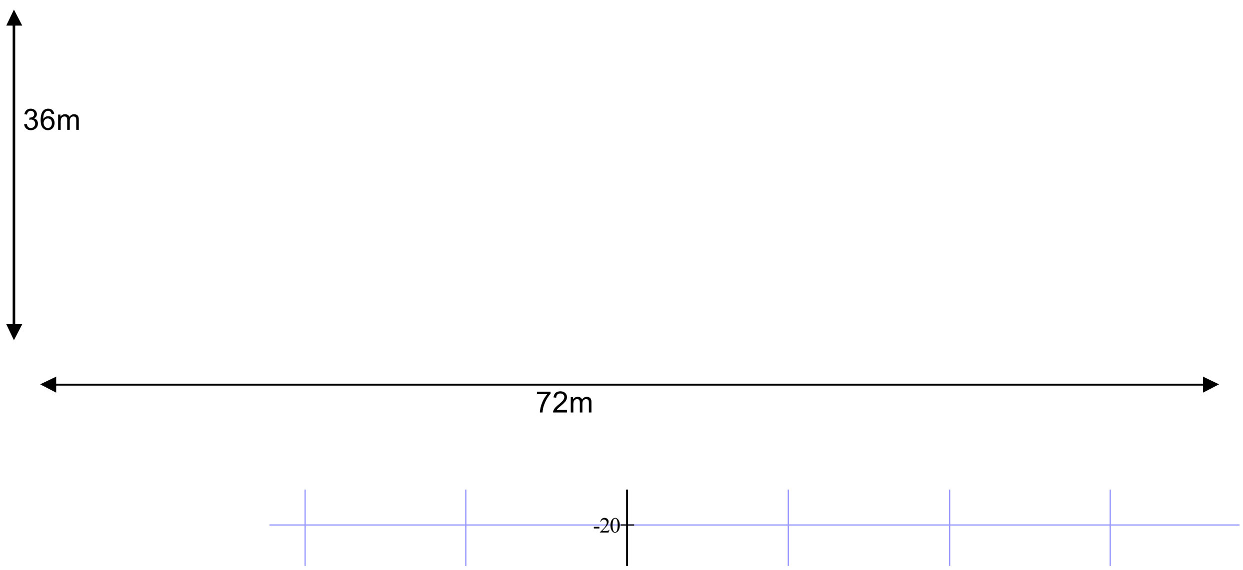
$$y = -\frac{h}{1296}x^2 + h$$

Modelling at Minimum Height:

Below in Fig 1 is a model of the roof structure with the minimum height of 36m, where the façade is designed at the width.

Fig 2. Model of 36m high roof structure at the façade

$$\boxed{f(x) = -0.027777778x^2 + 36; R^2 = 1}$$



3 points were plotted on the graphing package, *Graph*, and a function was given to best fit the points. The function given was $y \approx -0.028 + 36$, where $R^2 = 1$, meaning that the function given perfectly fits the points. The closer R^2 is to 1, the more accurate the function is. Now, to find this function algebraically.

Method 2. Finding the function at $h = 36$:

To determine the function in Fig 1 above algebraically, sub in $h = 36$ into $y = -\frac{h}{1296}x^2 + h$

- The function is:

$$y = -\frac{36}{1296}x^2 + 36$$

$$= -\frac{1}{36}x^2 + 36$$

To show that $a = -\frac{h}{1296}$ is true:

According to *Graph* in Fig 1,

$$a \approx -0.028$$

and $-\frac{1}{36} \approx -0.028$

 The function is correct

Maximum Cuboid Volume Dimensions:

After determining the function, we must find the dimensions of a cuboid which has the maximum volume under the curve.

Method 3. Finding the dimensions of a cuboid with the maximum volume under the curve:

To find the dimensions of a cuboid with maximum volume under the curve, we find the area of the face of the cuboid:

Let $A = Area$

$$y = -\frac{1}{36}x^2 + 36$$

$A(rectangle) = xy$ {sub in y value}

$$= x\left(\frac{1}{36}x^2 + 36\right)$$

$$= \frac{-1}{36}x^3 + 36x$$

Differentiate to find maximum area at $A'(rectangle) = 0$,

$$A'(rectangle) = \frac{-3}{36}x^2 + 36$$

$$0 = \frac{-3}{36}x^2 + 36$$

$$\frac{3}{36}x^2 = 36$$

$$3x^2 = 1296$$

$$x^2 = 432$$

$$x = \pm\sqrt{432}$$

$$= (\pm\sqrt{144})(\sqrt{3})$$

$$= \pm 12\sqrt{3}$$

$\approx \pm 20.794$ {these values are the roots}

Find the second derivative to prove that this is a maximum curve:

$$A'(rectangle) = \frac{-3}{36}x^2 + 36$$

$$A''(rectangle) = \frac{-5}{36}x$$

$$A'' < 0$$

 The function concaves downwards when $x > 0$

Sub in x value (from the first derivative) into original function to find y value

$$y = \frac{-1}{36}x^2 + 36$$

$$= \frac{-1}{36}(\pm 12\sqrt{3})^2 + 36$$

$$= 24$$

 Dimensions of the maximum volume of the cuboid are:

The width is found by finding the difference between the 2 points as the line of symmetry is designed at the y axis (refer to Fig 2 below).

$$width = \left(12\sqrt{3}\right) - \left(-12\sqrt{3}\right)$$

$$= 24\sqrt{3}$$

$$\approx 41.569\text{m}$$

$$height = 24\text{m}$$

$$length = 150\text{m}$$

And the volume (V) of the cuboid:

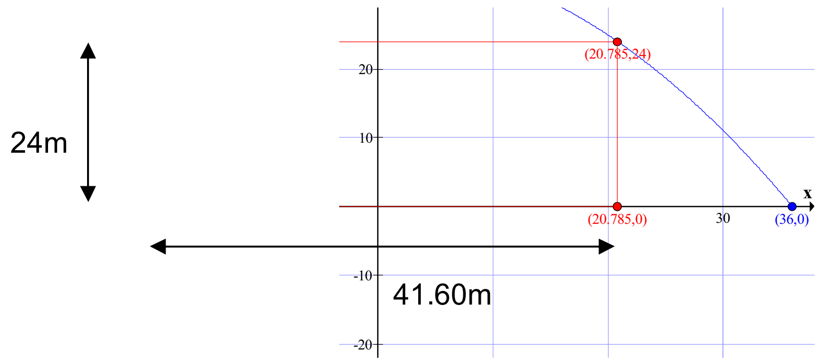
$$V_{max}(Cuboid) = w \times h \times l$$

$$\approx 41.569 \times 24 \times 150$$

$$\approx 149649.12$$

 These dimensions produce the cuboid with the maximum volume under the graph (also shown in Fig 2 below).

Fig 3. Dimensions of cuboid with maximum volume when $h = 36$ (front view)



Varying Height of Roof:

After determining the dimensions of the maximum cuboid, we test to see how our independent variable (the maximum height of the roof) affects our dependant variable (the cuboid).

Using the same methodology in Method 3, the following table was constructed and calculated using the GDC.

The maximum and minimum height of the roof was chosen. 2 consecutive heights near the maximum where then chose to see the relationship between a 1 meter increases. And the medium height of the roof was chosen.

Table 1. How height of roof affects maximum cuboid dimensions and volume

Maximum height of roof (m)	Function $\left(y = \frac{-h}{1296}x^2 + h\right)$	Width of cuboid (m) ($2x$ value)	Height of cuboid (m) (y value)	Length of cuboid (m)	Max volume of cuboid (m^3)
36	$y = \frac{-1}{36}x^2 + 36$	41.569	24	150	149649.190
45	$y = \frac{-5}{144}x^2 + 45$	41.569	30	150	187061.487
52	$y = \frac{-13}{324}x^2 + 52$	41.569	34.667	150	216162.019
53	$y = \frac{-53}{1296}x^2 + 53$	41.569	35.333	150	220314.784
54	$y = \frac{-1}{24}x^2 + 54$	41.569	36	150	224473.785

Through inspection, the width remains the same for all tested heights. It is evident that as the height of the roof increases by 1m, the height of the also cuboid increases, by $\frac{2}{3}$m. For this type of model function, the height of the cuboid can be found by $y = \frac{2}{3}h$; where y is the height of the cuboid and h is the maximum height of the roof.

Method 4. Finding the increase in height

Let $height = h$

Using values of roof height at 52m and 45m

$$h\ (cuboid) = \frac{h_1 - h_2}{height_{roof}\ difference}$$

$$= \frac{34.667 - 30}{4}$$

$$= \frac{2}{3}$$

The width of the maximum volume of the cuboid remains the same, the length is fixed at 150m, and the height of the cuboid increases as the maximum height of the roof increases. Therefore, the volume should also increase in proportion to the height of the roof as $volume = l \times w \times h$.

Ratio of Wasted Space to Office Block:

After calculating the maximum volume of the cuboid, it is important to determine how efficient the volume of the cuboid is by comparing it to the volume of wasted space. The volume of wasted space is found by, $Volume\ of\ Under\ the\ Curve - \ Volume\ of\ cuboid$.

Method 5:

To find volume under the curve algebraically, we integrate:

$$\int_{-36}^{36} \left(\frac{-1}{36}x^2 + 36\right) dx$$

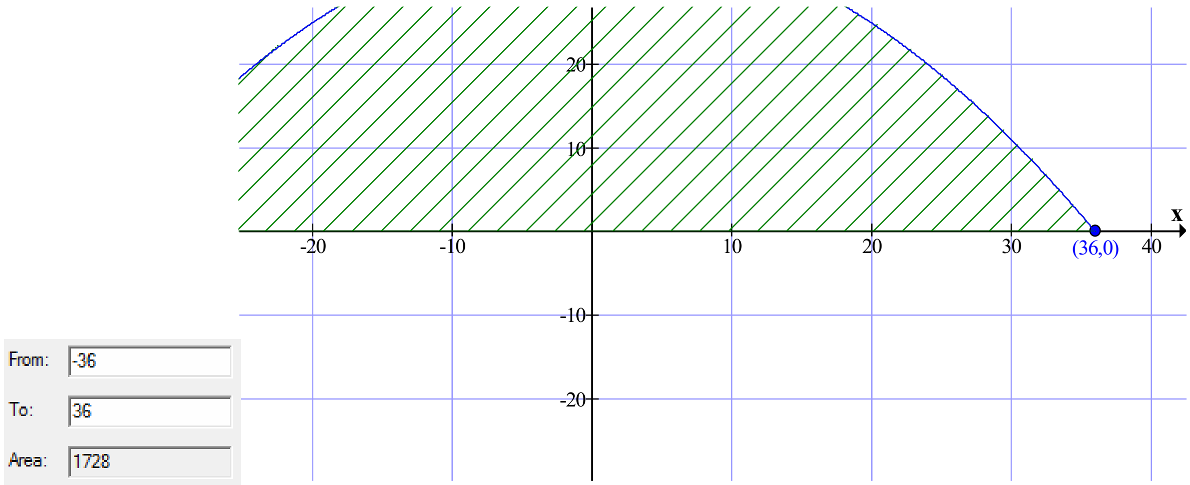
Using GDC to find the answer:

$$= 1728\ m^2$$

Fig 4 below shows the area under the curve which was calculated. The area under the curve was also determined by using *Graph* which supports the answer that was found.

Fig 5. Area under the graph (and between the x axis)

$$\boxed{f(x) = -\frac{1}{36}x^2 + 36}$$



Multiply by length to find the volume:

$$Volume\ (under\ curve) = 1728\ \times 150$$
$$= 259200\ m^3$$

$$Volume\ (Wasted\ space) = Volume\ (under\ curve) - \ Volume\ (cuboid)$$
$$= \ 259200 - \ 149649.190$$
$$= 109550.810\ m^3$$

The following table was constructed by following the same methodology in Method 4, and the maximum volume was obtained from table 1.

Table 2. How the maximum height of the roof affects ratio of volume wasted

Maximum height of roof (m)	Volume of wasted space (m^3)	Max volume of cuboid (m^3)	Ratio Wasted space : Volume cuboid
36	109550.810	149649.190	0.732 : 1
45	136938.513	187061.487	0.732 : 1
52	158237.981	216162.019	0.732 : 1
53	161285.216	220314.784	0.732 : 1
54	164326.215	224473.785	0.732 : 1

Table 2 above shows that ratio of wasted space to the volume of the cuboid does not change as the height varies. This tells us that the efficiency of the space cannot be improved by varying the height. The table also tells us that approximately the value of 73.2% of the volume of the cuboid is equivalent to the wasted space.

Maximum Cuboid Floor Area:

We now investigate the maximum cuboid floor area to determine how much space we are able to use. Each level of the cuboid is at a fixed height of 2.5m.

Method 6. Finding the maximum floor area of the cuboid

The office floor area of one level is:

$Area\,(floor) = l \times w$

As the length and width are at fixed dimensions of 150m and 41.569m (or $24\sqrt{3}$m)

$Area\,(floor) = 150 \times 24\sqrt{3}$

- The area of each floor is:

$\approx 6235.383\ m^2$

The number of levels is found by:

$No.(levels) = \frac{y}{2.5}$; where y is the height of the office block

- The total floor area can be found by:

$6235.383 \times \frac{y}{2.5}$

Method 7. Finding the maximum floor level when the roof height = 36m

At $h = 36$m

$$A_{total}(floor) = 6235.383 \times \frac{y}{2.5}$$

$= 6235.383 \times \frac{24}{2.5}$

$$= 6235.383 \times 9.60$$

Cannot contain 10 floors, but can contain 9 floors. Therefore we round the number of levels to the lowest positive integer:

$= 6235.383\ \times 9$

$= 56118.42$

Using Method 7, the following table was constructed.

Table 4. Maximum area of cuboid floor:

Maximum height of roof (m)	Number of levels	Maximum area of cuboid floor (m^2)
36	9	56118.447
42	16	99766.128
46	18	112236.894
52	20	124707.660
53	21	130943.043
54	21	130943.043

Switching the Side of the Façade

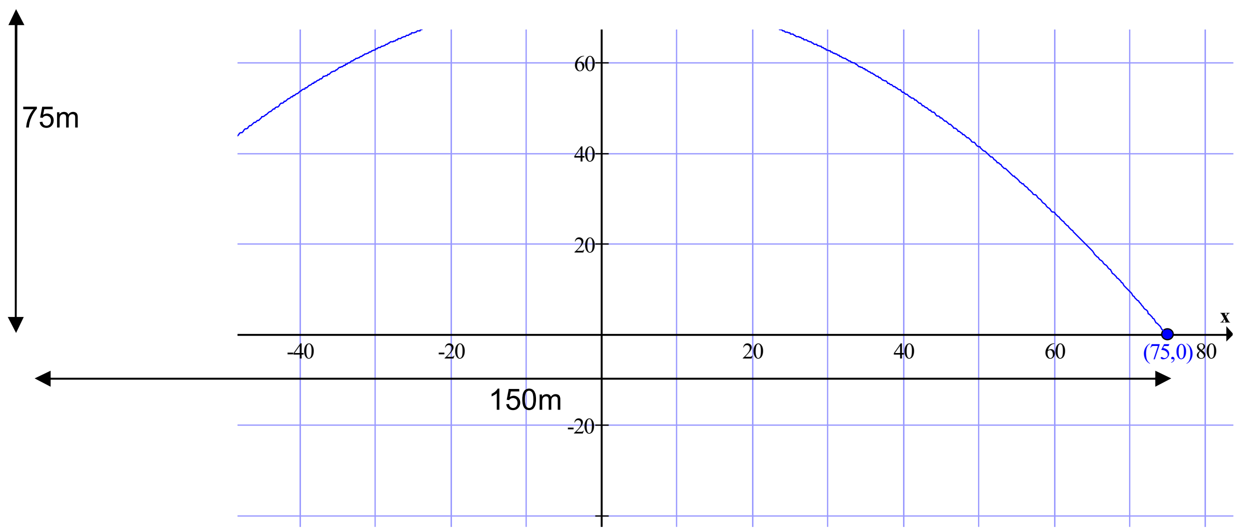
In order to test for more efficient possibilities, the face of the façade will be on the longer side of the base, where the length is 150m.

Using the same methodology in Method 1, we obtain the standard formula when the façade is on the length:

$$y = \frac{-h}{5625}x^2 + h$$

Fig 6. Model of roof structure at minimum height when the façade switches

$f(x) = -0.001333333x^2 + 36; R^2 = 1$



Using the methodology shown in Method 3, the follow table was constructed, where the maximum and minimum height was found. 2 consecutive heights near the maximum, and a few points near the medium was found.

Table 5. dimension and maximum volume of cuboid when the façade switches

Maximum height of roof (m)	Function $\left(y = \frac{-h}{5625}x^2 + h\right)$	Width of cuboid (m)	Height of cuboid (m)	Length of cuboid (m)	Max volume of cuboid (m^3)
75	$y = \frac{-1}{75}x^2 + 75$	86.603	50	72	311769.145
85	$y = \frac{-17}{1125}x^2 + 85$	86.603	56.667	72	353340.443
95	$y = \frac{-19}{1125}x^2 + 95$	86.603	63.333	72	394905.506
110	$y = \frac{-22}{1125}x^2 + 110$	86.603	73.333	72	457259.335

111	$y = \frac{-37}{1875}x^2 + 111$	86.603	74	72	461418.335
112.5	$y = \frac{-112.5}{5625}x^2 + 112.5$	86.603	75	72	467653.718

Through analysis of Table 5, the width still remains the same which is in the same case when the façade was experimented on the shorter side of the base. Also, through using Method 4, the increase in height was found, where for every 1m increase in the height of the roof, height of the cuboid increases by $\frac{2}{3}$m.

Now to find the efficiency when the façade switches sides, we implement the same methodology shown in Method 5.

Fig 7. Area under the curve when the façade switches sides and the curve is at minimum height

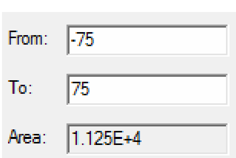


Table 6. Efficiency table when the façade switches sides

Maximum height of roof (m)	Volume of wasted space (m^3)	Max volume of cuboid (m^3)	Ratio Volume wasted : Volume cube
75	228230.855	311769.145	0.732 : 1
85	258659.557	353340.443	0.732 : 1
95	28909.494	394905.506	0.732 : 1

110	334740.665	457259.335	0.732 : 1
111	337781.665	461418.335	0.732 : 1
112.5	342346.282	467653.718	0.732 : 1

The efficiency is seen to be the same as when the façade was on the shorter side of the base.

After determining the efficiency, we find the area of the cuboid floor once again.

Table 7. Area of cuboid floor when the façade switches sides

Maximum height of roof (m)	Number of levels	Maximum area of cuboid floor (m^2)
75	30	187061.487
85	34	212003.019
95	38	236944.551
110	44	274356.848
111	44	274356.848
112.5	45	280592.231

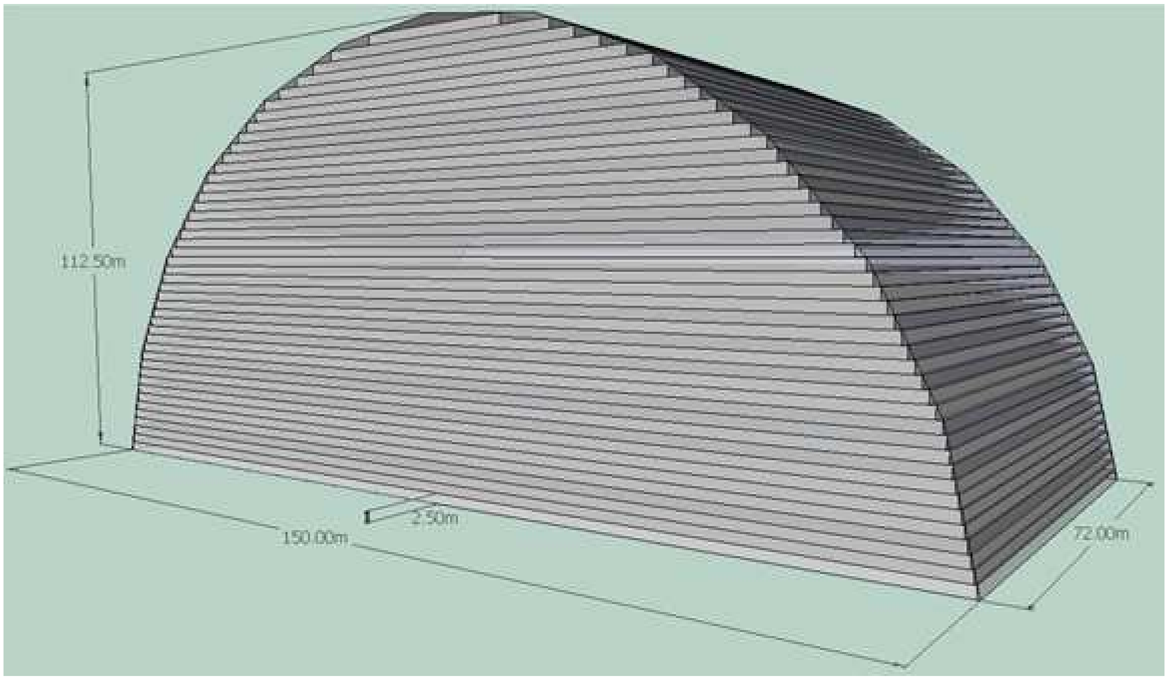
Comparing the side of the Façade:

Through analysis of both situations, the façade at the longer length of the base will be chosen to be the design at a maximum height where the roof is 112.5m tall. This due to the fact that provides a larger volume and it provides a larger area floor area.

Increasing the efficiency:

To improve the efficiency, instead of having a single cuboid under the curve, we will have multiple cuboids with each at a fixed height of 2.5m (refer to Fig 8 below)

Fig 8. Multiple cuboids



Method 8. Finding the volume of the multiple cuboids:

To find the volume of the multiple cuboids:

Let $f = \text{the floor height measured from the ground}$

e.g. At level 1:

$$f = 2.5$$

At level 2:

$$f = 5$$

At level 3:

$$f = 7.5$$

And etc.

Maximum height $(h) = 112.5$

$$y = \frac{-h}{1296}x^2 + h$$

$$f = \frac{-112.5}{1296}x^2 + 112.5$$

$$f - 112.5 = -\frac{112.5}{1296}x^2$$

$$1296\,(f - 112.5) = -112.5x^2$$

$$-50(f - 112.5) = x^2$$

$$x = \pm\sqrt{-50(f - 112.5)}$$

The difference between these 2 points is the width

Therefore, multiply the positive root by 2 to obtain the width

$$Width = 2\sqrt{-50(f - 112.5)}$$

The volume:

$$V = l \times w \times h$$

$$= 72 \times 2.5 \times 2\sqrt{-50(f - 112.5)}$$

$$= 360\sqrt{-50(f - 112.5)}$$

This can be further generalised by:

$f = 2.5n$ Where $n =$ number of floors

There are 45 floors when the maximum height of the roof is 112.5m.

- The equation to finding the volume of multiple cuboids is

$$\sum_{n=1}^{45} 360\sqrt{-50(2.5n - 112.5)}$$

$$\approx 795688.274\, m^3$$

Method 9. Using Excel:

To now further validate this statement through the use of Microsoft Excel to construct the follow table below. Column C was found by applying the equation, $width = 2\sqrt{-50(f - 112.5)}$, to Column C, where f is the value directly left in Column B. This rule was extended until the floor level 45.

Column F was obtained by multiplying the cells in Column C, Column D and Column E in the same row. This rule was also extended down until level 45.

By using the 'AutoSum' tool in Excel, the total volume was found and supports the equation above.

Column G was found by multiplying the width and length of each row and once again, the rule was extended until level 45. The 'AutoSum' tool was used once again to determine the total floor area.

Table 8. Dimensions and volume of multiple cuboids:

Floor level (Column A)	Floor height at certain level (Column B)	Width (Column C)	Length (Column D)	Minimum height for each floor (Column E)	Volume (Column F)	Floor area (Column G)
1	2.5	148.324	72	2.5	26698.315	10679.32582
2	5	146.629	72	2.5	26393.181	10557.27238
3	7.5	144.914	72	2.5	26084.478	10433.79126
4	10	143.178	72	2.5	25772.078	10308.83117
5	12.5	141.421	72	2.5	25455.844	10182.33765
6	15	139.642	72	2.5	25135.632	10054.25283
7	17.5	137.841	72	2.5	24811.288	9924.515102
8	20	136.015	72	2.5	24482.647	9793.058766
9	22.5	134.164	72	2.5	24149.534	9659.813663
10	25	132.288	72	2.5	23811.762	9524.70472
11	27.5	130.384	72	2.5	23469.129	9387.651463
12	30	128.452	72	2.5	23121.419	9248.567
13	32.5	126.491	72	2.5	22768.399	9107.360
14	35	124.499	72	2.5	22409.819	8963.928
15	37.5	122.475	72	2.5	22045.408	8818.163
16	40	120.416	72	2.5	21674.870	8669.948
17	42.5	118.322	72	2.5	21297.887	8519.155
18	45	116.190	72	2.5	20914.110	8365.644
19	47.5	114.018	72	2.5	20523.158	8209.263
20	50	111.803	72	2.5	20124.612	8049.845
21	52.5	109.545	72	2.5	19718.012	7887.205
22	55	107.238	72	2.5	19302.845	7721.140
23	57.5	104.881	72	2.5	18878.559	7551.424
24	60	102.470	72	2.5	18444.511	7377.805
25	62.5	100	72	2.5	18000	7200
26	65	97.468	72	2.5	17544.230	7017.692
27	67.5	94.868	72	2.5	17076.300	6830.520
28	70	92.195	72	2.5	16595.180	6638.072
29	72.5	89.443	72	2.5	16099.690	6439.876
30	75	86.603	72	2.5	15588.457	6235.383

31	77.5	83.666	72	2.5	15059.880	6023.952
32	80	80.623	72	2.5	14512.064	5804.826
33	82.5	77.460	72	2.5	13942.740	5577.096
34	85	74.162	72	2.5	13349.157	5339.663
35	87.5	70.712	72	2.5	12727.922	5091.169
36	90	67.082	72	2.5	12074.767	4829.907
37	92.5	63.246	72	2.5	11384.200	4553.680
38	95	59.161	72	2.5	10648.944	4259.577
39	97.5	54.772	72	2.5	9859.006	3943.602
40	100	50	72	2.5	9000	3600
41	102.5	44.721	72	2.5	8049.845	3219.938
42	105	38.730	72	2.5	6971.370	2788.548
43	107.5	31.623	72	2.5	5692.100	2276.840
44	110	22.361	72	2.5	4024.922	1609.969
45	112.5	0	72	2.5	0	0
Total:					795688.274	318275.310

Now, after determining the total volume of the multiple cuboids, we must determine the efficiency by calculating the ratio of wasted space volume to the volume of the cuboids. By implementing Method 5, the ratio was found to be:

$$14311.726 : 795688.274$$

$$= 0.0179 : 1$$

This value tells us that the wasted space volume is equivalent to 1.79% of the volume of the multiple cuboids. This is considered to be fairly efficient compared with only have a single cuboid with 73.2%.

The floor is also more efficiently used as a single cuboid has a floor area of 280592.231m^2, where as multiple cuboids has a total floor area of 318275.310

Limitations:

Some limitations to this design is of the calculations of the floor area, it does not consider the placements for the stairs and elevators. Also, the designs do not account for thickness of each floor. The sum of each floor may lead to less levels being able to fit if each level must be 2.5m tall.

The efficiency calculations only accounts for the space that is wasted. The other main factor which contributes to this is the financial costs. For example, it may be more financially beneficial by building a single cuboid rather than multiple cuboids due to the amount of materials and the costs of the contractors. Disregarding the aesthetical purposes of the building, it would be more efficient to build a regular rectangular building as it would be able to fit a greater amount of cuboids will less wasted space.

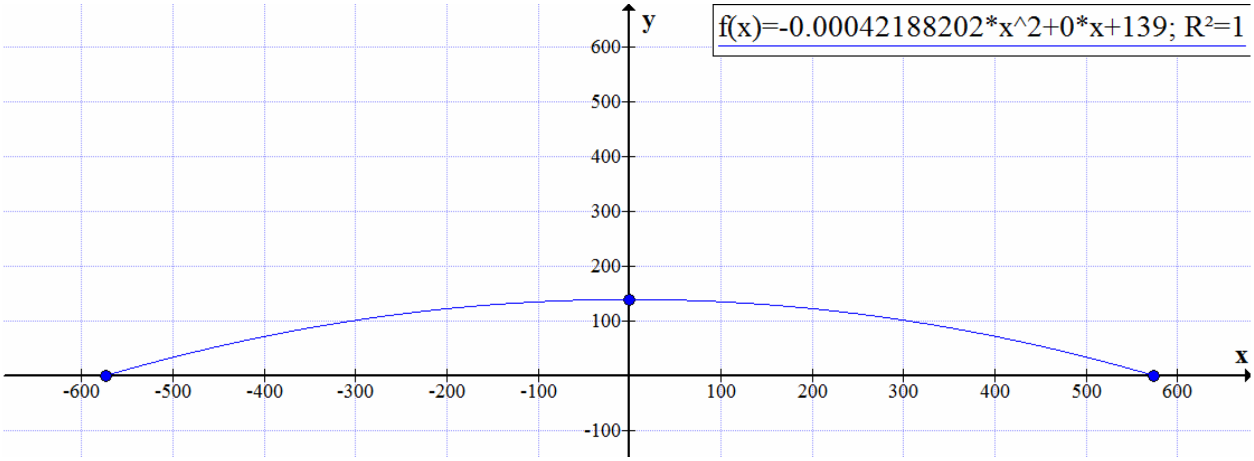
Comparison to Other Structures:

A real life example which follows a parabolic structural design, is the Sydney Harbour Bridge (Fig 9 below).

Fig 9.

The Sydney Harbour bridge is 1149m long, 139m tall and 49m in wide. It can be graphed such as in Fig 10 below.

Fig 10. Graph of Sydney Harbour Bridge



The design of this building is much different from the one chosen. The one chosen for the building follows the equation, $y = \frac{-112.5}{5625}x^2 + 112.5$. The height is 75% of the width, where as the Sydney Harbour Bridge has a height of about 12% of its width.