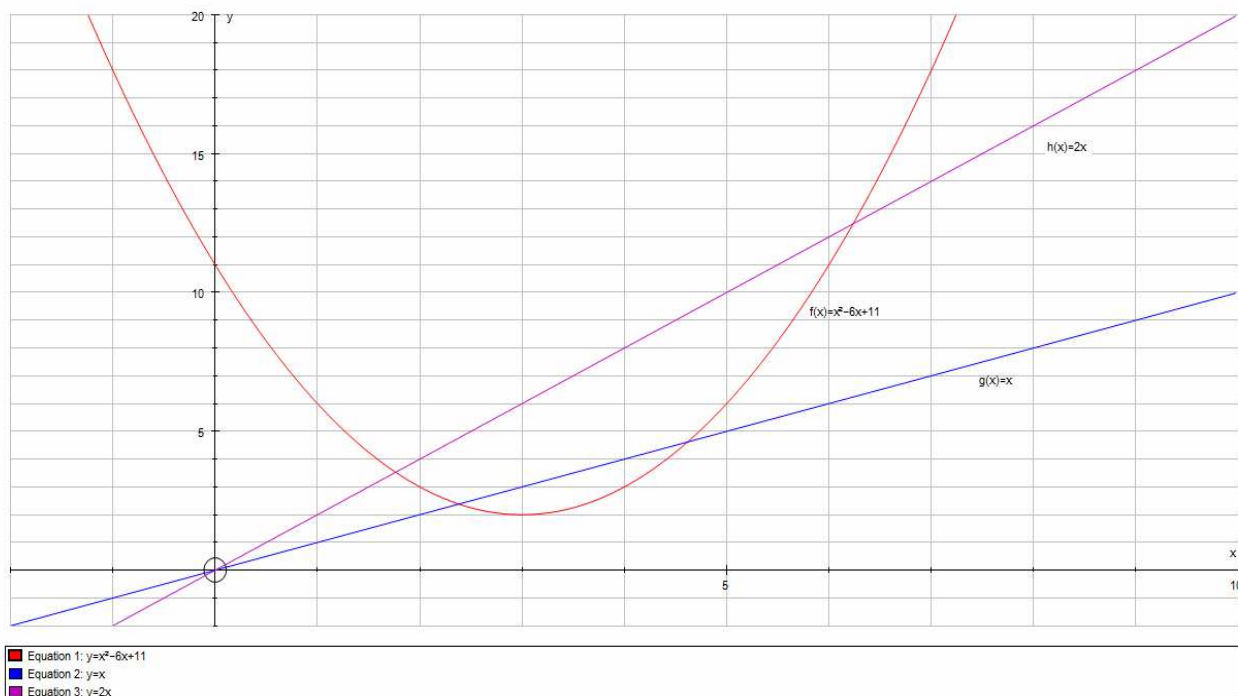


Maths Portfolio 1 HL Type 1

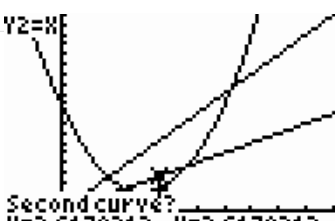
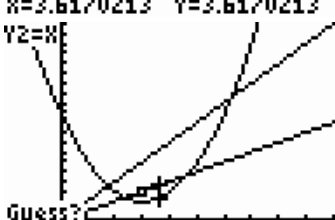

Parabola Investigation

1. Consider the function $f(x) = (x - 3)^2 + 2 = x^2 - 6x + 11$, $g(x) = x$ and $h(x) = 2x$


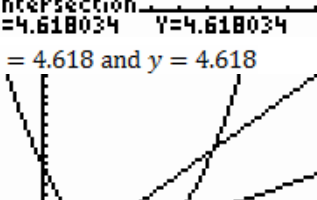
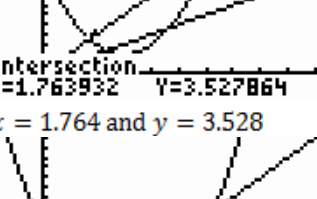


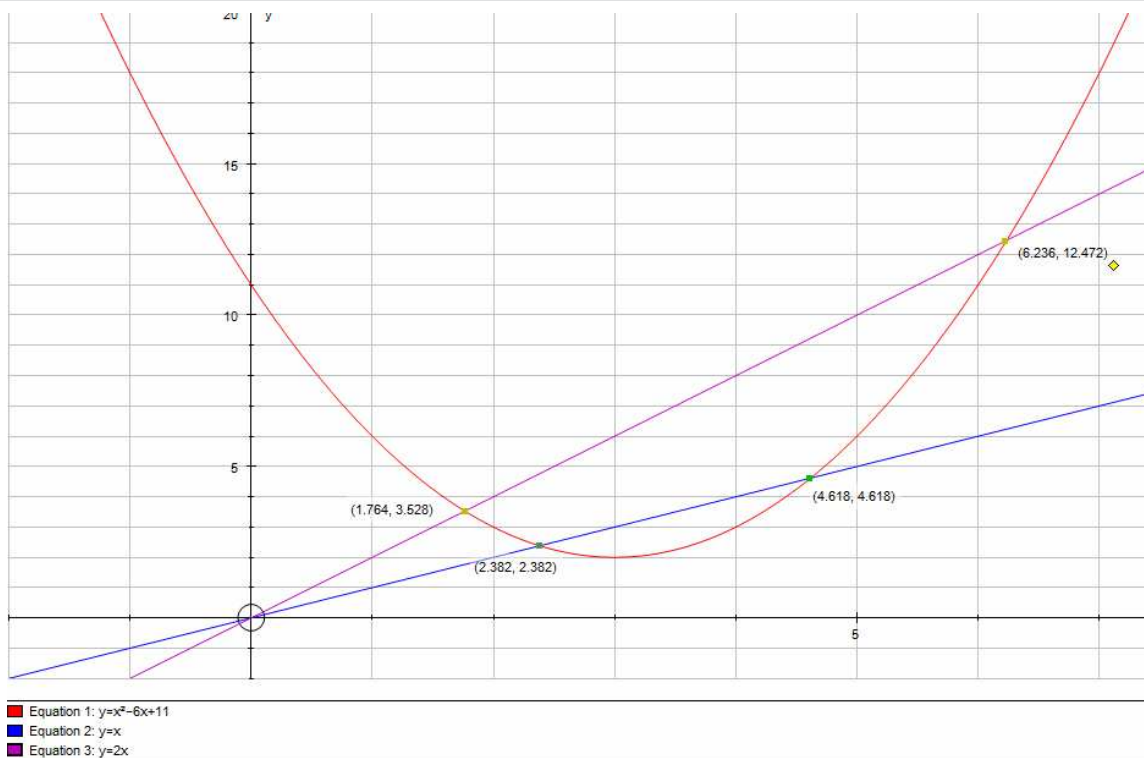
To find the four intersections in the graph shown above using the GDC,

- i. **CALCULATE**
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
- Press the 2nd button and then the TRACE button to select the CALC function. Select the intersect function by pressing button 5.
- Y1=(X-3)²+2
- First curve?
X=3.6170213 Y=2.3807153
- ii. Select the first curve of intersection and press ENTER.

- iii.  Select the second curve of intersection and press ENTER.
- iv.  Select the area of estimation of the intersection point and press ENTER.
- v.  The first intersection point between $f(x)$ and $g(x)$ is $x = 2.382$ and $y = 2.382$

Repeat the above steps i-iv, to obtain the other intersection points between $f(x)$ and $g(x)$ and between $f(x)$ and $h(x)$.

-  The second intersection point between $f(x)$ and $g(x)$ is $x = 4.618$ and $y = 4.618$
-  The first intersection point of $f(x)$ and $h(x)$ is $x = 1.764$ and $y = 3.528$
-  The second intersection point of $f(x)$ and $h(x)$ is $x = 6.236$ and $y = 12.472$



The x - values of these intersection from left to right are

$$x_1 = 1.764$$

$$x_2 = 2.382$$

$$x_3 = 4.618$$

$$x_4 = 6.236$$

To find the values of $S_L = x_2 - x_1$ and $S_R = x_4 - x_3$,

$$S_L = x_2 - x_1 = 2.382 - 1.764 = 0.618$$

$$S_R = x_4 - x_3 = 6.236 - 4.618 = 1.618$$

To calculate the value of $D = |S_L - S_R|$,

$$D = |S_L - S_R|$$

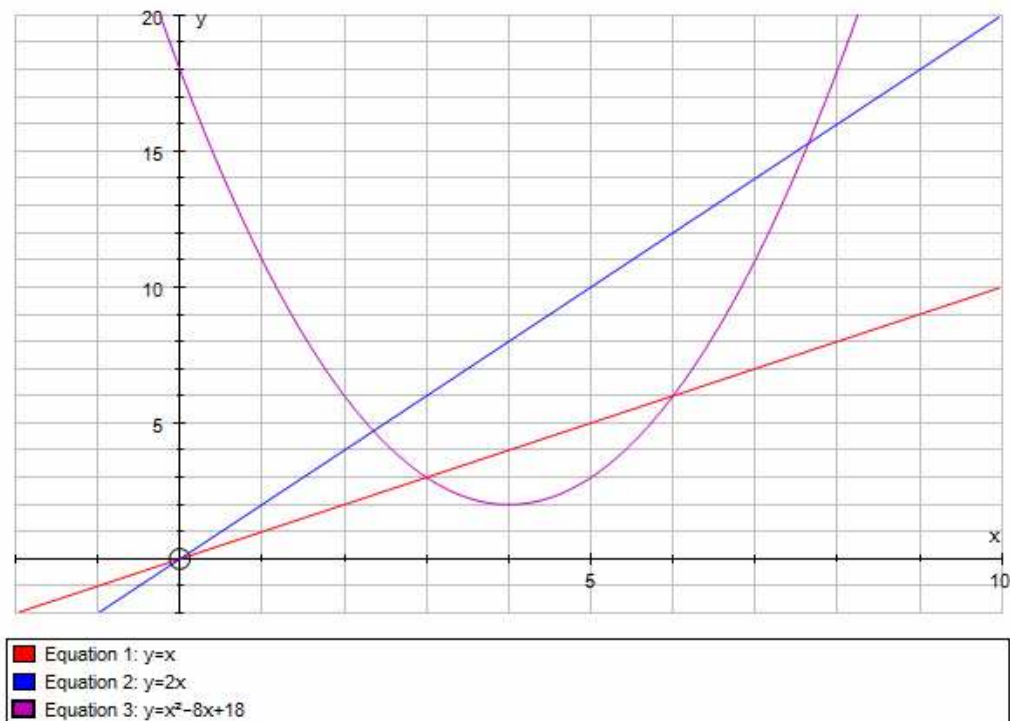
$$= |0.618 - 1.618|$$

$$= |-1|$$

$$= 1$$

2. To find other values of D for other parabolas of the form $y = ax^2 + bx + c, a > 0$, with vertices in quadrant 1, intersected by the lines $y = x$ and $y = 2x$.

Consider the parabola $y = x^2 - 8x + 18$ and the lines $y = x$ and $y = 2x$,



The intersections of the parabola with the lines $y = x$ and $y = 2x$ can be calculated using both the GDC and manual calculation.

By manual calculation,

To calculate the intersection between $y = x^2 - 8x + 18$ and $y = x$,

$$y = x^2 - 8x + 18 \dots\dots (1)$$

$$y = x \dots\dots\dots (2)$$

Sub (2) into (1),

$$x^2 - 8x + 18 = x$$

$$x^2 - 9x + 18 = 0$$

$$(x - 3)(x - 6) = 0$$

$$x = 3 \text{ or } x = 6$$

Sub $x = 3$ into (2),

$$y = 3$$

$$\therefore (x, y) = (3, 3)$$

Sub $x = 6$ into (2),

$$y = 6$$

$$\therefore (x, y) = (6, 6)$$

\therefore The intersections between the parabola $y = x^2 - 8x + 18$ and $y = x$ are (3,3) and (6,6).

By using the GDC,

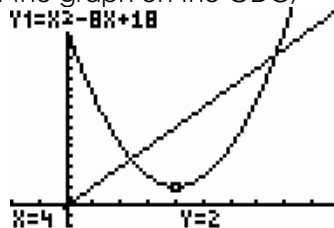
To calculate the intersection between $y = x^2 - 8x + 18$ and $y = 2x$,

i. Key in equations of $y = x^2 - 8x + 18$ and $y = 2x$ into the GDC,

```

Plot1 Plot2 Plot3
Y1=X^2-8X+18
Y2=2X
Y3=
Y4=
Y5=
Y6=
Y7=
    
```

ii. Press the TRACE button to plot the graph on the GDC,

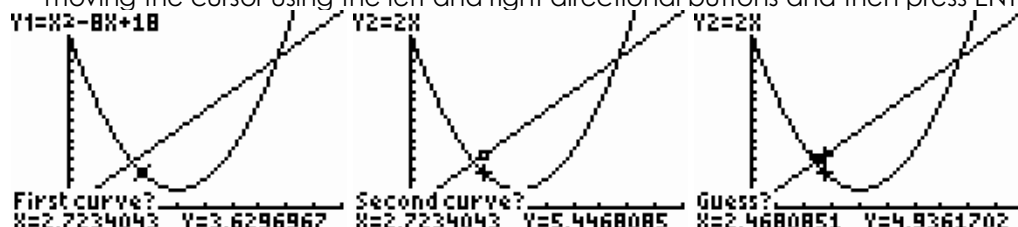


iii. Press the 2nd button and the TRACE button to select the CALC function. Select the intersect function by pressing 5 to calculate the intersection between the parabola and the linear line.

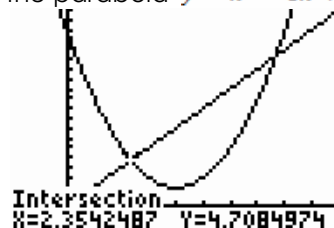
```

MATH>
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
    
```

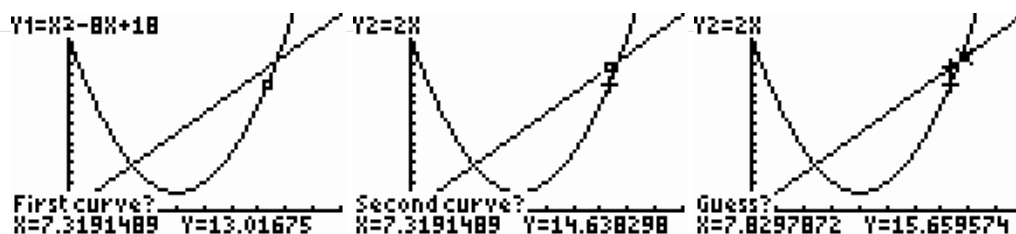
iv. Select the first curve which is the parabola by pressing ENTER and then the second curve which is the line $y = 2x$ by pressing ENTER, then estimate the location of the intersection by moving the cursor using the left and right directional buttons and then press ENTER.



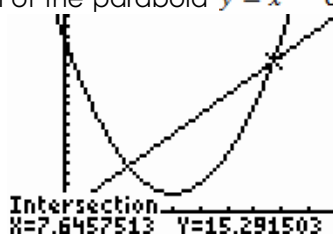
v. Hence, the first intersection of the parabola $y = x^2 - 8x + 18$ and $y = 2x$ is (2.354, 4.708).



vi. Repeat step iii and iv to find the second intersection by moving the cursor closer to the second intersection,



vii. Hence, the second intersection of the parabola $y = x^2 - 8x + 18$ and $y = 2x$ is (7.646, 15.292)



The x -values from left to right are:

$$x_1 = 2.354$$

$$x_2 = 3$$

$$x_3 = 6$$

$$x_4 = 7.646$$

Calculation of S_L and S_R

$$S_L = x_2 - x_1$$

$$= 3 - 2.354$$

$$= 0.646$$

$$S_R = x_4 - x_3$$

$$= 7.646 - 6$$

$$= 1.646$$

Calculation of $D = |S_L - S_R|$

$$D = |S_L - S_R|$$

$$= |0.646 - 1.646|$$

$$= |-1|$$

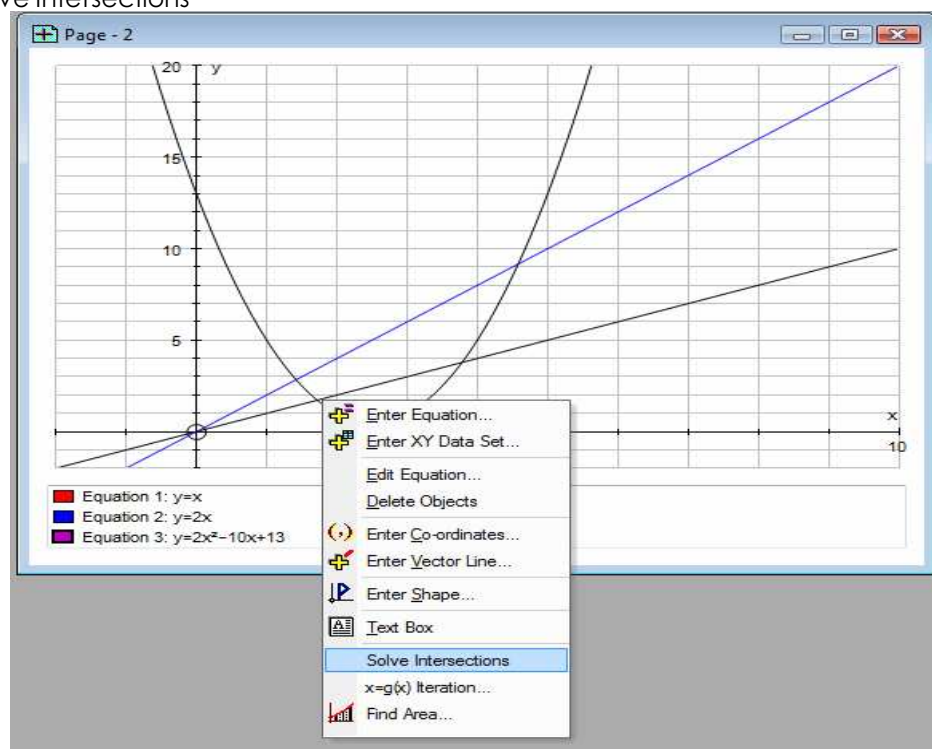
$$= 1$$

Consider the parabola $y = 2x^2 - 10x + 13$ and the lines $y = x$ and $y = 2x$.



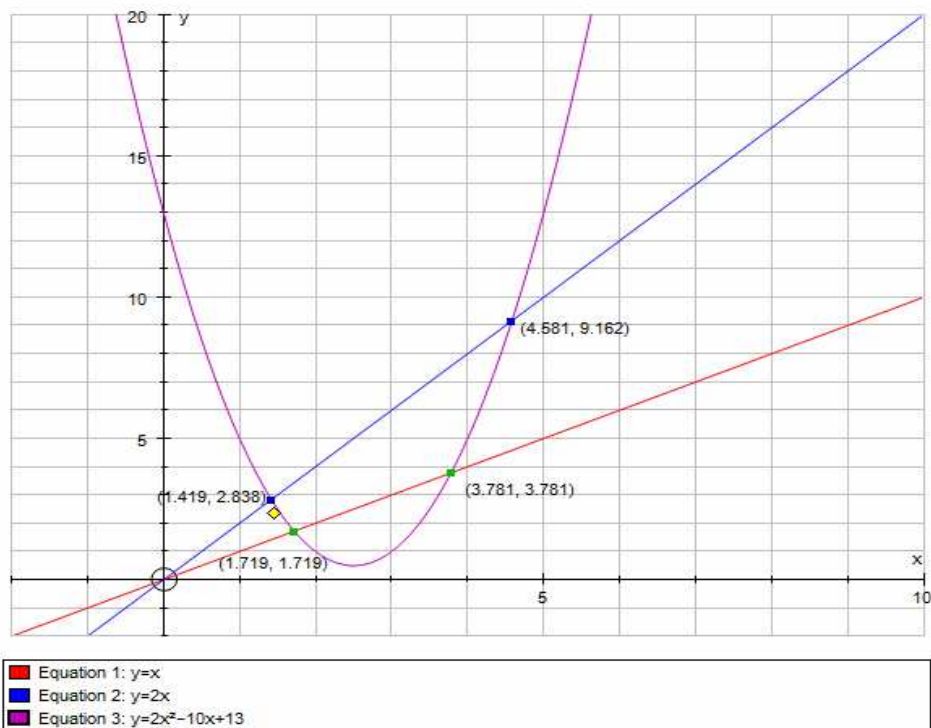
By using the software Autograph to calculate the intersection points,

- Select the parabola and the line $y=x$ to calculate the intersection points. Select the option of "Solve Intersections"



- Repeat step i. to solve for intersections for the parabola and line $y=2x$ by highlighting the line $y=2x$ instead of the line $y=x$.

iii. Label the intersection



points.

The x -values from left to right along the x -axis are:

$$x_1 = 1.419$$

$$x_2 = 1.719$$

$$x_3 = 3.781$$

$$x_4 = 4.581$$

Calculation of S_L and S_R .

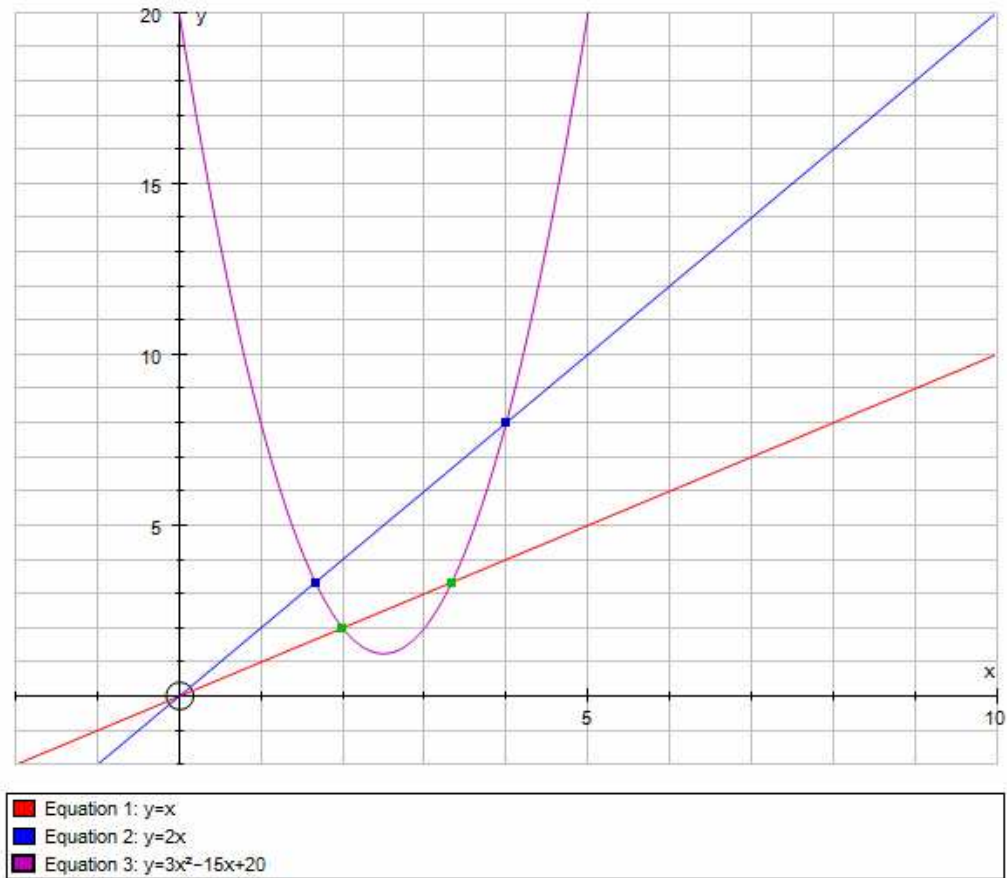
$$\begin{aligned} S_L &= x_2 - x_1 \\ &= 1.719 - 1.419 \\ &= 0.300 \end{aligned}$$

$$\begin{aligned} S_R &= x_4 - x_3 \\ &= 4.581 - 3.781 \\ &= 0.800 \end{aligned}$$

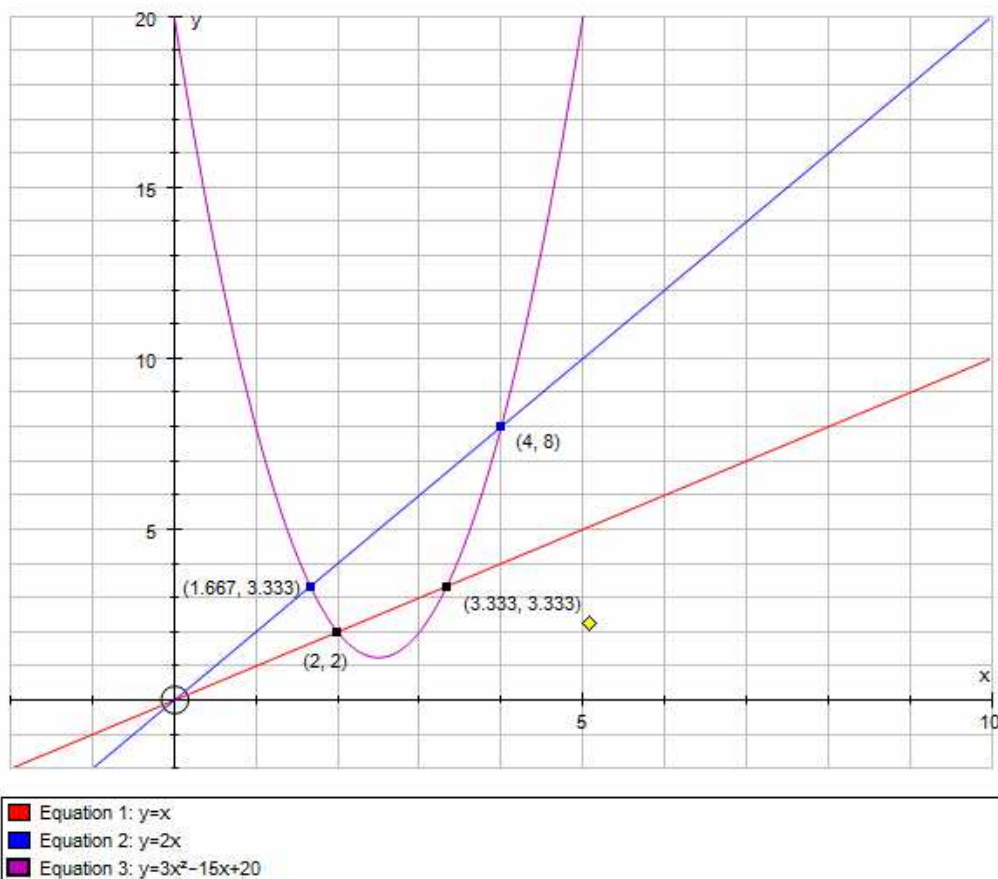
Calculation of D ,

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.300 - 0.800| \\ &= |-0.5| \\ &= 0.5 \end{aligned}$$

Consider the parabola $y = 3x^2 - 15x + 20$ and the lines $y = x$ and $y = 2x$.



By using the software Autograph,
The intersections between the parabola $y = 3x^2 - 15x + 20$ and the line $y = x$ and $y = 2x$ are shown in the graph below:



The intersections between the parabola and line $y=x$ are (2,2) and (3.333,3.333).
 The intersections between the parabola and line $y=2x$ are (1.667,3.333) and (4,8).
 The x-values from left to right along the x-axis are:

$$\begin{aligned}x_1 &= \frac{5}{3} \\x_2 &= 2 \\x_3 &= 3\frac{1}{3} \\x_4 &= 4\end{aligned}$$

Calculation of S_L and S_R .

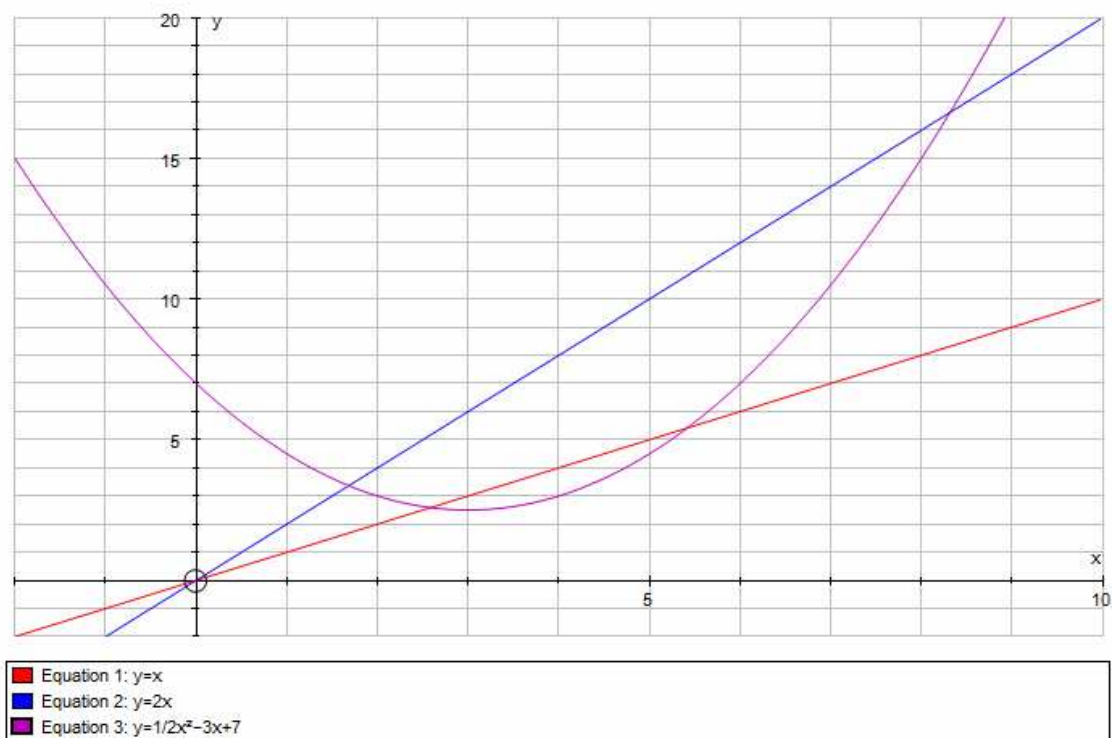
$$\begin{aligned}S_L &= x_2 - x_1 \\&= 2 - \frac{5}{3} \\&= \frac{1}{3}\end{aligned}$$

$$\begin{aligned}S_R &= x_4 - x_3 \\&= 4 - 3\frac{1}{3} \\&= \frac{2}{3}\end{aligned}$$

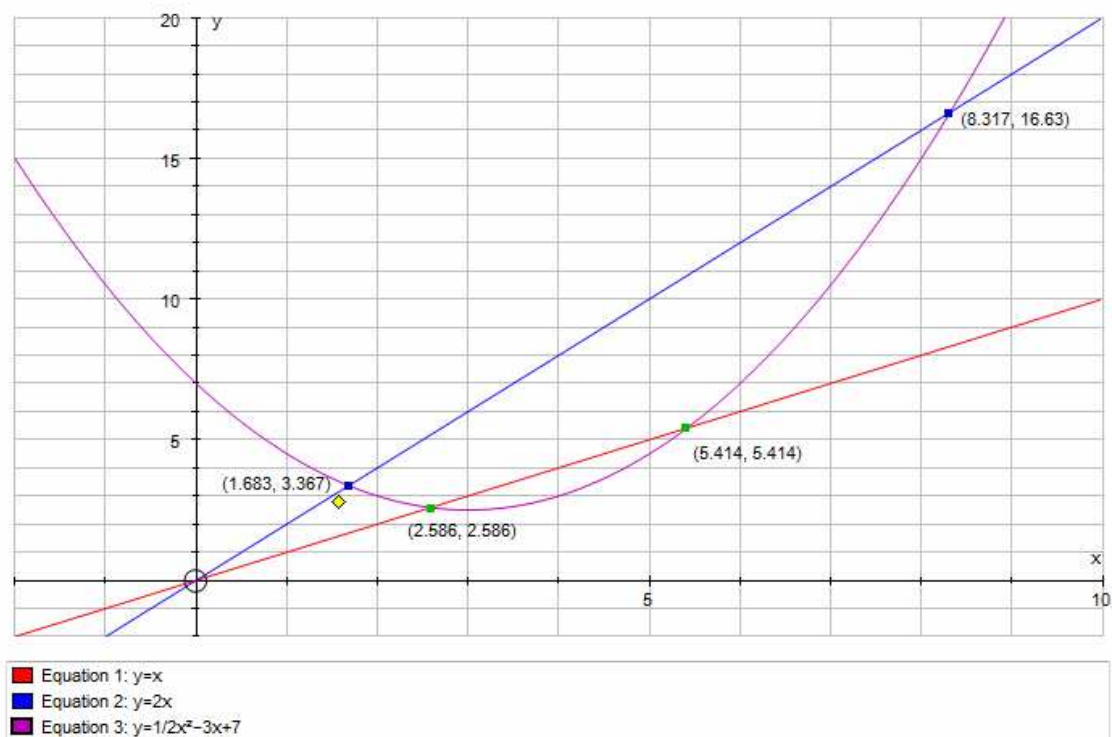
Calculation of D,

$$\begin{aligned}D &= |S_L - S_R| \\&= \left| \frac{1}{3} - \frac{2}{3} \right| \\&= \left| -\frac{1}{3} \right| = \frac{1}{3}\end{aligned}$$

Consider the parabola $y = \frac{1}{2}x^2 - 3x + 7$ and the line $y = x$ and $y = 2x$,



By using Autograph software, the four intersections between the parabola and the lines $y = x$ and $y = 2x$ can be found.



The intersections between the parabola and line $y=x$ are (2.586, 2.586) and (5.414, 5.414)
 The intersections between the parabola and line $y=2x$ are (1.683, 3.367) and (8.317, 16.63)

The x-values from left to right on the x-axis:

$$x_1 = 1.683$$

$$x_2 = 2.586$$

$$x_3 = 5.414$$

$$x_4 = 8.317$$

Calculation of S_L and S_R ,

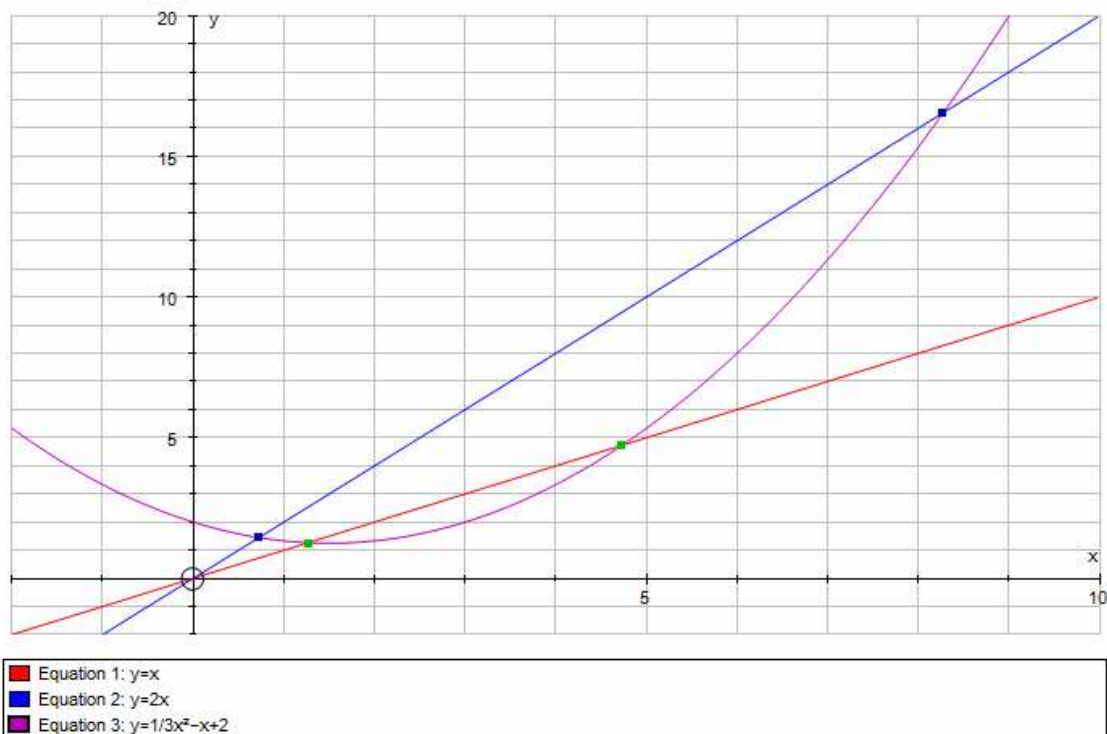
$$\begin{aligned} S_L &= x_2 - x_1 \\ &= 2.586 - 1.683 \\ &= 0.903 \end{aligned}$$

$$\begin{aligned} S_R &= x_4 - x_3 \\ &= 8.317 - 5.414 \\ &= 2.903 \end{aligned}$$

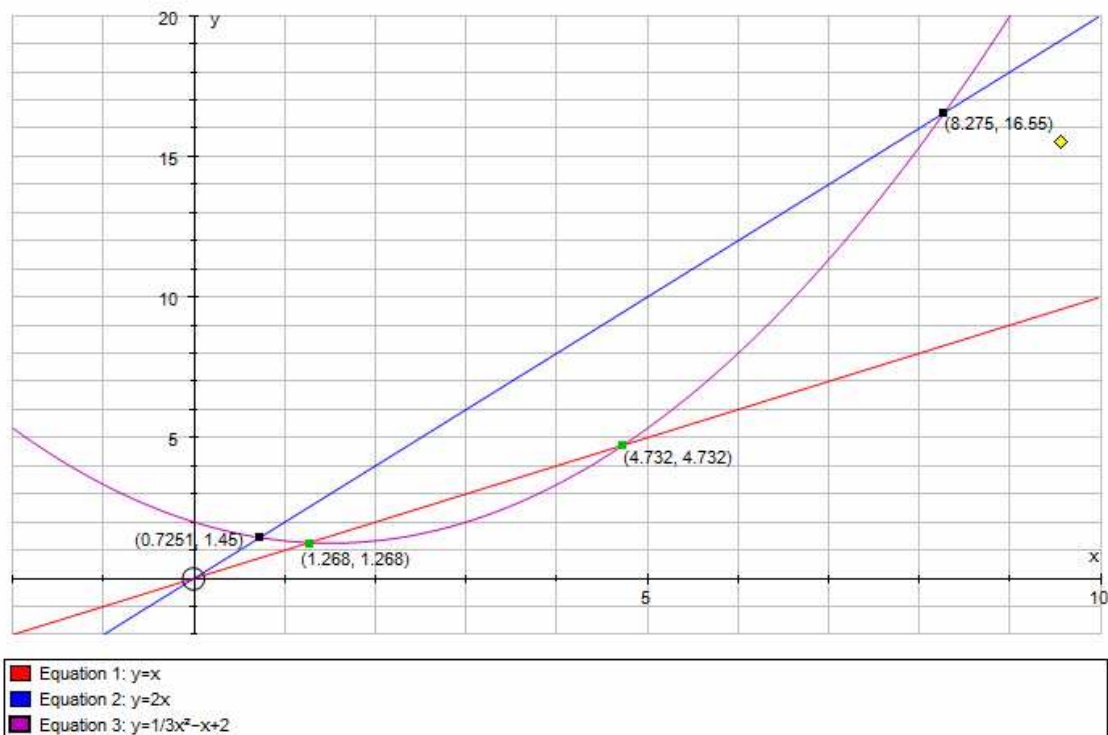
Calculation of D

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.903 - 2.903| \\ &= |-2| \\ &= 2 \end{aligned}$$

Consider the parabola $y = \frac{1}{3}x^2 - x + 2$ and the lines $y = x$ and $y = 2x$



The four intersections between the parabola and the lines $y=x$ and $y=2x$ can be found via Autograph software.



The intersections between the parabola and line $y = x$ are (1.268,1.268) and (4.732,4.732)
 The intersections between the parabola and line $y = 2x$ are (0.7251,1.45) and (8.275,16.55)

The x-values from left to right on the x-axis:

$$x_1 = 0.7251$$

$$x_2 = 1.268$$

$$x_3 = 4.732$$

$$x_4 = 8.275$$

Calculation of S_L and S_R ,

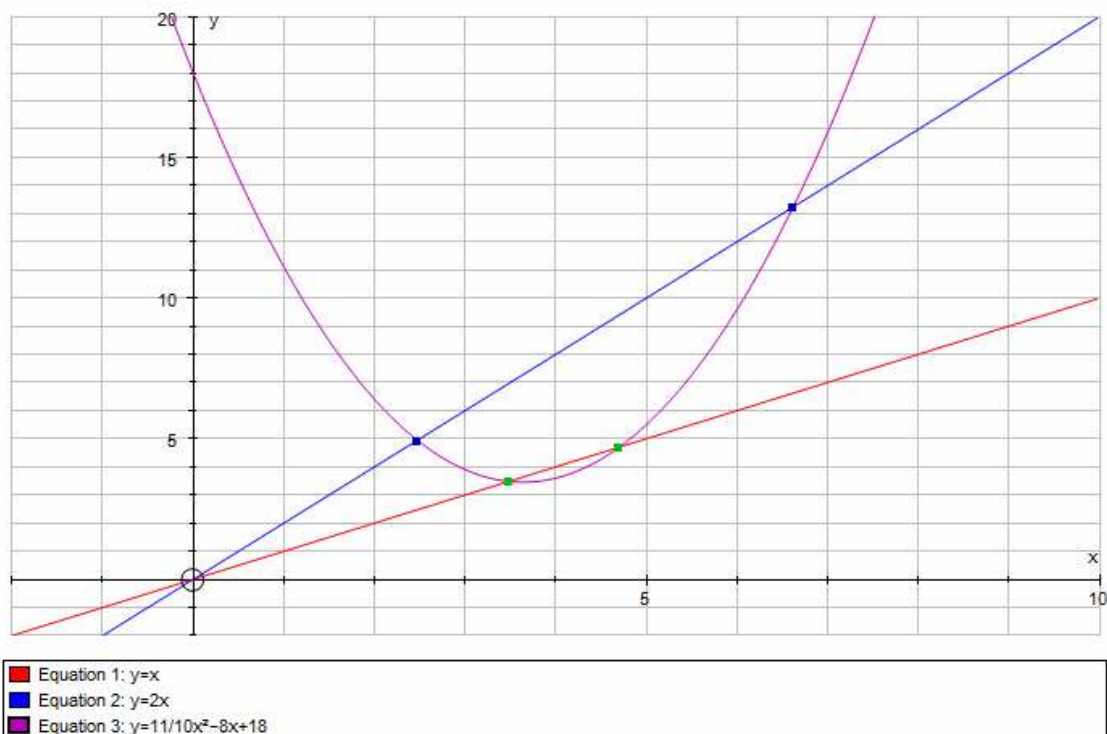
$$\begin{aligned} S_L &= x_2 - x_1 \\ &= 1.268 - 0.7251 \\ &= 0.5429 \approx 0.543 \end{aligned}$$

$$\begin{aligned} S_R &= x_4 - x_3 \\ &= 8.275 - 4.732 \\ &= 3.543 \end{aligned}$$

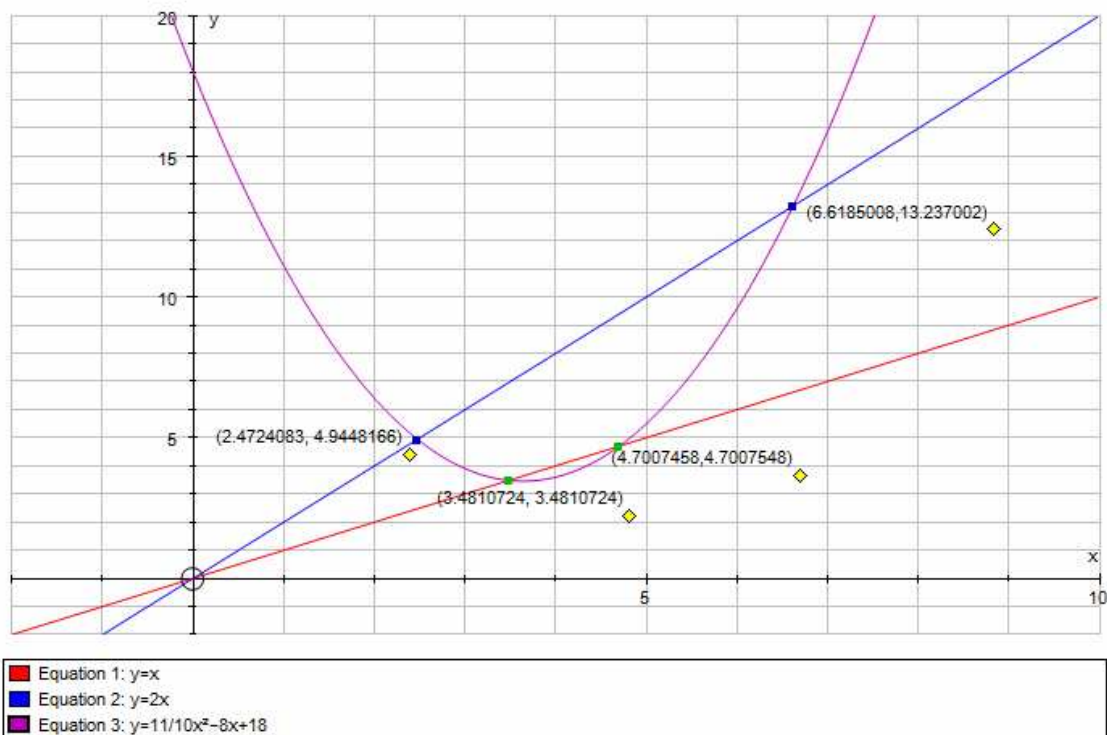
Calculation of D,

$$\begin{aligned} D &= |S_L - S_R| \\ &= |0.543 - 3.543| \\ &= |-3| \\ &= 3 \end{aligned}$$

Consider the parabola $y = \frac{11}{10}x^2 - 8x + 18$ and the lines $y = x$ and $y = 2x$.



The intersections between the parabola and the lines $y=x$ and $y=2x$ can be found using Autograph software,



The intersections between the parabola and the line $y=x$ are $(3.4810724, 3.4810724)$ and $(4.7007458, 4.7007548)$

The intersections between the parabola and the line $y=2x$ are $(2.4724083, 4.9448166)$ and $(6.6185008, 13.237002)$

The x-values from left to right on the x-axis:

$$x_1 = 2.4724083$$

$$x_2 = 3.4810724$$

$$x_3 = 4.7007458$$

$$x_4 = 6.6185008$$

Calculation of S_L and S_R :

$$S_L = x_2 - x_1$$

$$= 3.4810724 - 2.4724083$$

$$= 1.0086641$$

$$S_R = x_4 - x_3$$

$$= 6.6185008 - 4.7007458$$

$$= 1.917755$$

Calculation of D:

$$D = |S_L - S_R|$$

$$= |1.0086641 - 1.917755|$$

$$= |-0.9090909|$$

$$= 0.9090909$$

$$\approx \frac{10}{11}$$

Table showing the values of D for various parabolas of the form

$y = ax^2 + bx + c, a > 0$, with vertices in quadrant 1, intersected by the lines $y = x$ and $y = 2x$.

Parabola equation	a	b	C	$D = S_L - S_R $
$y = x^2 - 6x + 11$	1	-6	11	1
$y = x^2 - 8x + 18$	1	-8	18	1
$y = 2x^2 - 10x + 13$	2	-10	13	$\frac{1}{2}$
$y = 3x^2 - 15x + 20$	3	-15	20	$\frac{1}{3}$
$y = \frac{1}{2}x^2 - 3x + 7$	$\frac{1}{2}$	-3	7	2
$y = \frac{1}{3}x^2 - x + 2$	$\frac{1}{3}$	-1	2	3
$y = \frac{11}{10}x^2 - 8x + 18$	$\frac{11}{10}$	-8	18	$\frac{10}{11}$

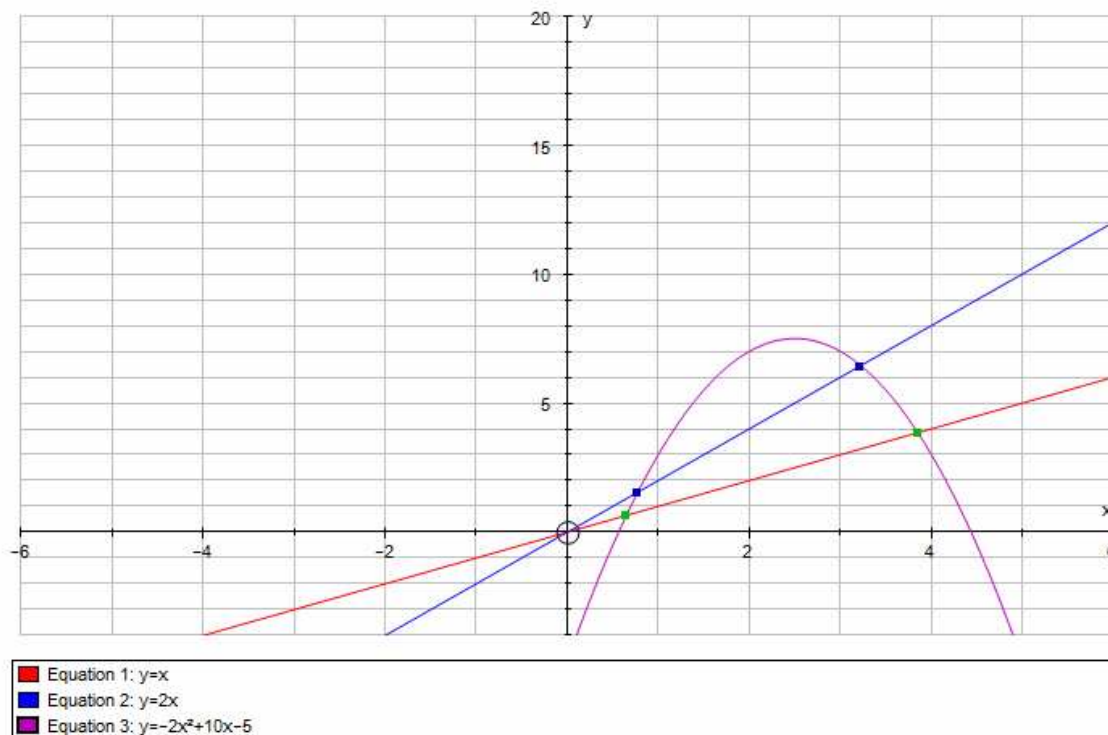
From the table, there is no relationship between b and c with $D = |S_L - S_R|$. However, a is inversely proportional to the value of $D = |S_L - S_R|$.

Conjecture: For parabolas with the form $y = ax^2 + bx + c, a > 0$, with vertices in quadrant 1, intersected by the lines $y=x$ and $y=2x$, the value of a is inversely proportional to the value of $D = |S_L - S_R|$, i.e. $D = \frac{1}{a}$.

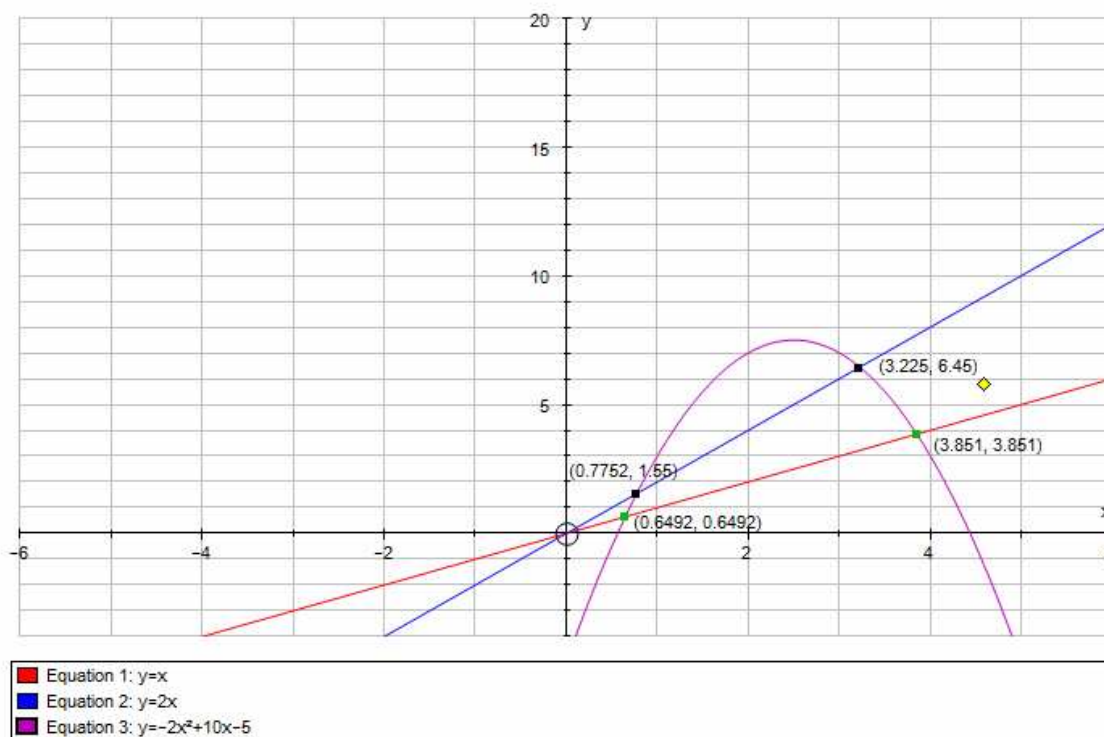
3. Investigation of the conjecture for any real value of a and any placement of the vertex.

To investigate my conjecture for any real value of a and any placement of the vertex, the conditions earlier attached to my conjecture that $a > 0$ and vertices in quadrant 1 and intersected by the lines $y=x$ and $y=2x$ should be discarded, but the conjecture that the value of a is inversely proportional to the value of $D = |S_L - S_R| = \frac{1}{a}$ should be kept.

First, consider the parabola $y = -2x^2 + 10x - 5$ and the lines $y = x$ and $y = 2x$.



The four intersections between the parabola and the two lines $y=x$ and $y=2x$ can be found once again by Autograph software.



The intersection between the parabola and the line $y=x$ are $(0.6492, 0.6492)$ and $(3.851, 3.851)$

The intersection between the parabola and the line $y=2x$ are (0.7752,1.55) and (3.225,6.45)

The x-values:

$$x_1 = 0.7752$$

$$x_2 = 0.6492$$

$$x_3 = 3.851$$

$$x_4 = 3.225$$

Calculation of $D = |S_L - S_R|$,

$$D = |S_L - S_R|$$

$$= |(x_2 - x_1) - (x_4 - x_3)|$$

$$= |(0.6492 - 0.7752) - (3.225 - 3.851)|$$

$$= |-0.126 - 0.626|$$

$$= |0.5|$$

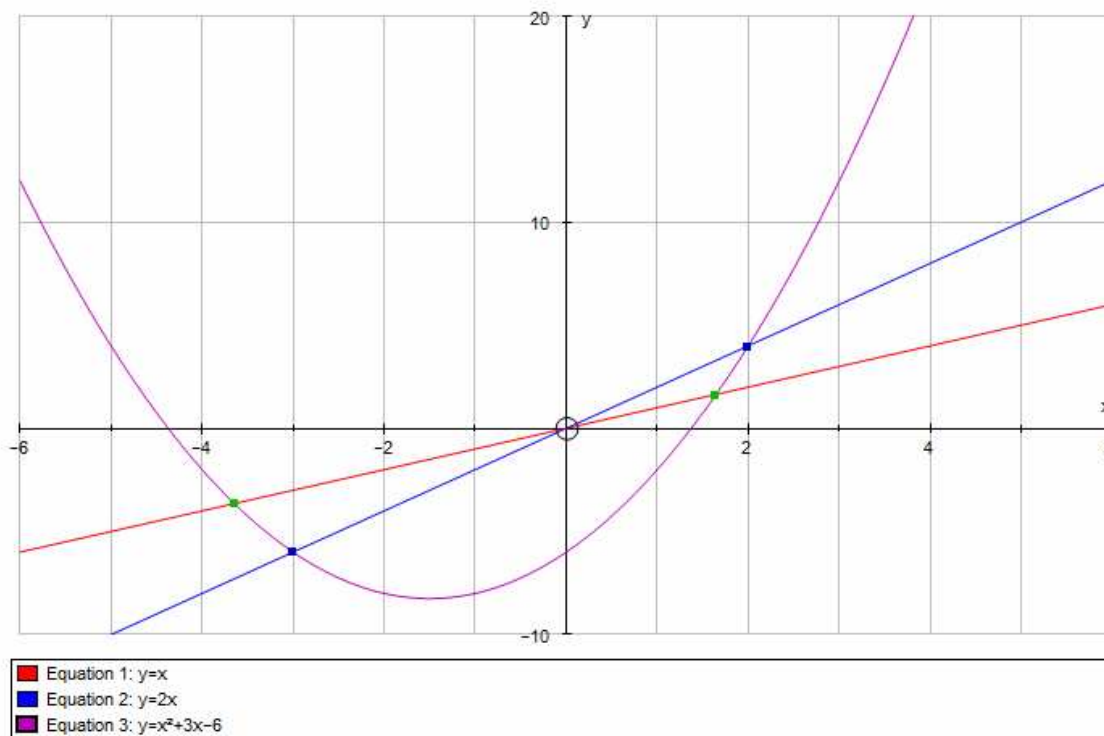
$$= 0.5$$

The conjecture does not hold when $a < 0$, because the value of a is -2 and according to my conjecture made in part 2, the value of D should be

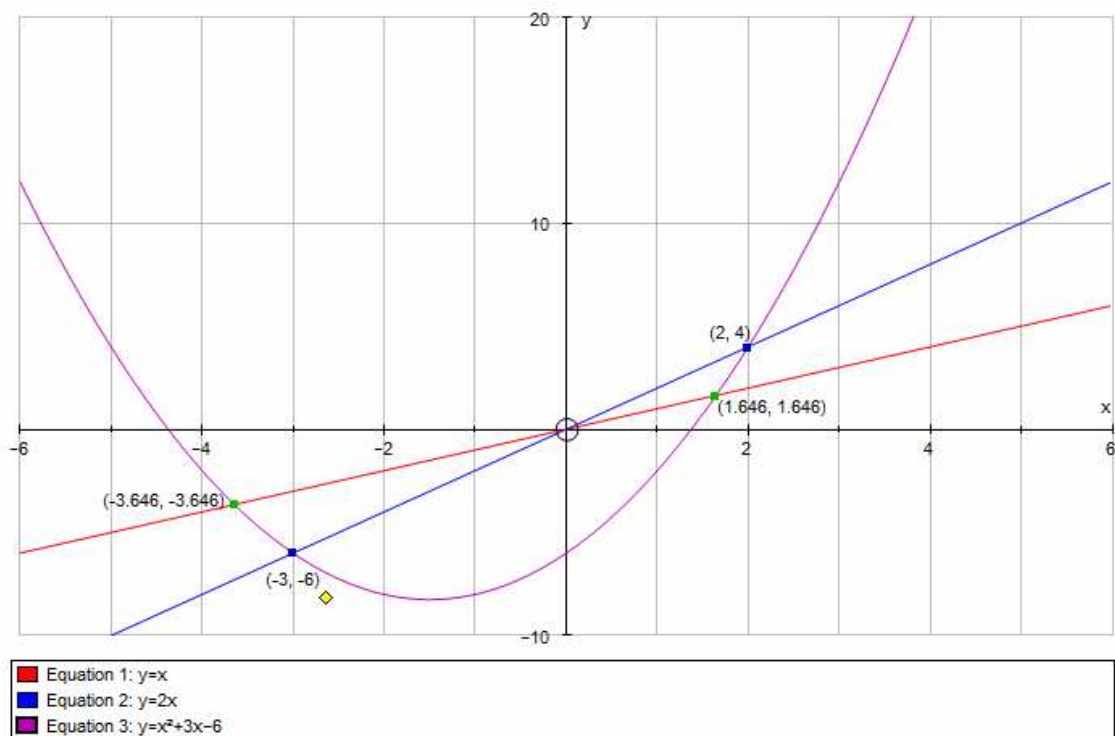
$$D = \frac{1}{a} = \frac{1}{-2} = -0.5$$

Hence, I would modify my conjecture to make $D = \frac{1}{a}$ always positive, i.e. $D = \frac{1}{|a|}, \neq 0$.

Consider the parabola $y = x^2 + 3x - 6$ and the lines $y = x$ and $y = 2x$.



The intersections between the parabola and the lines $y = x$ and $y = 2x$:



The intersection between the parabola and the line $y = x$ are $(-3.646, -3.646)$ and $(1.646, 1.646)$
 The intersection between the parabola and the line $y = 2x$ are $(-3, -6)$ and $(2, 4)$

x-values :

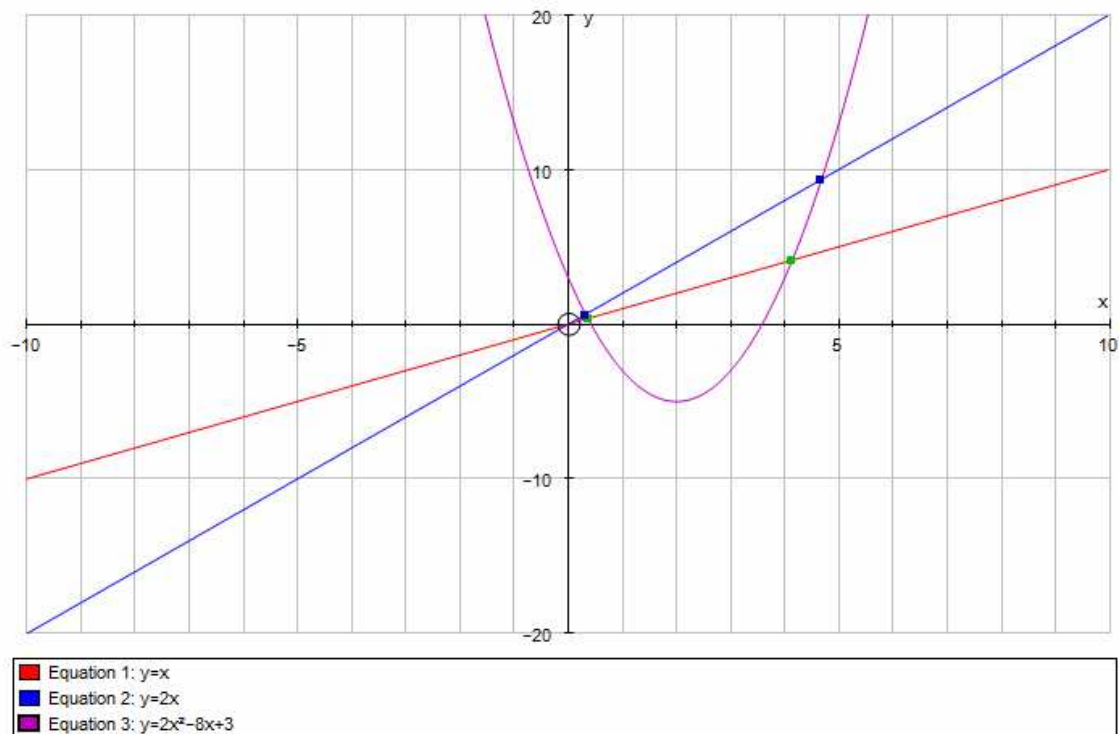
$$\begin{aligned} x_1 &= -3 \\ x_2 &= -3.646 \\ x_3 &= 1.646 \\ x_4 &= 2 \end{aligned}$$

Calculation of D,

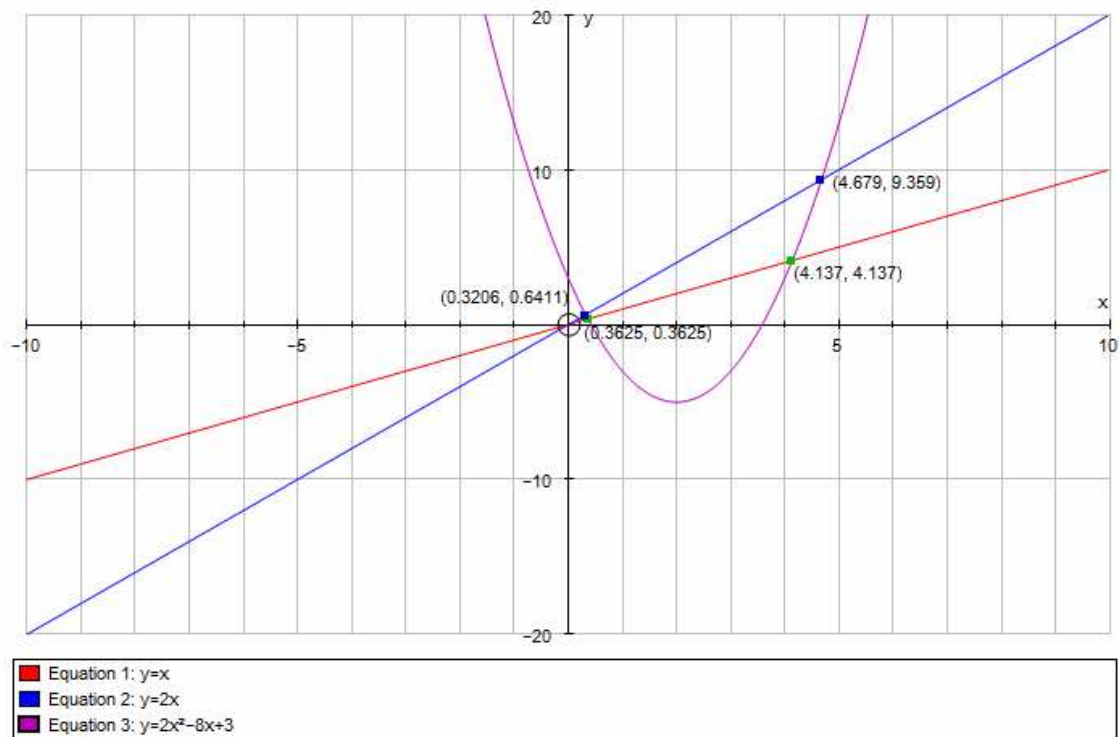
$$\begin{aligned} D &= |S_L - S_R| \\ &= |(x_2 - x_1) - (x_4 - x_3)| \\ &= |(-3.646 - -3) - (2 - 1.646)| \\ &= |-0.646 - 0.354| \\ &= |-1| \\ &= 1 \end{aligned}$$

In this case, the conjecture holds true when the vertex is in the third quadrant, $a > 0, a \in \mathbb{R}$, and the parabola intersects the lines $y = x$ and $y = 2x$ at two distinct points each.

Consider the parabola
 $y = 2x^2 - 8x + 3$ and the lines $y = x$ and $y = 2x$,



The intersections of the parabola and the lines $y = x$ and $y = 2x$:



The intersections between the parabola and the line $y=x$ are $(0.3625, 0.3625)$ and $(4.137, 4.137)$

The intersections between the parabola and the line $y=2x$ are $(0.3206, 0.6411)$ and $(4.679, 9.359)$

The x-values:

$$x_1 = 0.3206$$

$$x_2 = 0.3625$$

$$x_3 = 4.137$$

$$x_4 = 4.679$$

Calculation of D

$$D = |S_L - S_R|$$

$$= |(x_2 - x_1) - (x_4 - x_3)|$$

$$= |(0.3625 - 0.3206) - (4.679 - 4.137)|$$

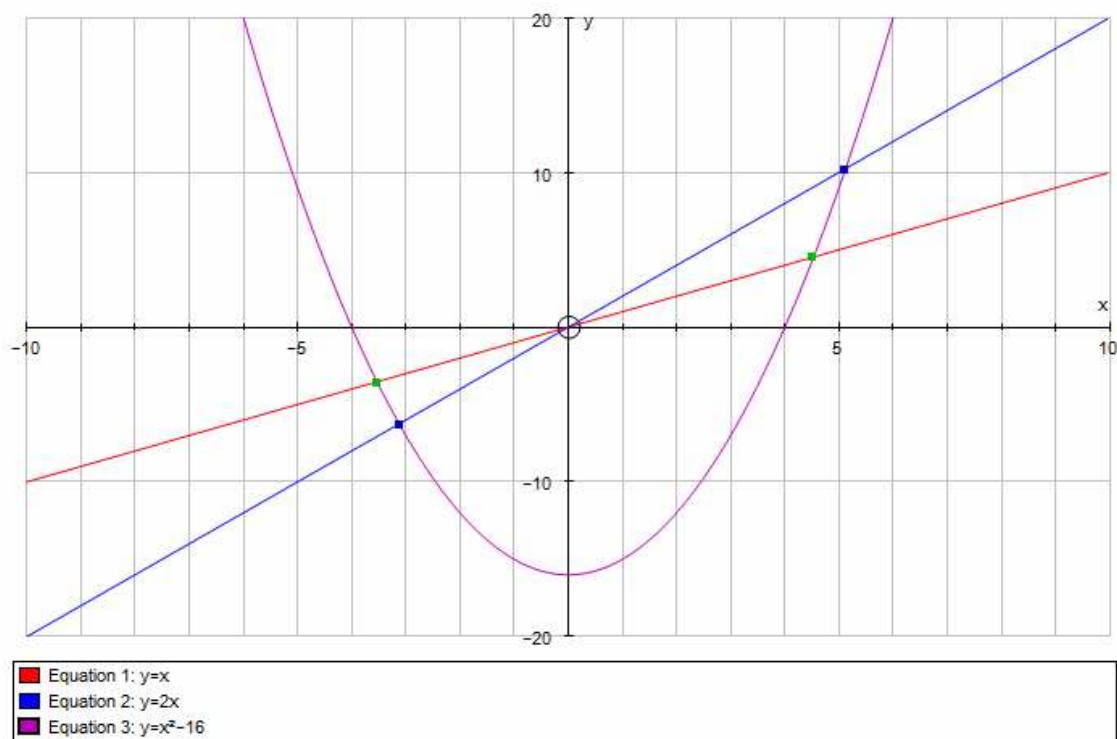
$$= |0.0419 - 0.542|$$

$$= |-0.5|$$

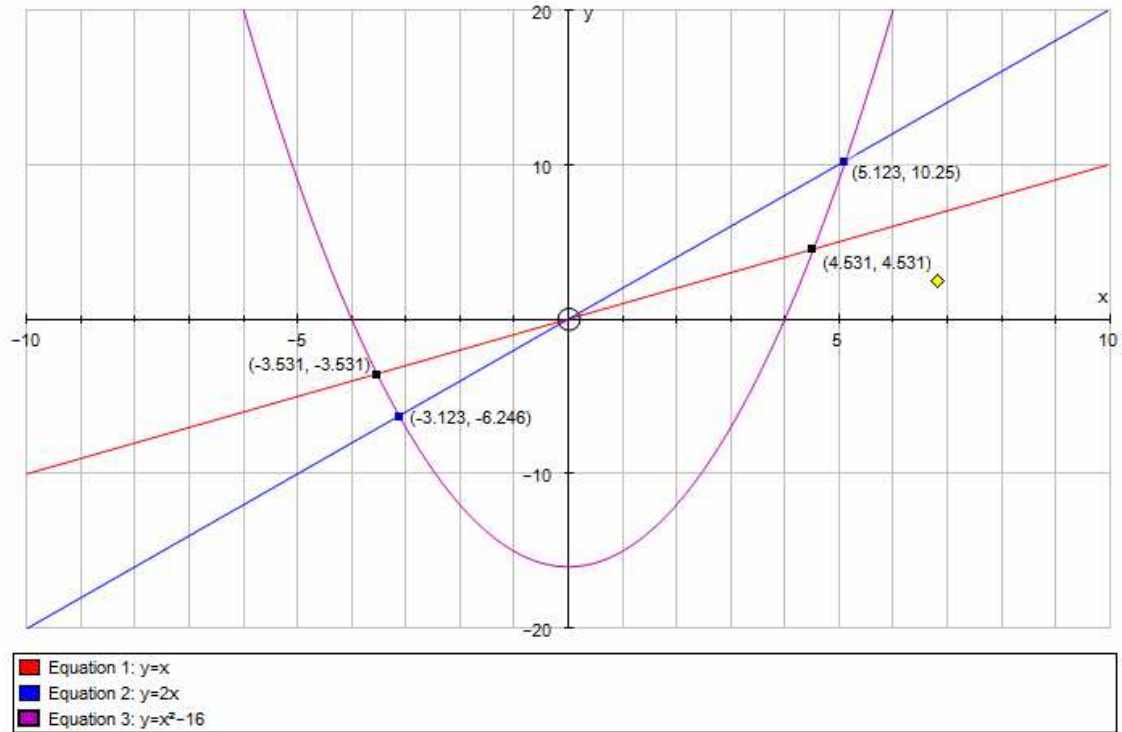
$$= 0.5$$

The conjecture holds for the case when the vertex is in quadrant 4, $a > 0, a \in \mathbb{R}$, and the parabola intersects the lines $y = x$ and $y = 2x$ at two distinct points each.

Consider the parabola $y = x^2 - 16$ and the lines $y = x$ and $y = 2x$.



The intersections of the parabola and the lines $y=x$ and $y=2x$ are:



The intersections between the parabola and the line $y = x$ are $(-3.531, -3.531)$ and $(4.531, 4.531)$

The intersections between the parabola and the line $y = 2x$ are $(-3.123, -6.246)$ and $(5.123, 10.25)$

The x-values :

$$x_1 = -3.123$$

$$x_2 = -3.531$$

$$x_3 = 4.531$$

$$x_4 = 5.123$$

Calculation of D

$$D = |S_L - S_R|$$

$$= |(x_2 - x_1) - (x_4 - x_3)|$$

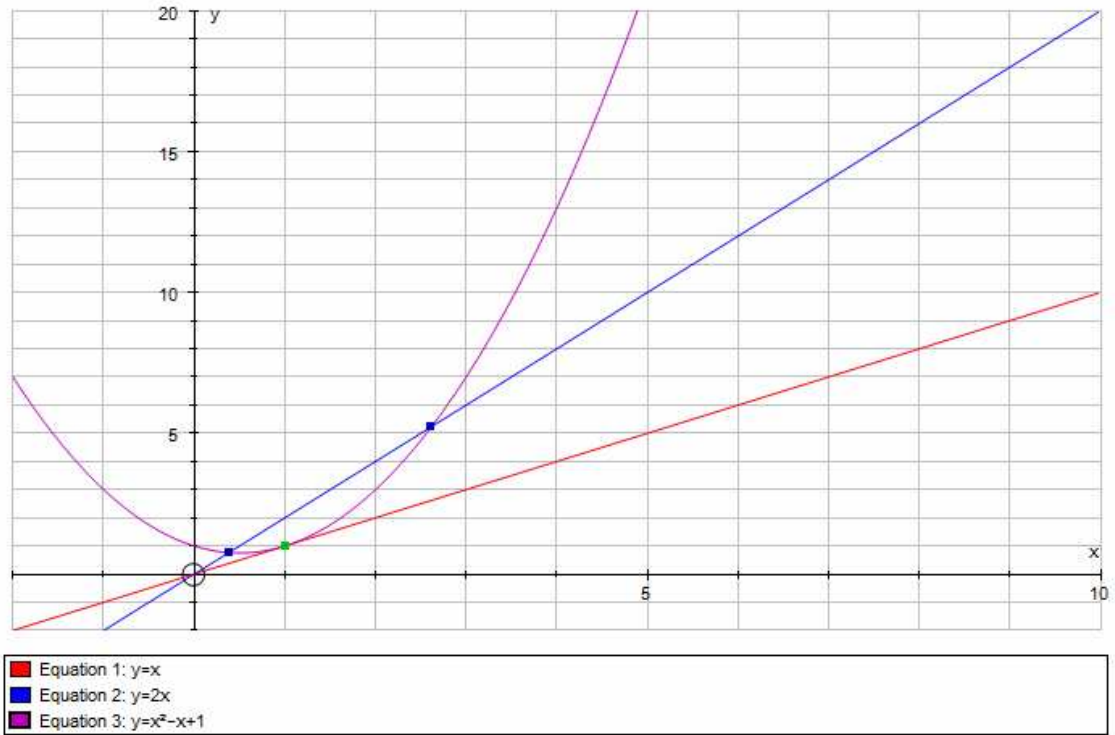
$$= |(-3.531 + 3.123) - (5.123 - 4.531)|$$

$$= |-0.408 - 0.592|$$

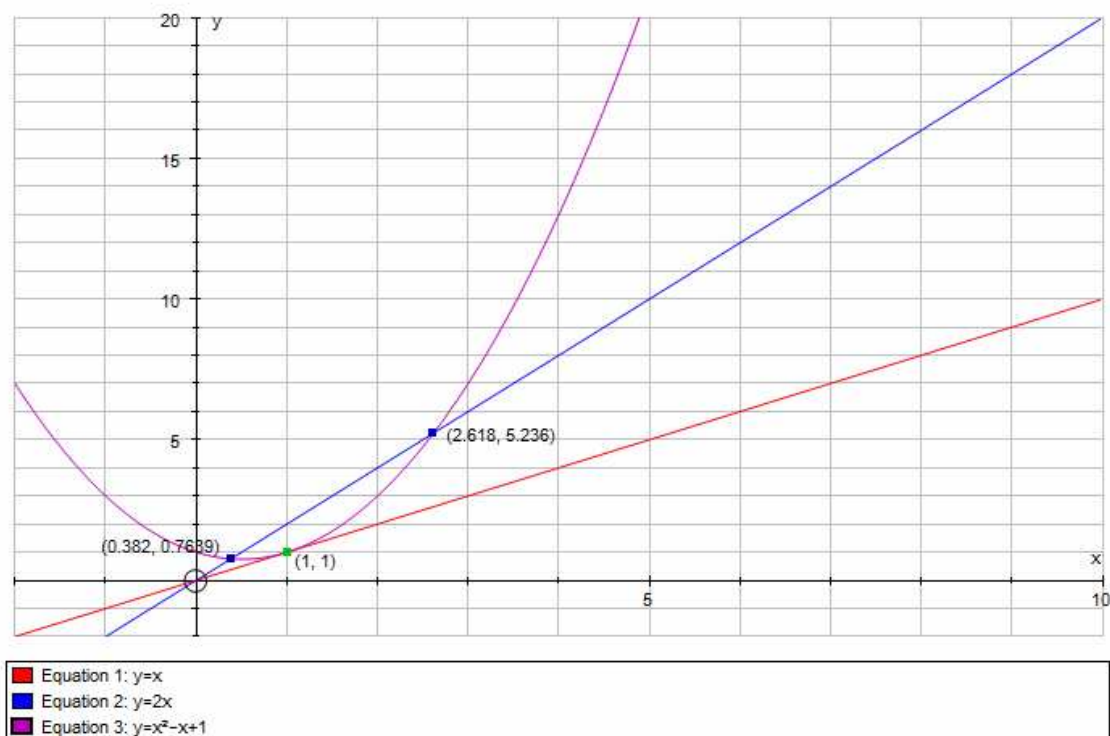
$$= |-1|$$

$$= 1$$

Consider the parabola $y = x^2 - x + 1$ and the lines $y = x$ and $y = 2x$



The intersections of the parabola and the lines $y=x$ and $y=2x$:



The intersection between the parabola and the line $y=x$ is $(1,1)$

The intersections between the parabola and the line $y=2x$ are $(0.382, 0.7639)$ and $(2.618, 5.236)$

The x-values:

Since there is only one real intersection between the parabola and $y = x$, the values of x_2 and x_3 will be repeated.

$$x_1 = 0.382$$

$$x_2 = 1$$

$$x_3 = 1$$

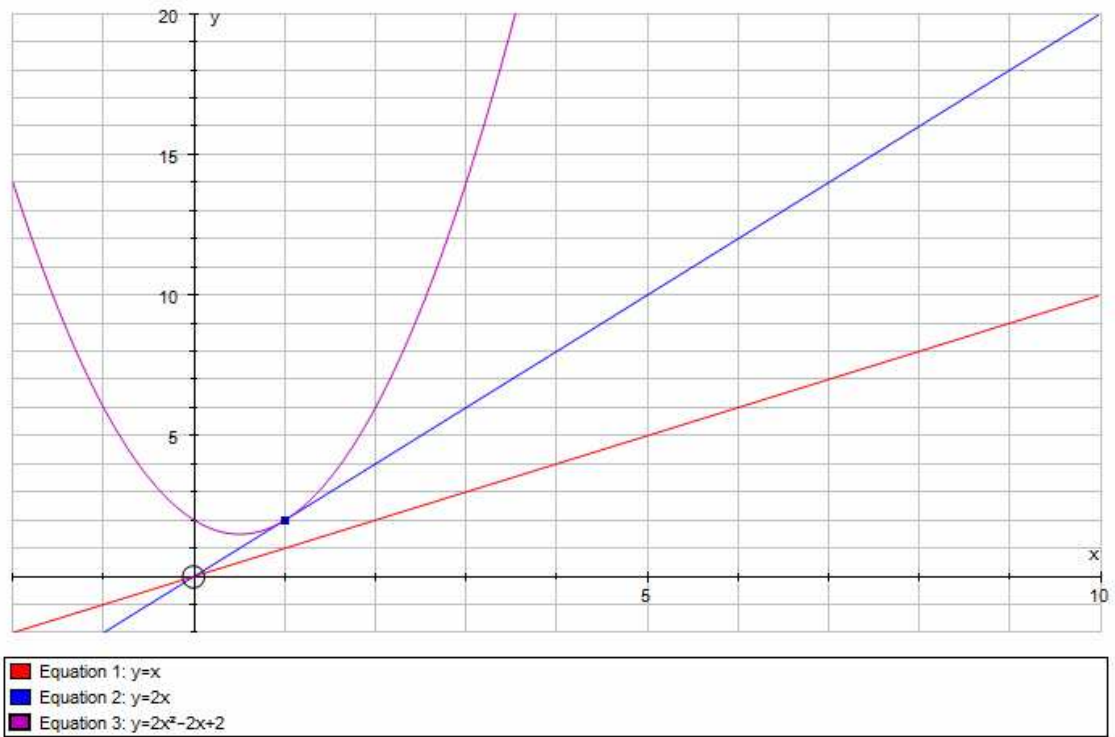
$$x_4 = 2.618$$

Calculation of D,

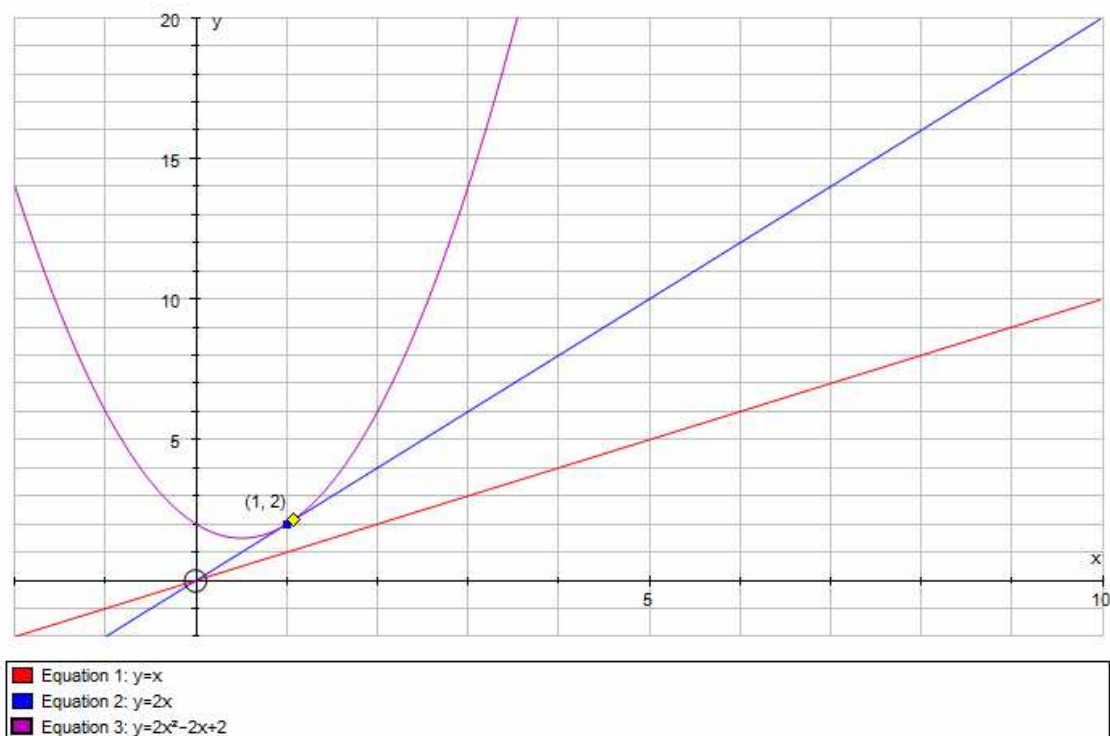
$$\begin{aligned} D &= |(x_2 - x_1) - (x_4 - x_3)| \\ &= |(1 - 0.382) - (2.618 - 1)| \\ &= |0.618 - 1.618| \\ &= |-1| \\ &= 1 \end{aligned}$$

The conjecture still holds true for when the intersect between the parabola and the line is repeated.

Consider the parabola $y = 2x^2 - 2x + 2$ and the lines $y = x$ and $y = 2x$.



The intersection between the parabola and the lines $y=x$ and $y=2x$.



There is only one point of real intersection between the parabola and the line $y=2x$ at $(1,2)$.

Hence, to find the intersections between the parabola and the line $y = x$, i.e. to find the imaginary intersections between the parabola and the line $y = x$.

$$y = 2x^2 - 2x + 2 \dots\dots\dots (1)$$

$$y = x \dots\dots\dots (2)$$

Substitute (2) into (1),

$$x = 2x^2 - 2x + 2$$

$$2x^2 - 3x + 2 = 0$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{3 \pm \sqrt{(-3)^2 - 4(2)(2)}}{2(2)} \\ &= \frac{3 \pm \sqrt{9 - 16}}{4} \\ &= \frac{3 \pm \sqrt{-7}}{4} \end{aligned}$$

Let $\sqrt{-1}$ be i .

$$\begin{aligned} &= \frac{3 \pm \sqrt{7}i}{4} \\ \therefore x &= \frac{3 + \sqrt{7}i}{4} \text{ or } x = \frac{3 - \sqrt{7}i}{4} \end{aligned}$$

Therefore, x-values:

$$x_1 = 1$$

$$x_2 = \frac{3 - \sqrt{7}i}{4}$$

$$x_3 = \frac{3 + \sqrt{7}i}{4}$$

$$x_4 = 1$$

Calculation of D,

$$D = |S_L - S_R|$$

$$= |(x_2 - x_1) - (x_4 - x_3)|$$

$$= \left| \left(\frac{3 - \sqrt{7}i}{4} - 1 \right) - \left(1 - \frac{3 + \sqrt{7}i}{4} \right) \right|$$

$$= \left| \frac{3 - \sqrt{7}i}{4} - 1 - 1 + \frac{3 + \sqrt{7}i}{4} \right|$$

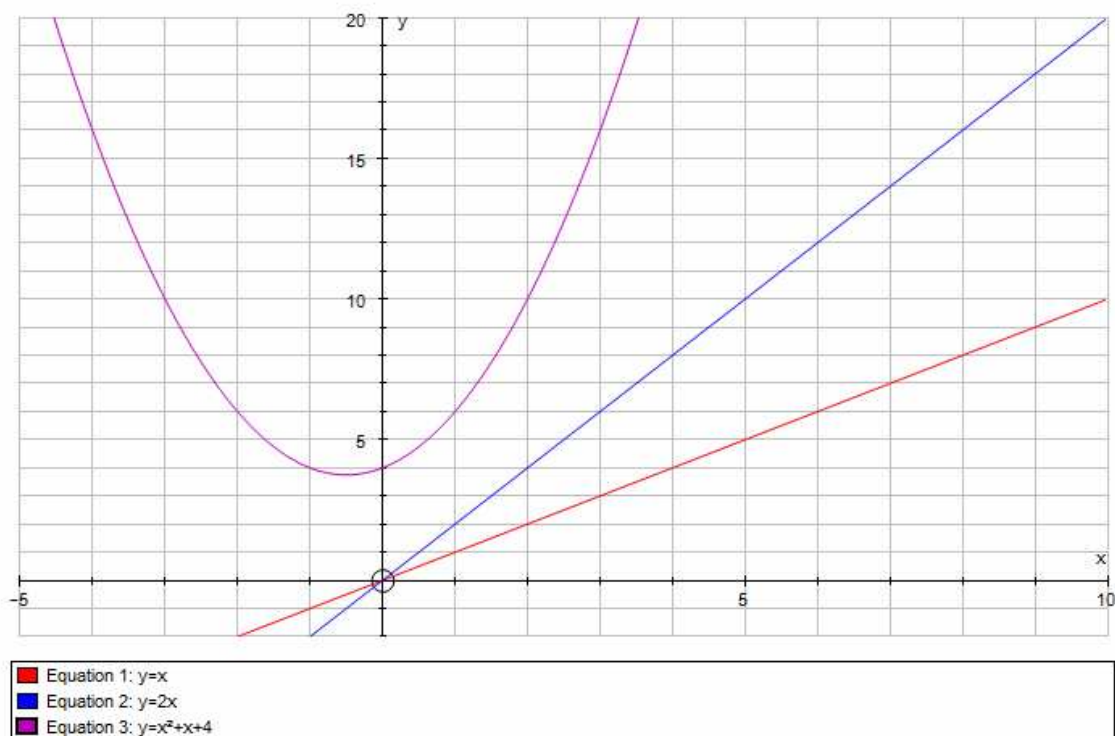
$$= \left| \frac{6}{4} - 2 \right|$$

$$= |-0.5|$$

$$= \frac{1}{2}$$

Therefore, the conjecture of $D = \frac{1}{|a|}$ holds true when the parabola only intersects one line.

Consider the parabola $y = x^2 + x + 4$ and the lines $y = x$ and $y = 2x$



There are no real intersections between the parabola and the lines $y = x$ and $y = 2x$.
 There are two distinct imaginary intersections between the parabola and the line $y = x$ and $y = 2x$ each.

To find the imaginary intersections between the parabola and the line $y = x$,

$$y = x^2 + x + 4 \dots \dots (1)$$

$$y = x \dots \dots \dots (2)$$

Substitute (2) into (1),

$$x^2 + x + 4 = x$$

$$x^2 + 4 = 0$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-0 \pm \sqrt{0^2 - 4(1)(4)}}{2(1)} \\ &= \frac{\pm \sqrt{-16}}{2} \end{aligned}$$

Let $\sqrt{-1}$ be i ,

$$x = \frac{\sqrt{16}i}{2} \text{ or } x = -\frac{\sqrt{16}i}{2}$$

$$x = 2i \text{ or } x = -2i$$

To find the imaginary intersection between the parabola and the line $y = 2x$,

$$y = x^2 + x + 4 \dots \dots (1)$$

$$y = 2x \dots \dots \dots (2)$$

Substitute (2) into (1),

$$x^2 + x + 4 = 2x$$

$$x^2 - x + 4 = 0$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2(a)} \\ &= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(4)}}{2(1)} \\ &= \frac{1 \pm \sqrt{-15}}{2} \end{aligned}$$

Let $\sqrt{-1}$ be i ,

$$x = \frac{1 \pm \sqrt{15}i}{2}$$

$$x = \frac{1 + \sqrt{15}i}{2} \text{ or } x = \frac{1 - \sqrt{15}i}{2}$$

Hence, the x-values are:

$$x_1 = \frac{1 - \sqrt{15}i}{2}$$

$$x_2 = -2i$$

$$x_3 = 2i$$

$$x_4 = \frac{1 + \sqrt{15}i}{2}$$

Calculation of D,

$$D = |S_L - S_R|$$

$$= |(x_2 - x_1) - (x_4 - x_3)|$$

$$= \left| \left(-2i - \frac{1 - \sqrt{15}i}{2} \right) - \left(\frac{1 + \sqrt{15}i}{2} - 2i \right) \right|$$

$$\begin{aligned}
 &= \left| -2i - \frac{1 - \sqrt{15}i}{2} - \frac{1 + \sqrt{15}i}{2} + 2i \right| \\
 &= \left| -\frac{2}{2} \right| \\
 &= |-1| \\
 &= 1
 \end{aligned}$$

The conjecture still holds true when the intersections are not real numbers, for real values of a and $a > 0$.

To prove the conjecture using the general equation of $y = ax^2 + bx + c$,

Let the roots of the general equation be α and β .

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

$$a(x - \alpha)(x - \beta) = 0$$

$$a(x^2 - (\alpha + \beta)x + \alpha\beta) = 0$$

$$ax^2 - a(\alpha + \beta)x + a(\alpha\beta) = 0$$

Hence, it can be deduced that,

$$-a(\alpha + \beta) = b$$

$$(\alpha + \beta) = -\frac{b}{a}$$

$$a(\alpha\beta) = c$$

$$\alpha\beta = \frac{c}{a}$$

Since $D = |S_L - S_R|$ and $S_L = x_2 - x_1$ and $S_R = x_4 - x_3$,

$$D = |S_L - S_R|$$

$$= |(x_2 - x_1) - (x_4 - x_3)|$$

$$= |x_2 - x_1 - x_4 + x_3|$$

$$= |(x_2 + x_3) - (x_1 + x_4)|, \text{ where}$$

$(x_2 + x_3)$ is the sum of the x -value of the intersections between the parabola $y = ax^2 + bx + c$ and the line $y = x$.

$(x_1 + x_4)$ is the sum of the x -value of the intersections between the parabola $y = ax^2 + bx + c$ and the line $y = 2x$.

To find the x -values of intersections between the parabola $y = ax^2 + bx + c$ and the lines

$$y = x \text{ and } y = 2x,$$

$$y = ax^2 + bx + c \dots \dots (1)$$

$$y = x \dots \dots \dots (2)$$

$$y = 2x \dots \dots \dots (3)$$

Substitute (2) into (1),

$$ax^2 + bx + c = x$$

$$ax^2 + (b - 1)x + c = 0$$

Hence, since the roots of the equation are x_2 and x_3 ,

The sum of the roots of the equation, i.e. $x_2 + x_3 = \frac{b-1}{a}$

Substitute (3) into (1),

$$ax^2 + bx + c = 2x$$

$$ax^2 + (b - 2)x + c = 0$$

Hence, since the roots of the equation are x_1 and x_4 ,
The sum of the roots of the equation, i.e. $x_1 + x_4 = \frac{b-2}{a}$

$$\begin{aligned} D &= |(x_2 + x_3) - (x_1 + x_4)| \\ &= \left| \frac{b-1}{a} - \frac{b-2}{a} \right| \\ &= \left| \frac{1}{a} \right| \\ &= \frac{1}{|a|}, a \neq 0 \end{aligned}$$

Hence, the conjecture is proven for all real values of a , $a \neq 0$.

4. Investigating the conjecture when the intersecting lines are changed.

To investigate whether the conjecture still works when the intersecting lines are changed, I will be using the same parabola while varying the intersecting lines.

To vary intersecting lines, the intersecting lines all follow the general equation of $y = mx + c$, where m is the gradient and c is the constant. Hence, for the two intersecting lines, I will be varying the m value and the c value.

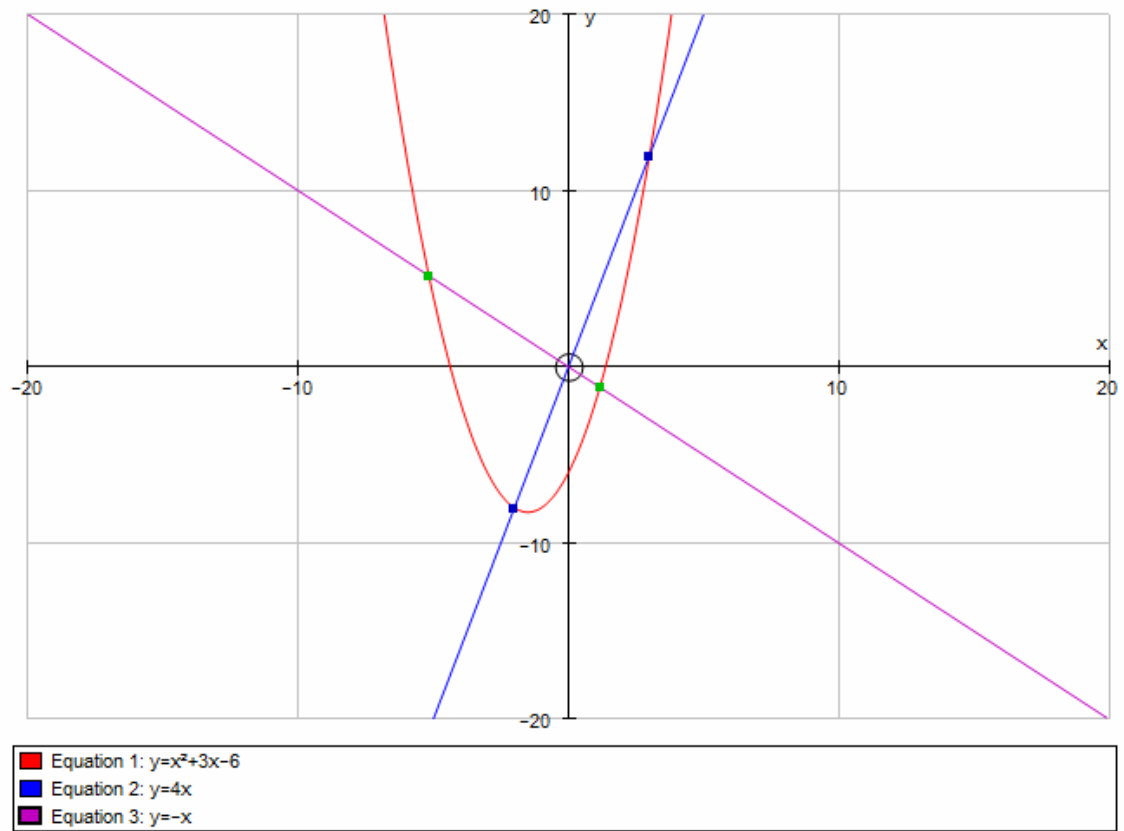
The equation of the two intersecting lines will be as follows:

Line equation 1: $y = m_1x + c_1$

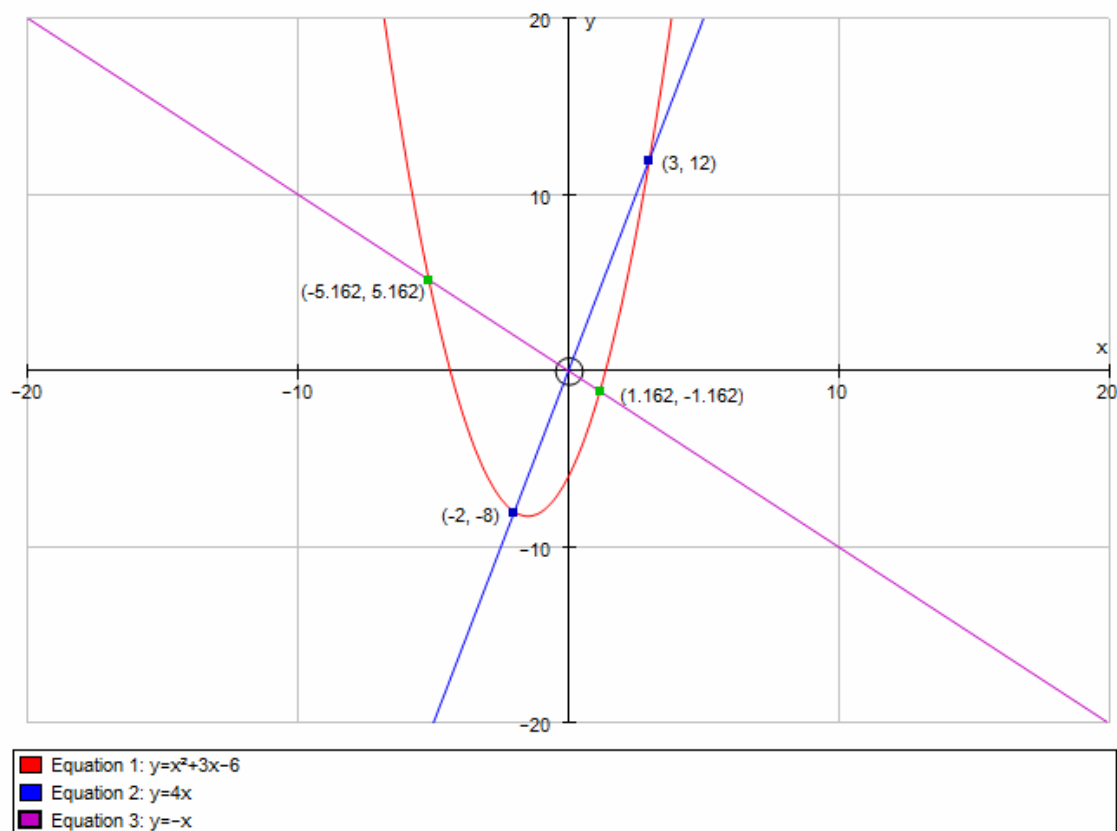
Line equation 2: $y = m_2x + c_2$

The values of m_1, m_2, c_1 and c_2 will be varied.

Consider the parabola $y = x^2 + 3x - 6$ and the intersecting lines of $y = 4x$ and $y = -x$,



The intersections between the parabola and the intersecting lines can then be found via Autograph software:



The intersections between the parabola and the line $y = 4x$ are $(-2, -8)$ and $(3, 12)$

The intersections between the parabola and the line $y = -x$ are $(-5.162, 5.162)$ and $(1.162, -1.162)$

Let the x-values of the intersections between the parabola and the line $y = 4x$ be x_2 and x_3 .
 Let the x-values of the intersections between the parabola and the line $y = -x$ be x_1 and x_4 .

Hence, the x-values:

$$x_1 = -5.162$$

$$x_2 = -2$$

$$x_3 = 3$$

$$x_4 = 1.162$$

Calculation of D:

$$D = |S_L - S_R|$$

$$= |(x_2 - x_1) - (x_4 - x_3)|$$

$$= |(-2 + 5.162) - (1.162 - 3)|$$

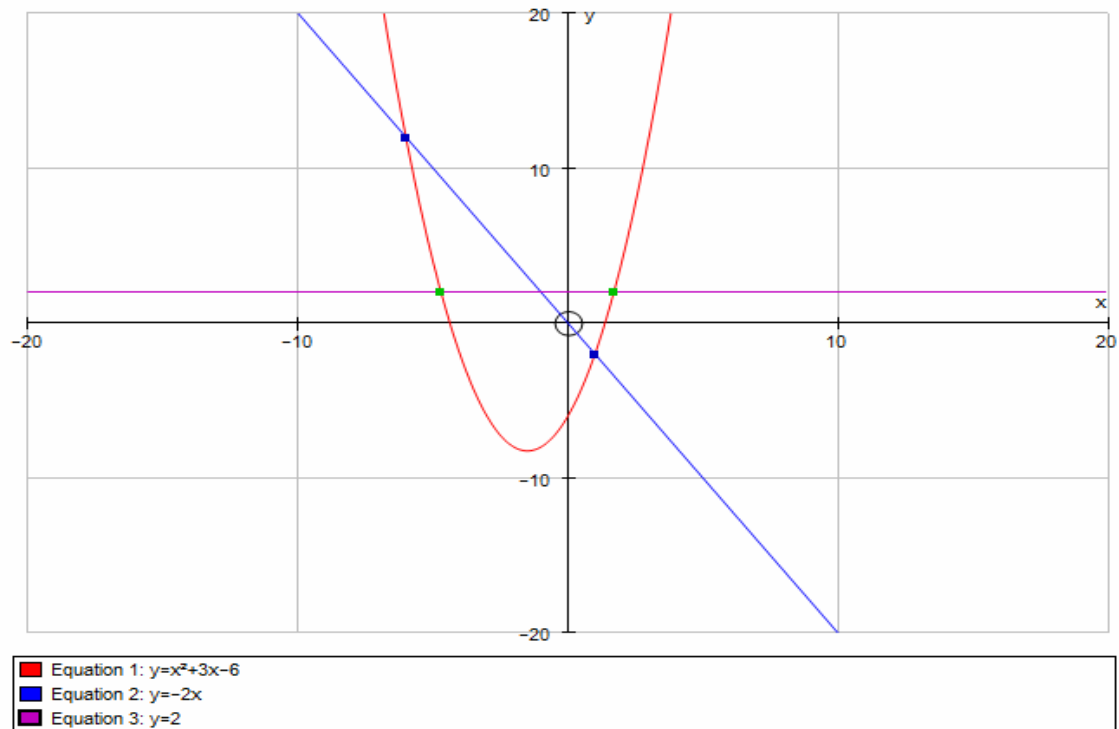
$$= |3.162 + 1.838|$$

$$= |5|$$

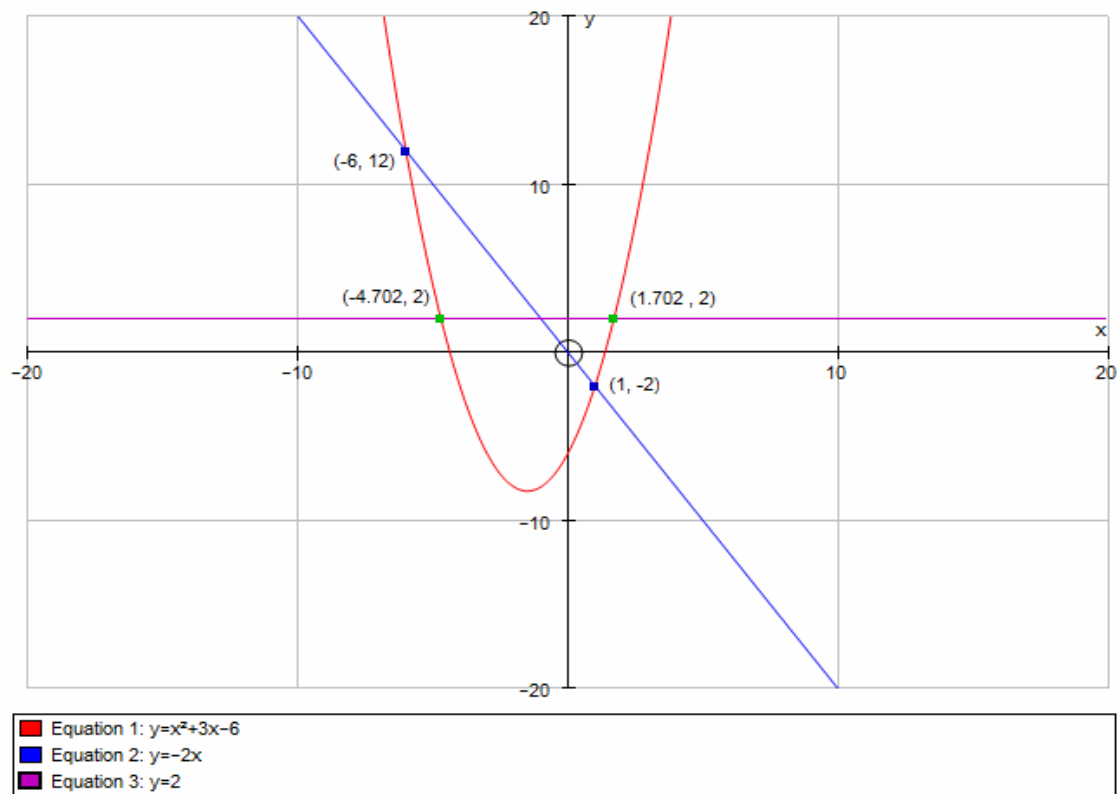
$$= 5$$

The conjecture does not hold when the intersecting lines are changed. The D value was suppose to be 1 with the conjecture that $D = \frac{1}{|a|}$. However, I cannot modify my conjecture yet as there is insufficient cases to be able to come up with a conjecture that can suit the purpose.

Consider the parabola $y = x^2 + 3x - 6$ and the lines $y = -2x$ and $y = 2$.



The intersections of the parabola and the lines can then be found by the Autograph software.



The intersections between the parabola and the line $y = -2x$ are $(-6, 12)$ and $(1, -2)$.
The intersections between the parabola and the line $y = 2$ are $(-4.702, 2)$ and $(1.702, 2)$.

Let the x-values of the intersections between the parabola and the line $y = -2x$ be x_1 and x_4 .
Let the x-values of the intersections between the parabola and the line $y = 2$ be x_2 and x_3 .

The x-values:

$$x_1 = -6$$

$$x_2 = -4.702$$

$$x_3 = 1.702$$

$$x_4 = 1$$

Calculation of D:

$$D = |S_L - S_R|$$

$$= |(x_2 - x_1) - (x_4 - x_3)|$$

$$= |(-4.702 + 6) - (1 - 1.702)|$$

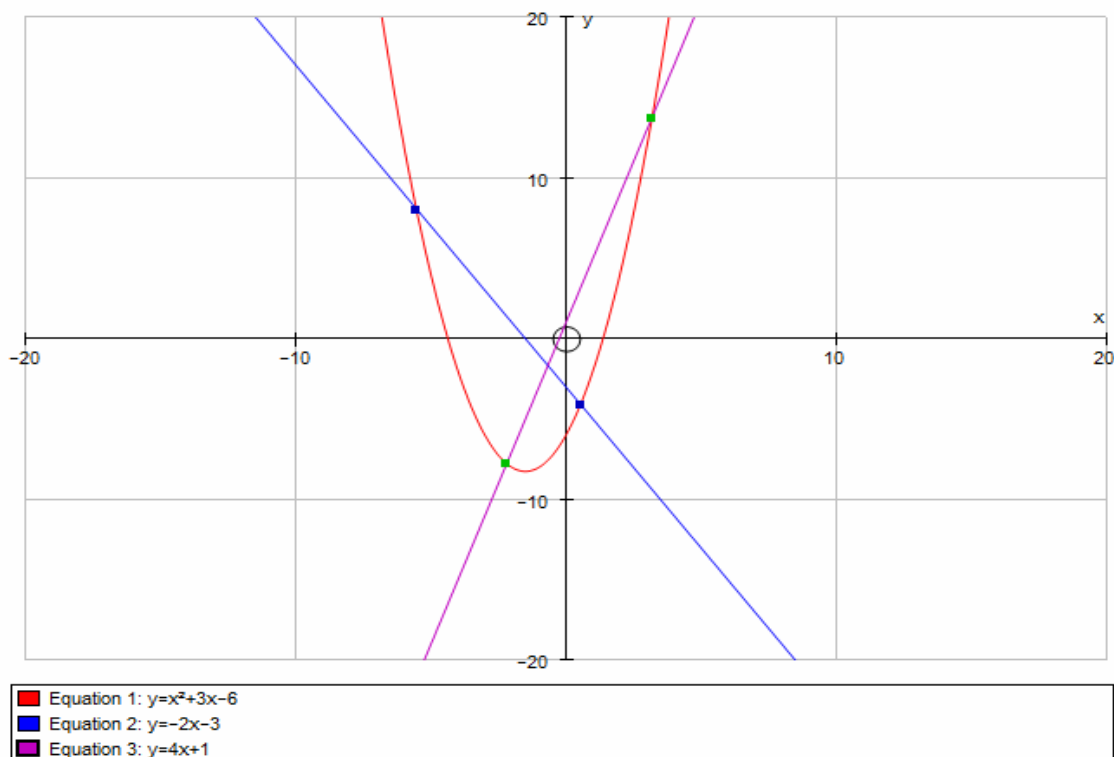
$$= |1.298 + 0.702|$$

$$= |2|$$

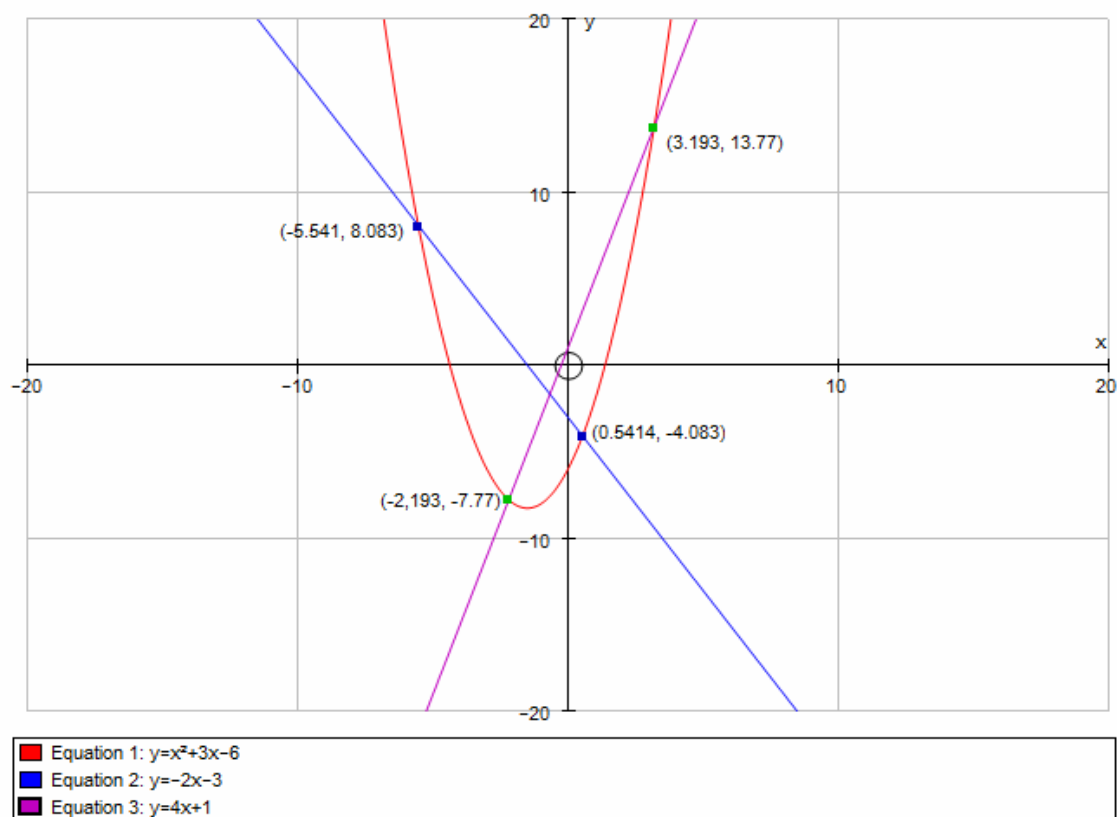
$$= 2$$

Once again the conjecture does not hold when the intersecting lines are changed as the D value was suppose to be 1 according to the conjecture that $D = \frac{1}{|a|}$.

Consider the parabola $y = x^2 + 3x - 6$ and the lines $y = -2x - 3$ and $y = 4x + 1$.



The intersections between the parabola and the lines can be found via Autograph software and is shown below:



The intersections between the parabola and the line $y = -2x - 3$ are $(-5.541, 8.083)$ and $(0.5414, -4.083)$.

The intersections between the parabola and the line $y = 4x + 1$ are $(-2.193, -7.77)$ and $(3.193, 13.77)$.

Let the x-values of the intersections of the parabola and the line $y = -2x - 3$ be x_1 and x_4 .

Let the x-values of the intersections of the parabola and the line $y = 4x + 1$ be x_2 and x_3 .

The x-values :

$$x_1 = -5.541$$

$$x_2 = -2.193$$

$$x_3 = 3.193$$

$$x_4 = 0.5414$$

Calculation of D:

$$D = |S_L - S_R|$$

$$= |(x_2 - x_1) - (x_4 - x_3)|$$

$$= |(-2.193 + 5.541) - (0.5414 - 3.193)|$$

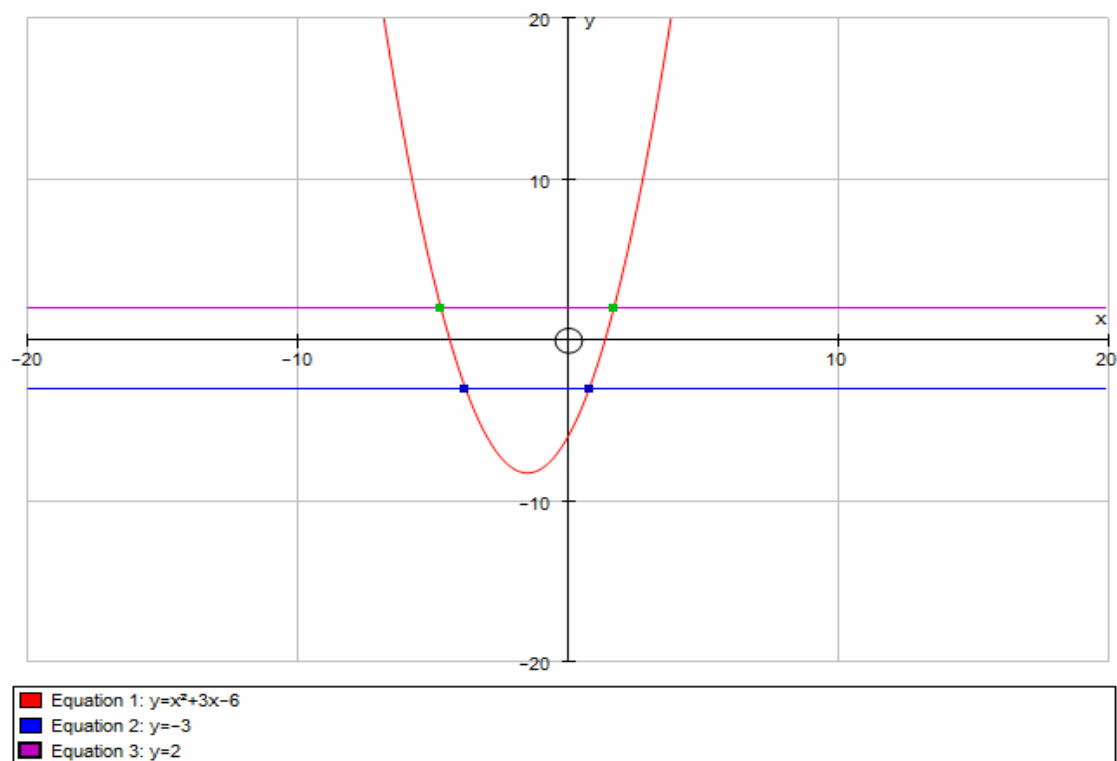
$$= |3.348 + 2.652|$$

$$= |6|$$

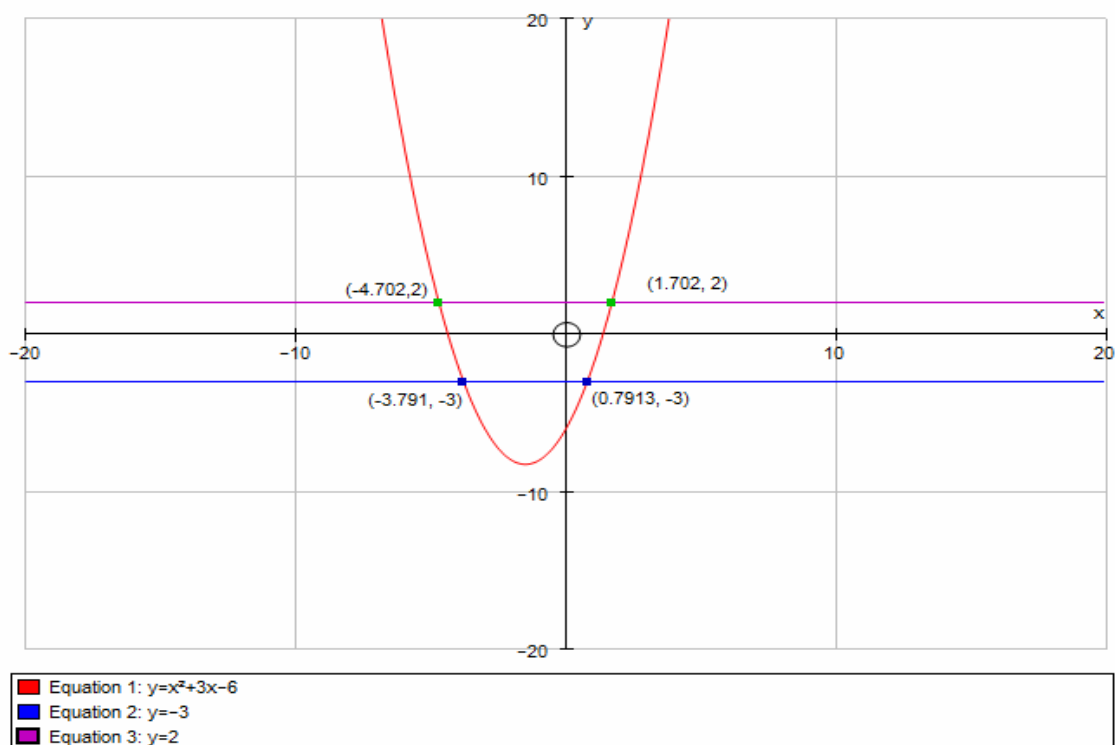
$$= 6$$

The conjecture does not hold once again as the value is supposed to be 1 according to the conjecture of $D = \frac{1}{|a|}$.

Consider the parabola $y = x^2 + 3x - 6$ and the line $y = 2$ and $y = -3$.



The intersections between the parabola and the lines can then be found via Autograph software and its shown below:



The intersections between the parabola and the line $y = 2$ are $(-4.702, 2)$ and $(1.702, 2)$.
 The intersections between the parabola and the line $y = -3$ are $(-3.791, -3)$ and $(0.7913, -3)$.

Let the x-values of the intersections between the parabola and the line $y = 2$ be x_1 and x_4 .
 Let the x-values of the intersections between the parabola and the line $y = -3$ be x_2 and x_3 .

The x-values:

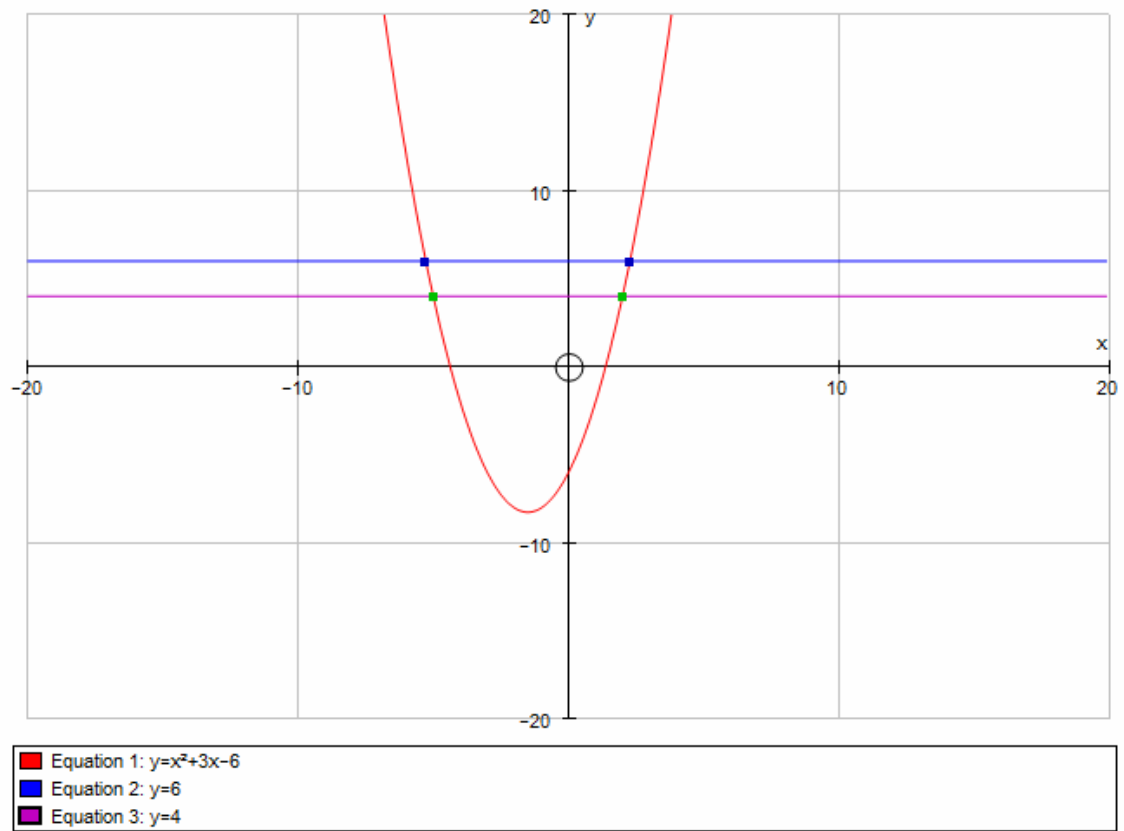
$$\begin{aligned}x_1 &= -4.702 \\x_2 &= -3.791 \\x_3 &= 0.7913 \\x_4 &= 1.702\end{aligned}$$

Calculation of D

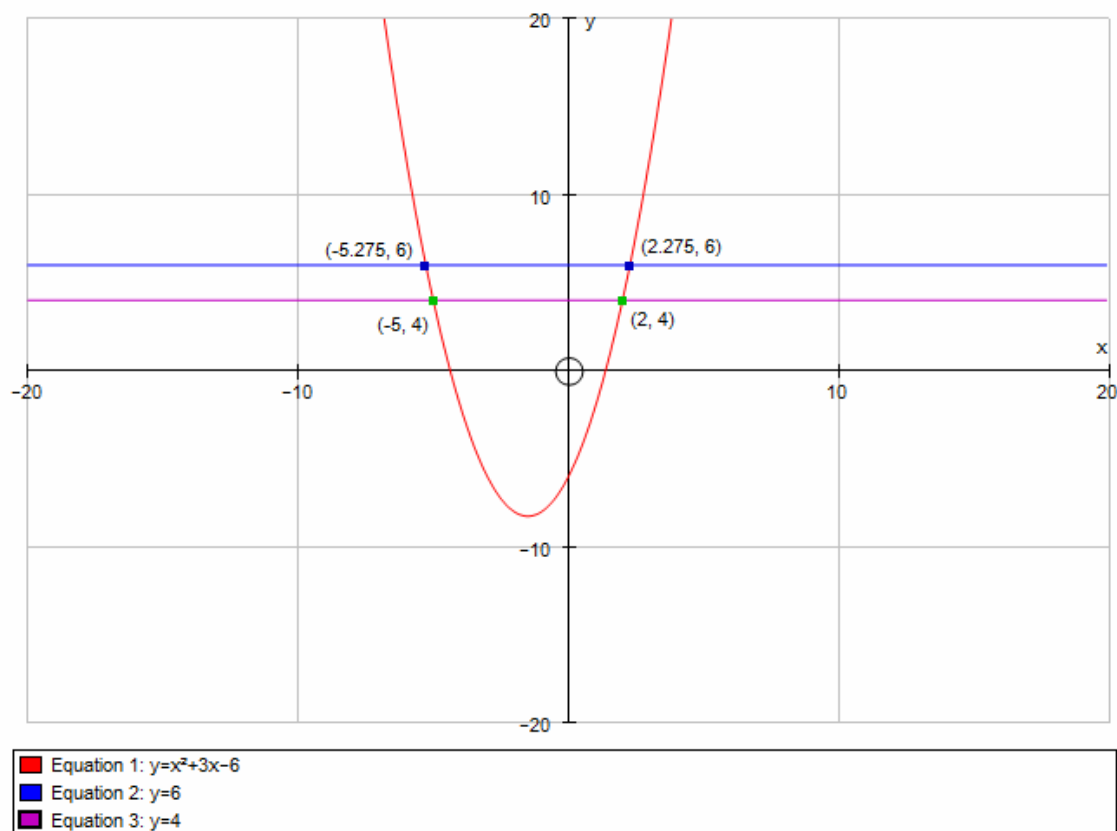
$$\begin{aligned}D &= |S_L - S_R| \\&= |(x_2 - x_1) - (x_4 - x_3)| \\&= |(-3.791 + 4.702) - (1.702 - 0.7913)| \\&= |0.911 - 0.911| \\&= |0| \\&= 0\end{aligned}$$

Hence the conjecture of $D = \frac{1}{|a|}$ is once again not proved with this case. The D value is supposed to be 1 according to the conjecture.

Consider the parabola $y = x^2 + 3x - 6$ and the lines $y = 6$ and $y = 4$.



The intersections between the parabola and the intersecting lines can then be found via Autograph software and is shown below:



The intersections between the parabola and the line $y = 6$ are $(-5.275, 6)$ and $(2.275, 6)$.
The intersections between the parabola and the line $y = 4$ are $(-5, 4)$ and $(2, 4)$.

Let the x-values of the intersections between the parabola and the line $y = 6$ to be x_1 and x_4 .
Let the x-values of the intersections between the parabola and the line $y = 4$ to be x_2 and x_3 .

The x-values :

$$\begin{aligned}x_1 &= -5.275 \\x_2 &= -5 \\x_3 &= 2 \\x_4 &= 2.275\end{aligned}$$

Calculation of D:

$$\begin{aligned}D &= |S_L - S_R| \\&= |(x_2 - x_1) - (x_4 - x_3)| \\&= |(-5 + 5.275) - (2.275 - 2)| \\&= |0.275 - 0.275| \\&= |0| \\&= 0\end{aligned}$$

Hence the conjecture of $D = \frac{1}{|a|}$ is once again proven wrong in this case.

I will now form a table to show the parabolas and different values of m_1, m_2, c_1 and c_2 and the value of D.

In this table, I will also include two cases from part 3 to show when the intersecting lines were $y = x$ and $y = 2x$.

Parabola	Equation of first intersecting line	Equation of second intersecting line	m_1	m_2	c_1	c_2	$D = S_L - S_R $
$y = x^2 + 3x - 6$	$y = -x$	$y = 4x$	-1	4	0	0	5
$y = x^2 + 3x - 6$	$y = -2x$	$y = 2$	-2	0	0	2	2
$y = x^2 + 3x - 6$	$y = -2x - 3$	$y = 4x + 1$	-2	4	-3	1	6
$y = x^2 + 3x - 6$	$y = 2$	$y = -3$	0	0	2	-3	0
$y = x^2 + 3x - 6$	$y = 6$	$y = 4$	0	0	6	4	0
$y = x^2 - 6x + 11$	$y = x$	$y = 2x$	1	2	0	0	1
$y = 2x^2 - 8x + 3$	$y = x$	$y = 2x$	1	2	0	0	$\frac{1}{2}$

Observing the table above, it can be observed that conjecture from earlier is non - applicable when the intersecting lines are changed. It can also be observed when the m_1 and m_2 values are altered, the D value changes but when the c_1 and c_2 values are altered, there is no change in the D value. Hence, I will try to find the relationship between the m_1 and m_2 values and the D value and from there modify my conjecture.

It can be observed that the D value is the value of $|m_2 - m_1|$ when the value of $a = 1$ as shown in the first 6 cases in the table. Hence, I would modify my conjecture to be:

Conjecture : D is the absolute value of the difference of m_1 and m_2 divided by a in the general equation of $y = ax^2 + bx + c$, i.e. $D = \left| \frac{m_2 - m_1}{a} \right|$, where $a \neq 0, a \in \mathbb{R}$.

To prove this conjecture algebraically using general equation of $y = ax^2 + bx + c, y = m_1x + c_1$ and $y = m_2x + c_2$.

Let the roots of the general equation be α and β .

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

$$a(x - \alpha)(x - \beta) = 0$$

$$a(x^2 - (\alpha + \beta)x + \alpha\beta) = 0$$

$$ax^2 - a(\alpha + \beta)x + a(\alpha\beta) = 0$$

Hence, it can be deduced that,

$$-a(\alpha + \beta) = b$$

$$(\alpha + \beta) = -\frac{b}{a}$$

$$a(\alpha\beta) = c$$

$$\alpha\beta = \frac{c}{a}$$

Since $D = |S_L - S_R|$ and $S_L = x_2 - x_1$ and $S_R = x_4 - x_3$,

$$D = |S_L - S_R|$$

$$= |(x_2 - x_1) - (x_4 - x_3)|$$

$$= |x_2 - x_1 - x_4 + x_3|$$

$$= |(x_2 + x_3) - (x_1 + x_4)|, \text{ where}$$

$(x_2 + x_3)$ is the sum of the x - value of the intersections between the parabola $y = ax^2 + bx + c$ and the line $y = m_1x + c_1$.

$(x_1 + x_4)$ is the sum of the x - value of the intersections between the parabola $y = ax^2 + bx + c$

and the line $y = m_2x + c_2$.

To find the x-values of intersections between the parabola $y = ax^2 + bx + c$ and the lines

$$y = m_1x + c_1 \text{ and } y = m_2x + c_2,$$

$$y = ax^2 + bx + c \dots \dots (1)$$

$$y = m_1x + c_1 \dots \dots \dots (2)$$

$$y = m_2x + c_2 \dots \dots \dots (3)$$

Substitute (2) into (1),

$$ax^2 + bx + c = m_1x + c_1$$

$$ax^2 + (b - m_1)x + (c - c_1) = 0$$

Hence, since the roots of the equation are x_2 and x_3 ,

$$\text{The sum of the roots of the equation, i.e. } x_2 + x_3 = \frac{b - m_1}{a}$$

Substitute (3) into (1),

$$ax^2 + bx + c = m_2x + c_2$$

$$ax^2 + (b - m_2)x + (c - c_2) = 0$$

Hence, since the roots of the equation are x_1 and x_4 ,

$$\text{The sum of the roots of the equation, i.e. } x_1 + x_4 = \frac{b - m_2}{a}$$

$$D = |(x_2 + x_3) - (x_1 + x_4)|$$

$$= \left| \frac{b - m_1}{a} - \frac{b - m_2}{a} \right|$$

$$= \left| \frac{m_2 - m_1}{a} \right|, a \neq 0, a \in \mathbb{R}.$$

Hence, the conjecture is proven for all real values of a and $a \neq 0$.

5. Determine whether a similar conjecture can be made for cubic polynomials.

The general equation of a cubic polynomial is $y = ax^3 + bx^2 + cx + d$.

Consider the cubic function $ax^3 + bx^2 + cx + d = 0, a > 0$

The cubic function has three distinct roots which will be labelled as α, β and γ respectively.

$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma) = 0$$

$$a[(x^2 - (\alpha + \beta)x + \alpha\beta)(x - \gamma)] = 0$$

$$a[x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma] = 0$$

$$ax^3 - a(\alpha + \beta + \gamma)x^2 + a(\alpha\beta + \beta\gamma + \alpha\gamma)x - a(\alpha\beta\gamma) = 0$$

$$ax^3 - a(\alpha + \beta + \gamma)x^2 + a(\alpha\beta + \beta\gamma + \alpha\gamma)x - a(\alpha\beta\gamma) = ax^3 + bx^2 + cx + d$$

From the above equation:

$$-a(\alpha + \beta + \gamma)x^2 = bx^2$$

$$(\alpha + \beta + \gamma) = -\frac{b}{a}$$

$$a(\alpha\beta + \beta\gamma + \alpha\gamma)x = cx$$

$$(\alpha\beta + \beta\gamma + \alpha\gamma) = \frac{c}{a}$$

$$-a(\alpha\beta\gamma) = d$$

$$(\alpha\beta\gamma) = -\frac{d}{a}$$

To find the intersections between the cubic equation and the linear lines of

$$y = m_1x + c_1 \text{ and } y = m_2x + c_2,$$

$$y = ax^3 + bx^2 + cx + d \dots \dots (1)$$

$$y = m_1x + c_1 \dots \dots (2)$$

$$y = m_2x + c_2 \dots \dots (3)$$

Let the intersections between the cubic function and the line $y = m_1x + c_1$ to be x_2, x_3 and x_6 .

Let the intersections between the cubic function and the line $y = m_2x + c_2$ to be x_1, x_4 and x_5 .

To find the intersections between the cubic equation and the line $y = m_1x + c_1$,

Substitute (2) into (1),

$$ax^3 + bx^2 + cx + d = m_1x + c_1$$

$$ax^3 + bx^2 + (c - m_1)x + (d + c_1) = 0$$

Hence, since the x_2, x_3 and x_6 are roots to the equation of $ax^3 + bx^2 + (c - m_1)x + (d + c_1) = 0$,

The sum of roots, i.e. $x_2 + x_3 + x_6 = -\frac{b}{a}$

The sum of the product of two roots, i.e. $x_2x_3 + x_3x_6 + x_2x_6 = \frac{c - m_1}{a}$

The product of all three roots, i.e. $x_2x_3x_6 = -\frac{d + c_1}{a}$

To find the intersections between the cubic equation and the line $y = m_2x + c_2$,

Substitute (3) into (1),

$$ax^3 + bx^2 + cx + d = m_2x + c_2$$

$$ax^3 + bx^2 + (c - m_2)x + (d - c_2) = 0$$

Hence, since x_1, x_4 and x_5 are roots to the equation of $ax^3 + bx^2 + (c - m_2)x + (d - c_2) = 0$,

The sum of the roots, i.e. $x_1 + x_4 + x_5 = -\frac{b}{a}$

The sum of the product of two roots, i.e. $x_1x_4 + x_4x_5 + x_1x_5 = \frac{c - m_2}{a}$

The product of all three roots, i.e. $x_1x_4x_5 = -\frac{(d - c_2)}{a}$

To apply the conjecture to cubic polynomials,

$$D = |S_L - S_R - S_N|$$

$$= |(x_2 - x_1) - (x_4 - x_3) - (x_5 - x_6)|$$

$$= |x_2 - x_1 - x_4 + x_3 - x_5 + x_6|$$

$$= |(x_2 + x_3 + x_6) - (x_1 + x_4 + x_5)|$$

$$= |(\text{sum of intersections of the first line}) - (\text{sum of intersections of the second line})|$$

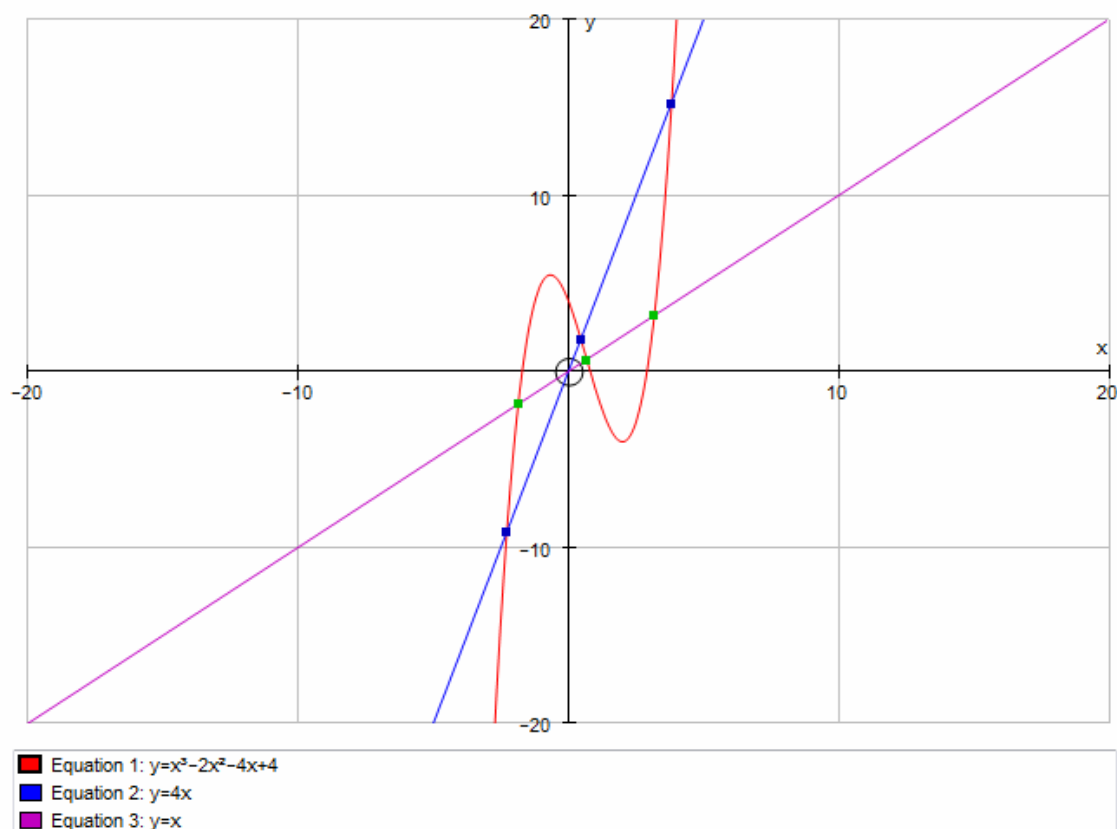
$$= \left| -\frac{b}{a} + \frac{b}{a} \right|$$

$$= |0|$$

$$= 0$$

I will also prove this conjecture graphically.

Consider the parabola $y = x^3 - 2x^2 - 4x + 4$ and the lines $y = x$ and $y = 4x$



The x-values of the intersections :

$$x_1 = -2.279 \quad x_2 = -1.856 \quad x_3 = 0.6783 \quad x_4 = 0.4594 \quad x_5 = 3.82 \quad x_6 = 3.177$$

Calculation of D:

$$\begin{aligned} D &= |S_L - S_R - S_N| \\ &= |(x_2 - x_1) - (x_4 - x_3) - (x_5 - x_6)| \\ &= |x_2 - x_1 - x_4 + x_3 - x_5 + x_6| \\ &= |(x_2 + x_3 + x_6) - (x_1 + x_4 + x_5)| \\ &= |(-1.856 + 0.6783 + 3.177) - (-2.279 + 0.4594 + 3.82)| \\ &= |2 - 2| \\ &= |0| \\ &= 0 \end{aligned}$$

However, the intersecting line may be quadratic in the case of cubic polynomials.

$$y = ax^3 + bx^2 + cx + d \dots\dots(1)$$

$$y = k_1x^2 + j_1x + l_1 \dots\dots\dots(2)$$

$$y = k_2x^2 + j_2x + l_2 \dots\dots\dots(3)$$

Let the intersections between the cubic function and the line $y = k_1x^2 + j_1x + l_1$ to be x_2, x_3 and x_6 .

Let the intersections between the cubic function and the line $y = k_2x^2 + j_2x + l_2$ to be x_1, x_4 and x_5 .

To find the intersections between the cubic function and the first quadratic curve,

Substitute (2) into (1),

$$ax^3 + bx^2 + cx + d = k_1x^2 + j_1x + l_1$$

$$ax^3 + (b - k_1)x^2 + (c - j_1)x + (d - l_1) = 0$$

Since the roots of the equation are x_2, x_3 and x_6 ,

$$\text{The sum of roots, i.e. } x_2 + x_3 + x_6 = -\frac{b-k_1}{a}$$

To find the intersections between the cubic function and the first quadratic curve,

Substitute (3) into (1),

$$ax^3 + bx^2 + cx + d = k_2x^2 + j_2x + l_2$$

$$ax^3 + (b - k_2)x^2 + (c - j_2)x + (d - l_2) = 0$$

Since the roots of the equation are x_1, x_4 , and x_5

$$\text{The sum of roots, i.e. } x_1 + x_4 + x_5 = -\frac{b-k_2}{a}$$

Since

$$D = |(\text{sum of intersections of the first line}) - (\text{sum of intersections of the second line})|$$

$$= \left| -\frac{b-k_1}{a} + \frac{b-k_2}{a} \right|$$

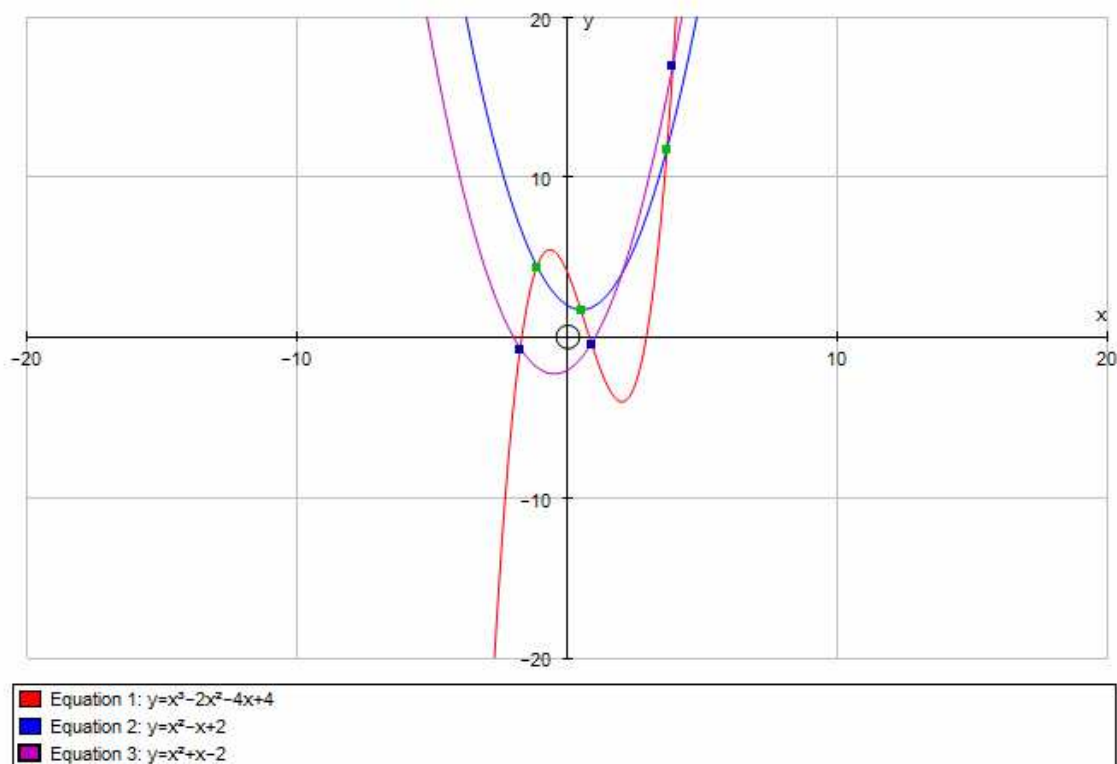
$$= \left| \frac{k_1 - k_2}{a} \right|$$

This can be proven graphically,

Consider the cubic function $y = x^3 - 2x^2 - 4x + 4$ and the quadratic equations

$$y = x^2 - x + 2 \text{ and}$$

$$y = x^2 + x - 2.$$



The x-values:

$$x_1 = -1.145 \quad x_2 = -1.764 \quad x_3 = 0.8748 \quad x_4 = 0.476 \quad x_5 = 3.669 \quad x_6 = 3.889$$

Calculation of D:

$$D = |S_L - S_R - S_N|$$

$$\begin{aligned}
 &= |(x_2 - x_1) - (x_4 - x_3) - (x_5 - x_6)| \\
 &= |(x_2 + x_3 + x_6) - (x_1 + x_4 + x_5)| \\
 &= |(-1.764 + 0.8748 + 3.889) - (-1.145 + 0.476 + 3.669)| \\
 &= |3 - 3| \\
 &= 0
 \end{aligned}$$

Alternatively,

$D = \frac{k_1 - k_2}{a}$, where k_1 is the coefficient of x^2 in the first quadratic equation and k_2 is the coefficient of x^2 in the second quadratic equation.

$$\begin{aligned}
 D &= \left| \frac{1 - 1}{1} \right| \\
 &= |0| \\
 &= 0
 \end{aligned}$$

Hence, the conjecture that can be made for cubic polynomials is that $D = 0$ for all cubic polynomials that are intersected with linear lines, however, when the cubic polynomial is intersected with quadratic equations, $D = \left| \frac{k_1 - k_2}{a} \right|$, where k_1 is the coefficient of x^2 in the first quadratic equation, k_2 is the coefficient of x^2 in the second quadratic equation.

6. Consider whether the conjecture might be modified to include higher order polynomials.

To consider higher order polynomials,

The general equation would be $y = a_0 + a_1x + \dots + a_nx^n$, where n is the highest degree.

Alternatively, it can be written as $y = ax^n + bx^{n-1} + \dots + rx + k$.

The roots of the equation would be $r_1, r_2, r_3, \dots, r_n$.

Hence,

$$\begin{aligned}
 ax^n + bx^{n-1} + \dots + rx + k &= a(x - r_1)(x - r_2)(x - r_3) \dots (x - r_n) = 0 \\
 a[(x^n - (r_1 + r_2 + r_3 + \dots + r_n)x^{n-1} + (r_1r_2 + r_2r_3 + r_1r_3 + \dots + r_{n-1}r_n)x^{n-2} \\
 - (r_1r_2r_3 + r_2r_3r_4 + \dots + r_{n-2}r_{n-1}r_n)x^{n-3} + \dots - (r_1r_2r_3 \dots r_n)] &= 0
 \end{aligned}$$

The sum of roots will be $-\frac{b}{a}$ as proven earlier.

When the polynomial $y = ax^n + bx^{n-1} + \dots + k$ intersects with a line that is at least two degrees lower than the polynomial, i.e. $y = kx^{n-2} + lx^{n-3} + \dots + p$, the two equations can then be equated to find the points of intersection which will be the roots of the new equation.

Hence the sum of roots will then be $-\frac{b}{a}$

Hence when the parabola is intersected with two linear lines, the value of D will be

$$\begin{aligned}
 D &= |(\text{Sum of intersections with the } y = k_1x^{n-2} + l_1x^{n-3} + \dots + p_1) \\
 &\quad - (\text{Sum of intersections with } y = k_2x^{n-2} + l_2x^{n-3} + \dots + p_2)| \\
 &= \left| -\frac{b}{a} + \frac{b}{a} \right| \\
 &= |0| \\
 &= 0
 \end{aligned}$$

Therefore, the conjecture that $D = 0$ will hold as long as the polynomial is intersected with a line that is at least two degrees lower than the polynomial.

However, when the polynomial is intersected with a line that is one degree lower, i.e.

Equation of polynomial: $y = ax^n + bx^{n-1} + \dots + k$

Equation of intersecting line: $y = kx^{n-1} + lx^{n-2} + \dots + p$,

The sum of roots of the new equation will be $-\frac{b-k}{a}$, as according to examples above.

When the polynomial is intersected by two lines that are one degree lower than the polynomial, the value of D will be

$$D = |(\text{Sum of intersections with } y = k_1x^{n-1} + l_1x^{n-2} + \dots + p_1) - (\text{Sum of intersections with } y = k_2x^{n-1} + l_2x^{n-2} + \dots + p_2)|$$

$$D = \left| -\frac{b-k_1}{a} + \frac{b-k_2}{a} \right|$$

$$= \left| \frac{k_1-k_2}{a} \right|$$

Hence when the higher order polynomials of degree n is intersected with lines that are one degree less, the conjecture will be $D = \left| \frac{k_1-k_2}{a} \right|$.