

# Practice Maths Portfolio –

## Fractals

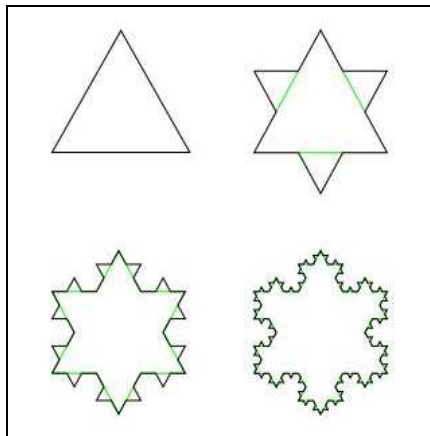
### Introduction

▲ **fractal** is generally "a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole", a property called self-similarity. The term was coined by Benoît Mandelbrot in 1975 and was derived from the Latin ~~fractus~~ <sup>fractus</sup> meaning "broken" or "fractured." ▲ mathematical fractal is based on an equation that undergoes iteration, a form of feedback based on recursion.

In 1904, Helge Von Koch gave a more geometric definition of a similar function, which is now called the Koch snowflake.

One can imagine that it was created by starting with a line segment, then recursively altering each line segment as follows:

1. Divide the line segment into three segments of equal length.
2. Draw an equilateral triangle that has the middle segment from step 1 as its base and points outward.
3. Remove the line segment that is the base of the triangle from step 2.



Koch Snowflake in progression going from stage 0 on the top left corner of the image on the left, to stage 3 on the bottom right of the image

In this Portfolio I am primarily going to see if there is any pattern in this fractal, and for that I am going to get the values for number of sides, length of each side, perimeter of the shape and the area of the shape. For the Number of sides I am basically going to count the number of sides. The length of each side can be calculated because each side gets split up into three parts. The perimeter will be measured using the simple formula,

Perimeter = length of one side x number of sides.

The area of the shape will be manually calculated for each of the smaller triangles and multiplied to the number of triangles. Formula

$$= s^2 \frac{\sqrt{3}}{4}$$

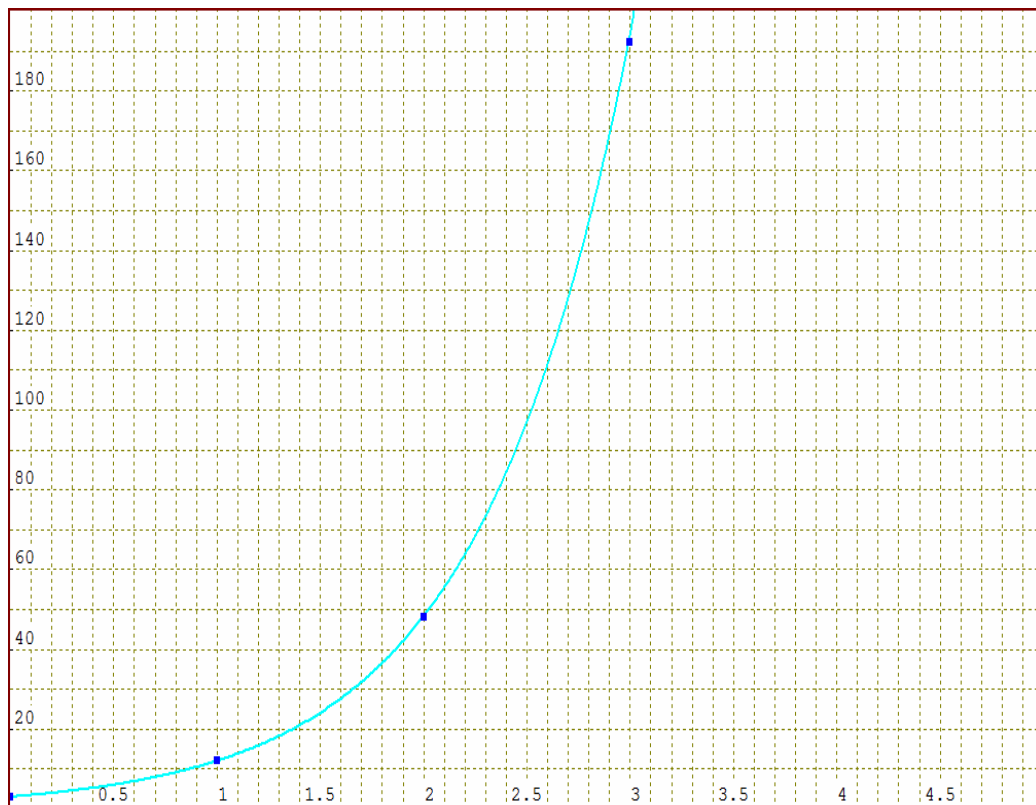
## Method

Taking the initial side length (length of one side of the triangle at Stage 0) to be 1, the following table shows the values for the number of sides, length for each side, perimeter and area for the stages 0 to 3.

Table 1

N <sub>n</sub> Number of sides		L <sub>n</sub> Length of each side		P <sub>n</sub> Perimeter of the shape		A <sub>n</sub> Area of the shape	
n (stage)	f(n)	n (stage)	f(n)	n (stage)	f(n)	n (stage)	f(n)
0	3	0	1	0	3	0	$\frac{\sqrt{3}}{4}$
1	12	1	$\frac{1}{3}$	1	4	1	$\frac{\sqrt{3}}{3}$
2	48	2	$\frac{1}{9}$	2	$5\frac{1}{3}$	2	$\frac{10\sqrt{3}}{27}$
3	192	3	$\frac{1}{27}$	3	$7\frac{1}{9}$	3	$\frac{94\sqrt{3}}{243}$

Using Graphmatica, I have plotted and drawn this Graph of  $n$  vs.  $N_n$



Scale

X axis – 1 step – 0.5 units

Y axis – 1 step – 20 units

### Relationship between n and Number of sides:

N <sub>n</sub> Number of sides	
n (stage)	f(n)
0	3
1	12
2	48
3	192

We can see from the table on the left that it is a geometric progression since every successive term after the 1<sup>st</sup> term is being multiplied by the common ratio 4.

We can confirm this common ratio by taking two values from the table on the left.

Ex. 1 N<sub>1</sub> = 3 and N<sub>2</sub> = 12

$$r = \frac{N_2}{N_1} = \frac{12}{3} = 4$$

Ex. 2 N<sub>2</sub> = 12 and N<sub>3</sub> = 48

$$r = \frac{N_3}{N_2} = \frac{48}{12} = 4$$

General formula for N<sub>n</sub>

$$3 \times 4^n$$

The general formula can be verified by putting in values from the table

$$n = 0$$

$$3 \times 4^0 \rightarrow 3 \times 1 = 3$$

$$n = 1$$

$$3 \times 4^1 \rightarrow 3 \times 4 = 12$$

$$n = 2$$

$$3 \times 4^2 \rightarrow 3 \times 16 = 48$$

$$n = 3$$

$$3 \times 4^3 \rightarrow 3 \times 64 = 192$$

Using Graphmatica, I have plotted and drawn this Graph of  $n$  vs.  $L_n$



### Scale

X-axis – 1 step = 0.5 units

Y-axis – 1 step = 0.1 units

### Relationship between n and Length of each side:

$L_n$

Length of each side	
n (stage)	f(n)
0	1
1	$\frac{1}{3}$
2	$\frac{1}{9}$
3	$\frac{1}{27}$

We can see from the table on the right that it is a geometric progression since every successive term after the 1<sup>st</sup> term is being multiplied by the common ratio  $\frac{1}{3}$ .

We can confirm this common ratio by taking two values from the table on the left.

Ex. 1  $L_1 = 1$  and  $L_2 = \frac{1}{3}$

$$r = \frac{L_2}{L_1} = \frac{\frac{1}{3}}{1} = \frac{1}{3}$$

Ex. 2  $L_2 = \frac{1}{3}$  and  $L_3 = \frac{1}{9}$

$$r = \frac{L_3}{L_2} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{3}{9} = \frac{1}{3}$$

General formula for  $L_n$

$$\frac{1}{3^n}$$

The general formula can be verified by putting in values from the table

$$n = 0$$

$$\frac{1}{3^n} = \frac{1}{3^0} \rightarrow \frac{1}{1} = 1$$

$$n = 1$$

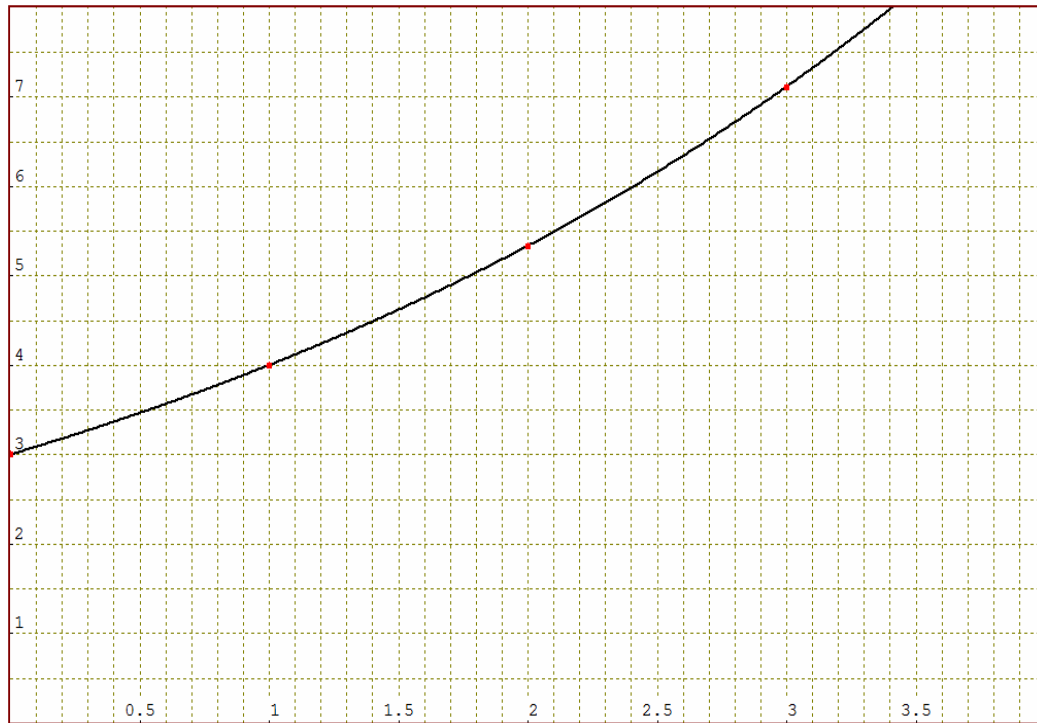
$$\frac{1}{3^n} = \frac{1}{3^1} \rightarrow \frac{1}{3}$$

$$n = 2$$

$$\frac{1}{3^n} = \frac{1}{3^2} \rightarrow \frac{1}{9}$$



Using Graphmatica, I have plotted and drawn this Graph of  $n$  vs.  $P_n$



Scale

X-axis – 1 step = 0.5 units

Y-axis – 1 step = 1 unit

### Relationship between n and Perimeter of the shape:

P <sub>n</sub> Perimeter of the shape	
n (stage)	f(n)
0	3
1	4
2	$5\frac{1}{3}$
3	$7\frac{1}{9}$

We can see from the table on the right that it is a geometric progression since every successive term after the 1<sup>st</sup> term is being multiplied by the common ratio  $\frac{4}{3}$ .

General formula for P<sub>n</sub>

$$(3 \times 4^n) \times \left(\frac{1}{3^n}\right)$$

$$3 \times \frac{4^n}{3^n} = 3 \times \left(\frac{4}{3}\right)^n$$

We can confirm this common ratio by taking two values from the table on the left.

Ex. 1 P<sub>1</sub> = 3 and P<sub>2</sub> = 4

$$r = \frac{P_2}{P_1} = \frac{4}{3} = \frac{4}{3}$$

Ex. 2 L<sub>2</sub> = 4 and L<sub>3</sub> =  $\frac{16}{3}$

$$r = \frac{L_3}{L_2} = \frac{\frac{16}{3}}{4} = \frac{16}{3} \times \frac{1}{4} = \frac{4}{3}$$

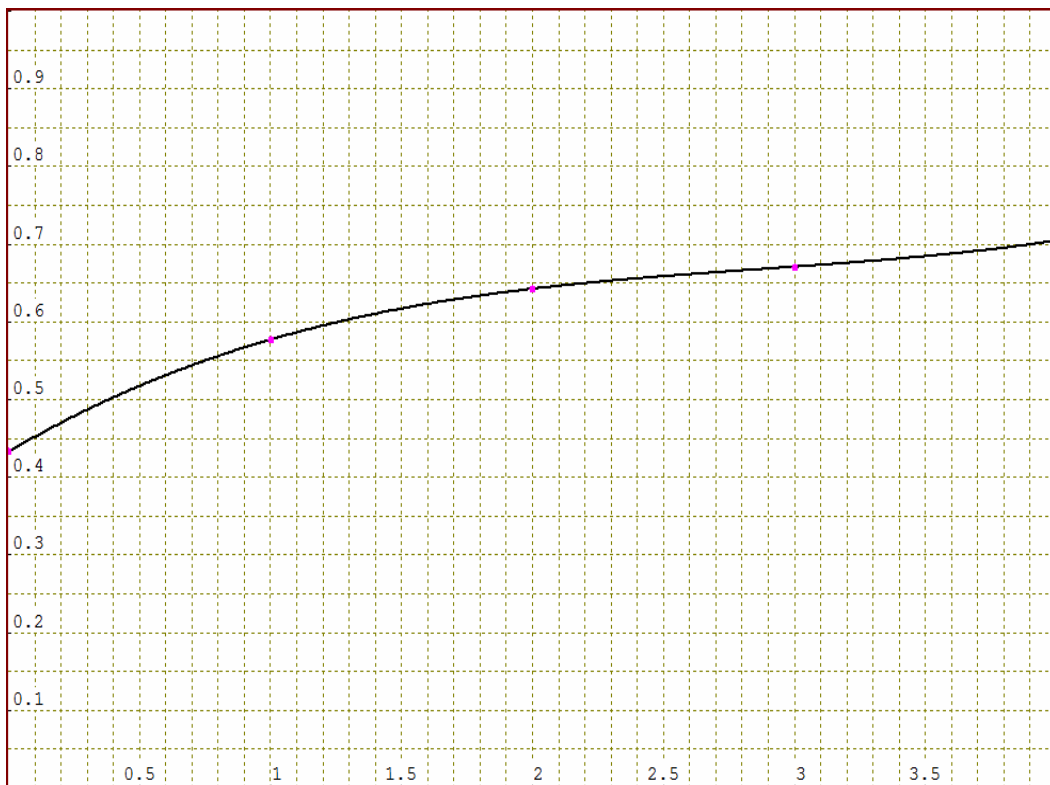
The general formula can be verified by putting in values from the table

$$n = 0 \rightarrow 3 \times \left(\frac{4}{3}\right)^0 \rightarrow 3 \times 1 = 3$$

$$n = 1 \rightarrow 3 \times \left(\frac{4}{3}\right)^1 \rightarrow 3 \times \frac{4}{3} = 4$$

$$n = 2 \rightarrow 3 \times \left(\frac{4}{3}\right)^2 \rightarrow 3 \times \frac{16}{9} = \frac{16}{3} = 5\frac{1}{3}$$

Using Graphmatica, I have plotted and drawn this Graph of  $n$  vs.  $P_n$



Scale

X-axis – 1 step = 0.5 units

Y-axis – 1 step = 0.1 units

### Relationship between n and Area of the shape:

$\blacktriangle_n$ Area of the shape	
n (stage)	f(n)
0	$\frac{\sqrt{3}}{4}$
1	$\frac{\sqrt{3}}{3}$
2	$\frac{10\sqrt{3}}{27}$
3	$\frac{94\sqrt{3}}{243}$

We can see from the table on the left we can clearly see that the initial number a is  $\frac{\sqrt{3}}{4}$ . Now since the denominator goes from 4 to 3 and then goes on increasing in odd powers of three I hypothesised the common ratio to be  $\frac{4}{9}$  which turned out to be correct. After subtracting  $\frac{\sqrt{3}}{4}$  and dividing the answers by  $\left(1 - \left(\frac{4}{9}\right)^n\right)$  the remainder was  $\frac{3\sqrt{3}}{20}$  and so I multiplied it by the same factor.

General formula for  $\blacktriangle_n$

$$\frac{\sqrt{3}}{4} + \left[ \frac{3\sqrt{3}}{20} \left( 1 - \left( \frac{4}{9} \right)^n \right) \right]$$

The general formula can be verified by putting in values from the table.

$$n = 0$$

$$\frac{\sqrt{3}}{4} + \left[ \frac{3\sqrt{3}}{20} \left( 1 - \left( \frac{4}{9} \right)^0 \right) \right]$$

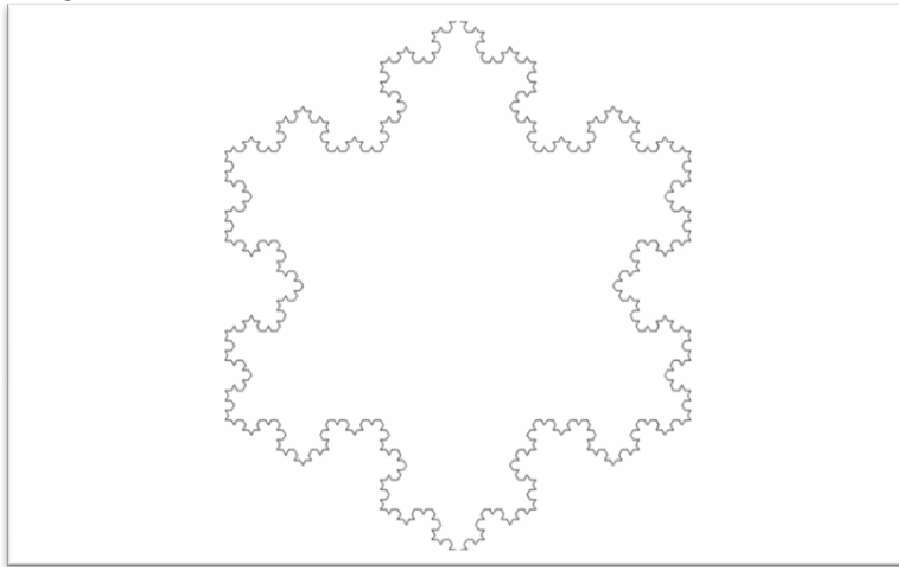
$$\frac{\sqrt{3}}{4} + \left[ \frac{3\sqrt{3}}{20} (1 - 1) \right] = \frac{\sqrt{3}}{4} + 0 = \frac{\sqrt{3}}{4}$$

$$n = 1$$

$$\frac{\sqrt{3}}{4} + \left[ \frac{3\sqrt{3}}{20} \left( 1 - \left( \frac{4}{9} \right)^1 \right) \right]$$

$$\frac{\sqrt{3}}{4} + \left[ \frac{3\sqrt{3}}{20} \left( 1 - \frac{4}{9} \right) \right] = \frac{\sqrt{3}}{4} + \frac{15\sqrt{3}}{180} = \frac{\sqrt{3}}{3}$$

**Stage 4**



For the diagram above I have used a software called Dr. Bill's software of the Von Koch snowflake simulation which I found online.

To investigate what happens at stage four, we will have to apply all of the above equations and substitute n as 4. we will have to substitute in these for

$$N_n \rightarrow 3 \times 4^4 = 3 \times 256 = 768$$

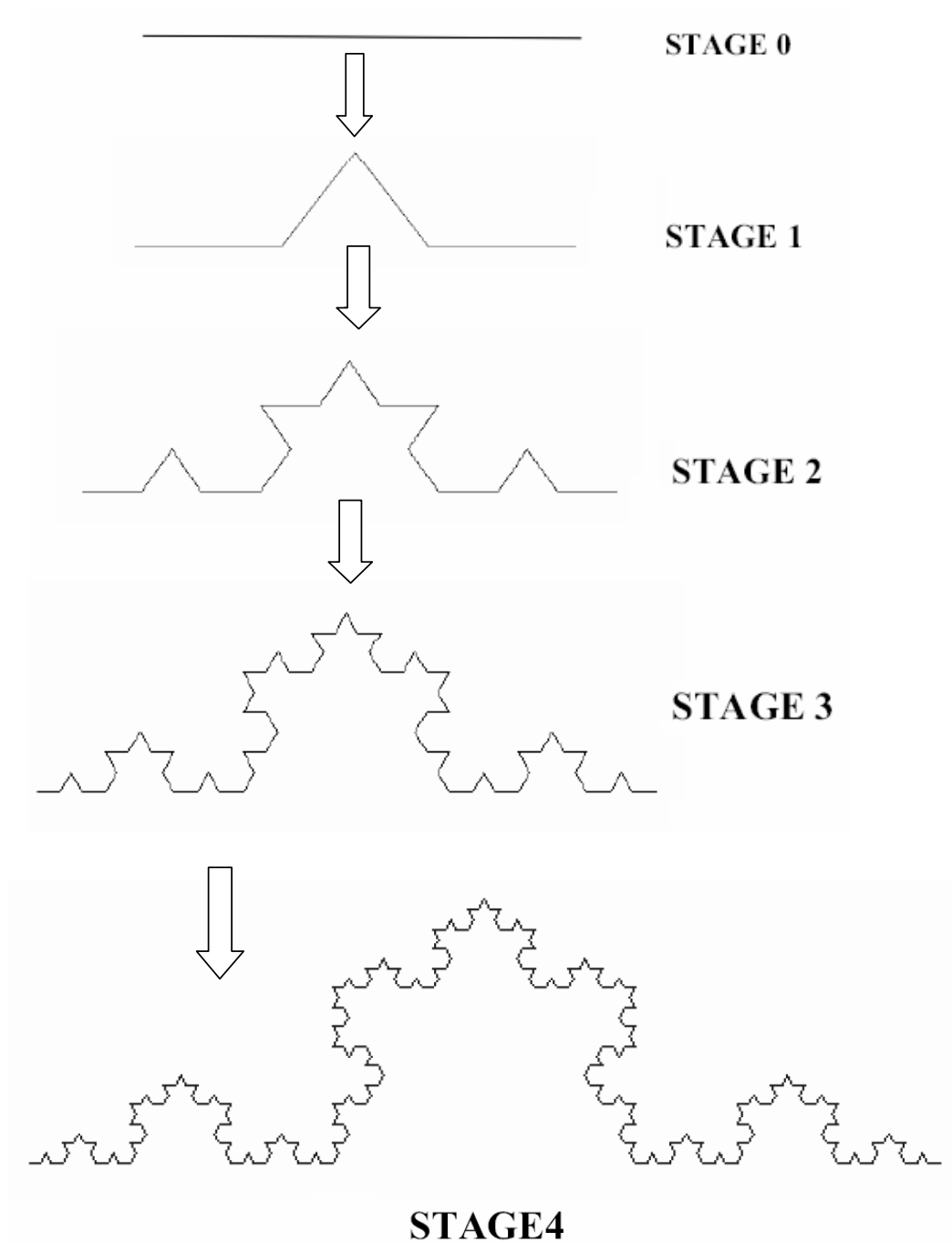
$$L_n \rightarrow \frac{1}{3^n} = \frac{1}{3^4} = \frac{1}{81}$$

$$P_n \rightarrow 3 \times \left( \frac{4}{3} \right)^4 = 3 \times \frac{256}{81} = \frac{256}{27}$$

$$A_n \rightarrow n = 4 \rightarrow \frac{\sqrt{3}}{4} + \left[ \frac{3\sqrt{3}}{20} \left( 1 - \left( \frac{4}{9} \right)^4 \right) \right] = \frac{\sqrt{3}}{4} + \frac{1261\sqrt{3}}{8748} = \frac{862\sqrt{3}}{2187}$$

Stage (n)	$N_n$	$L_n$	$P_n$	$A_n$
4	768	$\frac{1}{81}$	$\frac{256}{27}$	$\frac{862\sqrt{3}}{2187}$

This is how one side of the Koch snowflake would look at stage 4. I have used another free online software in order to simulate this.



Verifying the values using the diagrams:

Number of sides:

From the diagrams above we can see that each successive iteration has transformed each side into four different sides and of a smaller length.

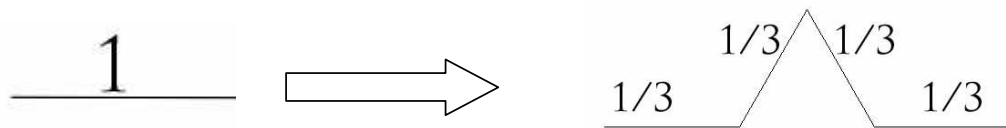
This proves my prediction for  $N_n = 3 \times 4^n$

Since  $\frac{N_{n+1}}{N_n} = 4$

$\therefore N_n \rightarrow 3 \times 4^4 = 3 \times 256 = 768$  is true,

Length of each side:

We can see that each side of the equilateral in the Koch snowflake gets divided into three parts. The middle third of the side is used as a base for another equilateral triangle having the same side length as the base.



The next shape would have a side length which would be a third of a third,

$$= \frac{\frac{1}{3}}{3} = \frac{1}{9}$$

Here we can see a pattern as,

Shape 1 – 1

Shape 2 –  $\frac{1}{3}$

Shape 3 –  $\frac{1}{9}$

And so I can say that my prediction about the relationship between length of side  $L_n$  and the stage  $n$  is true,

$$\therefore L_n \rightarrow \frac{1}{3^n} = \frac{1}{3^4} = \frac{1}{81} \text{ is true,}$$



Perimeter of each shape:

The perimeter of the first shape would be  $= 1 \times 1 = 1$

While the perimeter of the second shape would be  $= 4 \times \frac{1}{3} = \frac{4}{3}$

Since a triangle has three side we can say that perimeter of the triangle would be the perimeter of one side multiplied by three

$$\therefore \text{shape 1} = 3 \times 1 = 3 \quad \text{or} \quad 3 \times \left(\frac{4}{3}\right)^0 = 3$$

For the second shape, the perimeter would be,

$$3 \times \frac{4}{3} = 4 \quad \text{or} \quad 3 \times \left(\frac{4}{3}\right)^1 = 4$$

We can see a pattern from this which clearly indicates that my prediction about the relationship of  $n$  and  $P_n$  is true,

$$\therefore P_n \rightarrow 3 \times \left(\frac{4}{3}\right)^4 = 3 \times \frac{256}{81} = \frac{256}{27} \quad \text{is true,}$$

# Part 5.

	A	B	C	D	E
1	Stage	No. of Sides	Length of a side	Perimeter	Area
2	0	3	1	3	0.4330127
3	1	12	0.333333333	4	0.5773503
4	2	48	0.111111111	5.333333333	0.6415003
5	3	192	0.037037037	7.111111111	0.6700114
6	4	768	0.012345679	9.481481481	0.6826830
7	5	3072	0.004115226	12.64197531	0.6883149
8	6	12288	0.001371742	16.85596708	0.6908179
9	7	49152	0.000457247	22.47462277	0.6919304
10	8	196608	0.000152416	29.96616369	0.6924248
11	9	786432	5.08053E-05	39.95488493	0.6926445
12	10	3145728	1.69351E-05	53.2731799	0.6927422
13	11	12582912	5.64503E-06	71.03090654	0.6927856
14	12	50331648	1.88168E-06	94.70787538	0.6928049
15	13	201326592	6.27225E-07	126.2771672	0.6928135
16	14	805306368	2.09075E-07	168.3695562	0.6928173
17	15	3221225472	6.96917E-08	224.4927416	0.6928190
18	16	12884901888	2.32306E-08	299.3236555	0.6928197
19	17	51539607552	7.74352E-09	399.0982074	0.6928201
20	18	2.06158E+11	2.58117E-09	532.1309432	0.6928202
21	19	8.24634E+11	8.60392E-10	709.5079242	0.6928203
22	20	3.29853E+12	2.86797E-10	946.0105656	0.6928203

To obtain the above table, various formulas were used in the spreadsheet software (Office '07).

These are as following: -

i. A3:  $f(x) = A2 + 1$

ii. B2:  $f(x) = 3 \times 4^{A2}$

iii. C2:  $f(x) = \frac{1}{3^{A2}}$

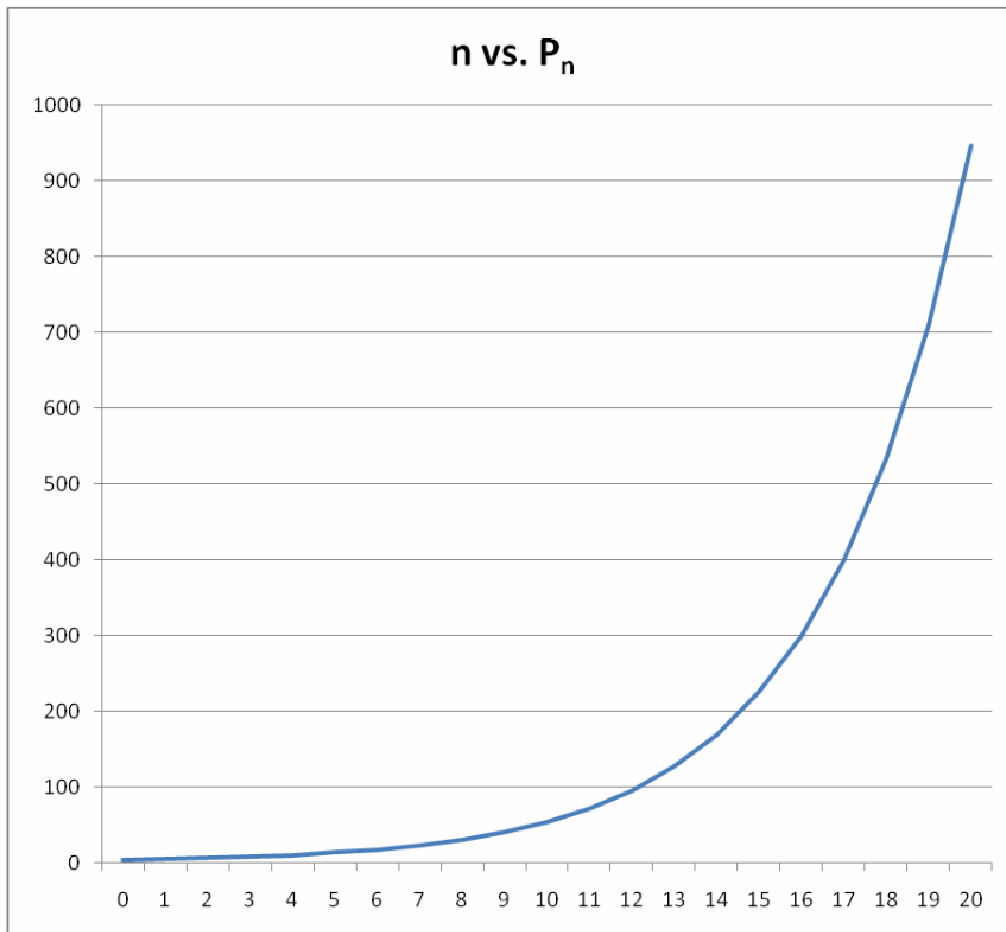
iv. D2:  $f(x) = B2 \times C2$

v. E2:  $f(x) = 0.5 * \sin(60 * \pi / 180)$  ( $= \sqrt{3}/4$ ) (fixed)

vi. E3:  $f(x) = E2 + (B2 * (0.5) * (C3^2) * \sin(60 * \pi / 180))$

## Part 6

The geometric pattern of the perimeter of the shape is a divergent series as the common ratio is  $> 1$ . Hence the perimeter at each stage increases exponentially. This is also proved by the graph.



From the graph we can see that as,

$$n \rightarrow \infty \quad P_n \rightarrow \infty$$

▲Area of the shape

Formula

$$\frac{\sqrt{3}}{4} + \left[ \frac{3\sqrt{3}}{20} \left( 1 - \left( \frac{4}{9} \right)^n \right) \right]$$

For the geometric progression in the formula above we can see that the common ratio is lesser than 1 and so this can be called a convergent series. When  $n$  increases towards  $\infty$ ,  $\left( \frac{4}{9} \right)^n$  becomes so small a value that the highlighted part of the equation becomes equal to one.

$$\therefore A_n = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} = \frac{8\sqrt{3}}{20} = \frac{2\sqrt{3}}{5}$$

$$\therefore \text{as } n \rightarrow \infty, A_n \rightarrow 2\sqrt{3}/5$$

Part 7

The following iterative formula was derived: -

$$A_{n+1} = A_n + \left[ \frac{\sqrt{3}}{12} \times \left( \frac{4}{9} \right)^{(n+1)-1} \right]$$

We can now substitute the general formula derived in this equation: -

$$\frac{\sqrt{3}}{4} \left[ 1 + \sum_{k=1}^{n+1} \left[ \left( \frac{1}{3} \right) \times \left( \frac{4}{9} \right)^{k-1} \right] \right] = \frac{\sqrt{3}}{4} \left[ 1 + \sum_{k=1}^n \left[ \left( \frac{1}{3} \right) \times \left( \frac{4}{9} \right)^{k-1} \right] \right] + \left[ \frac{\sqrt{3}}{12} \times \left( \frac{4}{9} \right)^{(n+1)-1} \right]$$

To prove this equation by induction...

**Step 1:** Assume statement true for  $n=1$ ,

$$LHS = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{27} = \frac{10\sqrt{3}}{27}$$

$$RHS = \frac{\sqrt{3}}{4} \left[ 1 + \sum_{k=1}^1 \left[ \left( \frac{1}{3} \right) \times \left( \frac{4}{9} \right)^{k-1} \right] \right] + \left[ \frac{\sqrt{3}}{12} \times \left( \frac{4}{9} \right)^{1+1-1} \right] = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} + \frac{10\sqrt{3}}{27}$$

$$\therefore LHS = RHS$$

Therefore, the statement is proven true for  $n=1$ .

**Step 2:** Assume statement true for  $n=k$ ,

$$\begin{aligned} P_k: \quad \frac{\sqrt{3}}{4} \left[ 1 + \sum_{k=1}^{k+1} \left[ \left( \frac{1}{3} \right) \times \left( \frac{4}{9} \right)^{k-1} \right] \right] &= \frac{\sqrt{3}}{4} \left[ 1 + \sum_{k=1}^k \left[ \left( \frac{1}{3} \right) \times \left( \frac{4}{9} \right)^{k-1} \right] \right] + \left[ \frac{\sqrt{3}}{12} \times \left( \frac{4}{9} \right)^k \right] \\ P_{k+1}: \quad \frac{\sqrt{3}}{4} \left[ 1 + \sum_{k=1}^{k+2} \left[ \left( \frac{1}{3} \right) \times \left( \frac{4}{9} \right)^{k-1} \right] \right] &= \frac{\sqrt{3}}{4} \left[ 1 + \sum_{k=1}^k \left[ \left( \frac{1}{3} \right) \times \left( \frac{4}{9} \right)^{k-1} \right] \right] + \left[ \frac{\sqrt{3}}{12} \times \left( \frac{4}{9} \right)^k \right] + \left[ \frac{\sqrt{3}}{12} \times \left( \frac{4}{9} \right)^{k+1} \right] \\ &= \frac{\sqrt{3}}{4} \left[ 1 + \sum_{k=1}^k \left[ \left( \frac{1}{3} \right) \times \left( \frac{4}{9} \right)^{k-1} \right] \right] + \left[ \frac{\sqrt{3}}{4} \times \frac{1}{3} \times \left( \frac{4}{9} \right)^k \right] + \left[ \frac{\sqrt{3}}{4} \times \frac{1}{3} \times \left( \frac{4}{9} \right)^{k+1} \right] \\ &= \frac{\sqrt{3}}{4} \left[ 1 + \sum_{k=1}^k \left[ \left( \frac{1}{3} \right) \times \left( \frac{4}{9} \right)^{k-1} \right] + \left[ \frac{1}{3} \times \left( \frac{4}{9} \right)^k \right] + \left[ \frac{1}{3} \times \left( \frac{4}{9} \right)^{k+1} \right] \right] \\ &= \frac{\sqrt{3}}{4} \left[ 1 + \sum_{k=1}^{k+1} \left[ \left( \frac{1}{3} \right) \times \left( \frac{4}{9} \right)^{k-1} \right] + \left[ \frac{1}{3} \times \left( \frac{4}{9} \right)^{k+1} \right] \right] \\ &= \frac{\sqrt{3}}{4} \left[ 1 + \sum_{k=1}^{k+2} \left[ \left( \frac{1}{3} \right) \times \left( \frac{4}{9} \right)^{k-1} \right] \right] \end{aligned}$$

$$\therefore LHS = RHS$$

$$\therefore P_k \Rightarrow P_{k+1}$$

We have already proven the statement true for  $n=1$  and since  $P_k \Rightarrow P_{k+1}$ , we can say the statement must be true for  $n=2, 3, 4, \dots n$ .