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Parallels and Parallelograms

Mathematics Coursework

This task will consider the number of parallelograms formed by intersecting m horizontal parallel lines with n parallel transversals; we are to deduce a formula that will satisfy the above.

Methodology

1. We started out the investigation with a pair of horizontal parallel lines and a pair of parallel transversals. One parallelogram (A_1) is formed (shown in *Figure 1*)

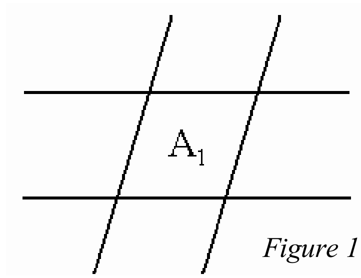


Figure 1

2. A third parallel transversal is added to the diagram as shown in *Figure 2*. Three parallelograms are formed: A_1, A_2, and $A_1 \cup A_2$

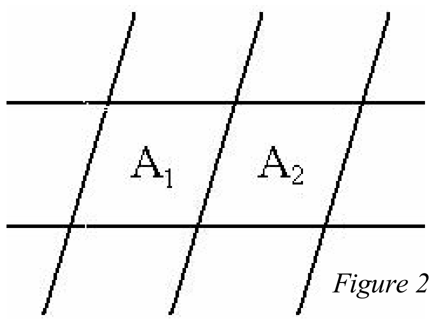


Figure 2

3. When a fourth transversal is added to *Figure 2* (*Figure 3*), six parallelograms are formed. A_1, A_2, A_3, $A_1 \cup A_2$, $A_2 \cup A_3$, $A_1 \cup A_3$

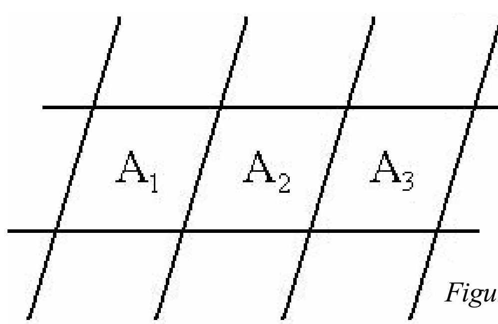


Figure 3

4. Figure 4 has 5 transversals cutting the pair of horizontal parallels, forming ten parallelograms. A_1, A_2, A_3, A_4, $A_1 \cup A_2$, $A_2 \cup A_3$, $A_3 \cup A_4$, $A_1 \cup A_3$, $A_2 \cup A_4$, $A_1 \cup A_4$

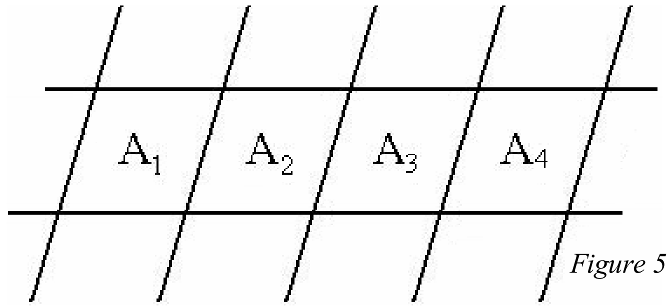


Figure 5

5. A sixth transversal was added to *Figure 5*, forming 15 parallelograms shown in Figure 6. A_1, A_2, A_3, A_4, A_5, $A_1 \cup A_2$, $A_2 \cup A_3$, $A_3 \cup A_4$, $A_4 \cup A_5$, $A_1 \cup A_3$, $A_2 \cup A_4$, $A_3 \cup A_5$, $A_1 \cup A_4$, $A_2 \cup A_5$, $A_1 \cup A_5$

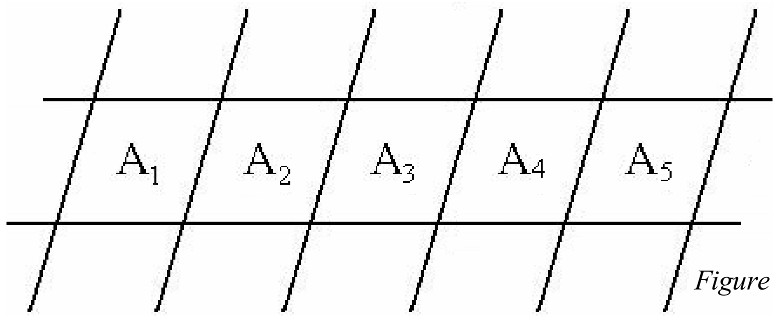


Figure 6

6. When a seventh transversal is added, twenty-one parallelograms are formed (*Figure 7*). A_1, A_2, A_3, A_4, A_5, A_6, $A_1 \cup A_2$, $A_2 \cup A_3$, $A_3 \cup A_4$, $A_4 \cup A_5$, $A_5 \cup A_6$, $A_1 \cup A_3$, $A_2 \cup A_4$, $A_3 \cup A_5$, $A_4 \cup A_6$, $A_1 \cup A_4$, $A_2 \cup A_5$, $A_3 \cup A_6$, $A_1 \cup A_5$, $A_2 \cup A_6$, $A_1 \cup A_6$.

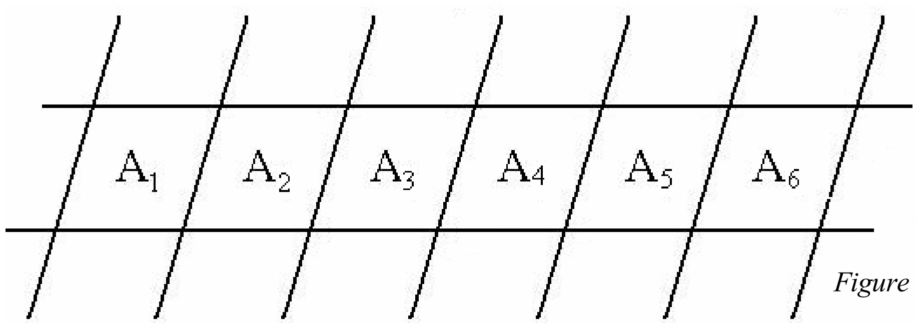


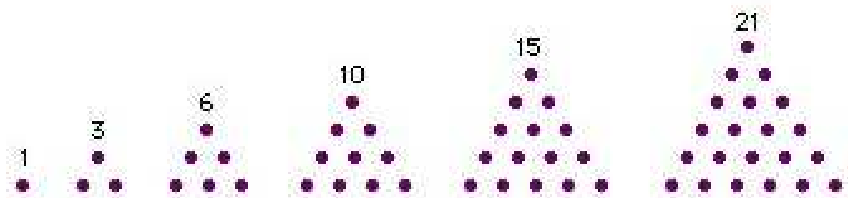
Figure 7

7. I then used technology (*table 1.0*) to record the above and calculate the differences between the parallelograms formed with each addition of a transversal.

Number of Horizontal Lines	Number of transversals	Number of Parallelograms formed	First difference between terms	Second difference between terms
2	2	1		
2	3	3	2	
2	4	6	3	1
2	5	10	4	1
2	6	15	5	1
2	7	21	6	1

Table 1.0

At this point, I realized that the 'First difference between terms' was somewhat similar to what happens with the terms in a triangle sequence:



http://www.visualstatistics.net/divertissements/saturn%20magic%20square.htm

Also since the number of parallelograms created as the number of transversals increased each had a Second Order difference of 1, it was immediately known that the general formula must be a quadratic equation. So to find the formula I followed the following steps:

$$ax^2 + bx + c$$

$x = 1 \quad a + b + c = 1$

$x = 2 \quad 4a + 2b + c = 3$

$x = 3 \quad 9a + 3b + c = 6$

Then solve them as simultaneous equations

$$\left.\begin{array}{l} a + b + c = 1 \\ 4a + 2b + c = 3 \end{array}\right. \longrightarrow 3a + b = 2$$

$$\left.\begin{array}{l} 4a + 2b + c = 3 \\ 9a + 3b + c = 6 \end{array}\right. \longrightarrow 5a + b = 3$$

Solve the two new equations as simultaneous equations

$3a + b = 2$

$5a + b = 3$

Answer: $a = \frac{1}{2}$ $b = \frac{1}{2}$ $c = 0$

Replace the values in the following formula $ax^2 + bx + c$

$$\frac{1}{2}x^2 + \frac{1}{2}x$$

$$\frac{x^2 + x}{2} = \frac{x\ x+1}{2}$$

But when I tried out the formula I found out that the answers didn't match:

$$U_2 = 3\frac{(3+1)}{2} = 6$$

$$U_3 = 4\frac{(4+1)}{2} = 10$$

The results we had have moved one term so instead of adding 1 to "n", we need to subtract 1 to "n" so the values can match. Now our final formula is:

$$\boxed{U_n = \frac{n\,(n-1)}{2}}$$

Once again I tested the above:

$$U_2 = 2\frac{(2-1)}{2} = 1$$

$$U_3 = 3\frac{(3-1)}{2} = 3$$

8. In order to find the general formula for the parallelograms formed by *m* horizontal parallel lines intersected by *n* parallel transversals; I decided to further the investigation by considering the number of parallelograms formed by <u>three</u> horizontal parallel lines intersected by a pair of parallel transversals (*Figure 8*). Three parallelograms were formed: A_1, A_2, and $A_1 \cup A_2$

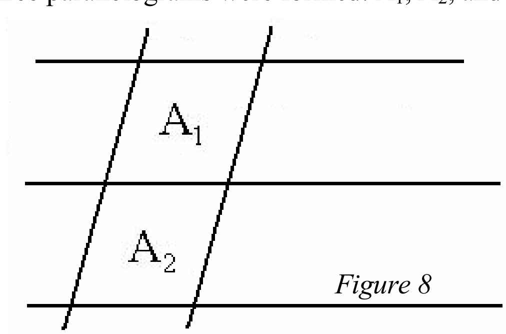


Figure 8

9. When a third transversal was added to the above figure, nine parallelograms were formed (*Figure 9*). A_1, A_2, A_3, A_4, $A_1 \cup A_2$, $A_3 \cup A_4$, $A_1 \cup A_3$, $A_2 \cup A_4$, $A_1 \cup A_4$

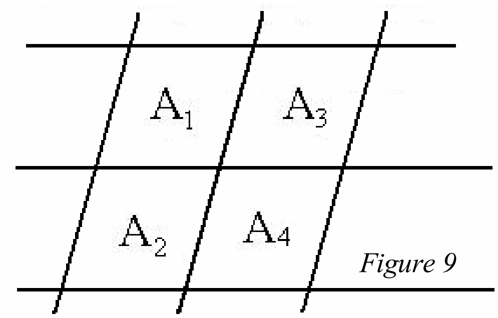


Figure 9

10. One more transversal was added to *Figure 9*, to form 18 parallelograms (*figure 10*). A_1, A_2, A_3, A_4, A_5, A_6, $A_1 \cup A_2$, $A_3 \cup A_4$, $A_5 \cup A_6$, $A_1 \cup A_3$, $A_3 \cup A_5$, $A_2 \cup A_4$, $A_4 \cup A_6$, $A_1 \cup A_5$, $A_2 \cup A_6$, $A_1 \cup A_4$, $A_3 \cup A_6$, $A_1 \cup A_6$

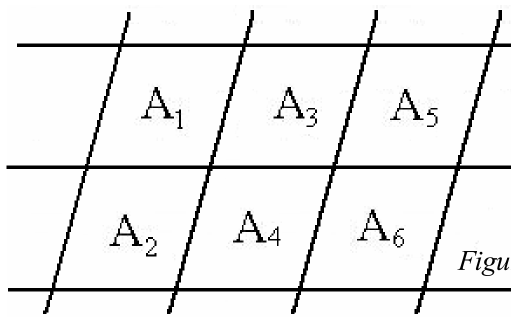


Figure 10

11. When I added the fifth transversal to Figure 10, it became incredibly difficult to count the parallelograms; luckily by then a pattern had emerged and I was able to predict the next few terms (recorded in *table 1.1*):

Number of Horizontal Lines	Number of transversals	Number of Parallelograms formed	Multiple	Difference between multiples	First different between terms	Second difference between terms
3	2	3	3 x 1			
3	3	9	3 x 3	2	6	
3	4	18	3 x 6	3	9	3
3	5	30	3 x 10	4	12	3
3	6	45	3 x 15	5	15	3
3	7	63	3 x 21	6	18	3

Table 1.1

Now the 'Second Order Difference' is 3 – triple the first set of parallelograms (pair of parallels intersecting with parallel transversals). Due to the second order being three, I deduced and found true that the number of parallelograms was increasing in multiples of three. The difference between the multiples is 2,3,4,5, and 6; which again follows the triangle number sequence. This meant that subsequently the formula for *m* horizontal parallel lines intersected by *n* parallel transversals would include the formula for the triangle sequence.

12. To make my statements broader I expanded my table to the following (table 1.2):

	Number of parallel transversals						
	2	3	4	5	6	7	n
2	1	3	6	10	15	21	
3	3	9	18	30	45	63	
4	6	18	36	60	90	126	
5	10	30	60	100	150	210	
6	15	45	90	150	225	315	
7	21	63	126	210	315	441	
m							

Table 1.2

See *graph 1* to get a more graphical view of how the number of parallelograms formed by *m* horizontal parallel lines intersected by *n* parallel transversals are very sequential, by looking at the graph, the sequence is much more evident.

13. When I compared all the formulas I got, I found the relationship which led me to find the general formula.

2 horizontal lines: $\frac{1n\,(n-1)}{2}$

3 horizontal lines: $\frac{3(n\,(n-1))}{2}$

4 horizontal lines: $\frac{6(n\,(n-1))}{2}$

5 horizontal lines: $\frac{10(n\,(n-1))}{2}$

The number and sequence repeat the formula $\frac{(n\,(n-1))}{2}$ and multiplies by the first term.

14. From this realization I was able to find that the final formula for calculating the number of parallelograms formed when *m* horizontal parallel lines are intersected by *n* parallel transversals:

$$\boxed{\frac{m(m-1)}{2} \times \frac{n(n-1)}{2}}$$

15. To test the validity of the formula I tested it against previously counted parallelograms (*Figure 10*), the intersection of 4 transversals with 3 horizontal parallel lines should form 18 parallelograms:

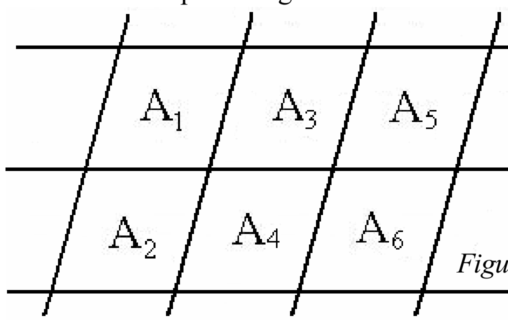


Figure 10

Using the formula:

$$\mathbf{\frac{m(m-1)}{2} \times \frac{n(n-1)}{2}}$$

$$\frac{3(3-1)}{2} \times \frac{4(4-1)}{2}$$

$$\frac{3(2)}{2} \times \frac{4(3)}{2}$$

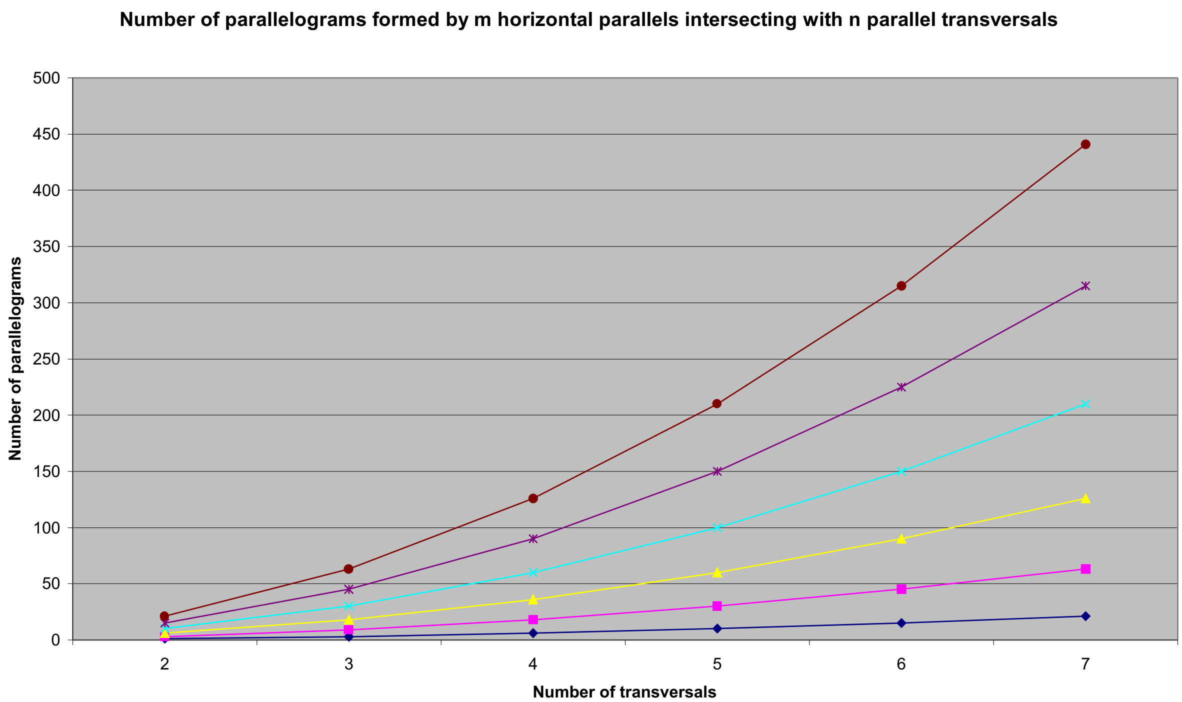
$$\frac{6}{2} \times \frac{12}{2}$$

$$3 \times 6$$

<u>= 18 parallelograms</u>

<u>Limitations</u>

It is very difficult to test the validity of the formula when there are lots of parallel transversals intersecting lots of horizontal parallels, because even though we will get a number, to prove there are so many parallelograms is confusing and difficult; therefore we can only assume that the answer might be right.


Number of parallelograms formed by m horizontal parallels intersecting with n parallel transversals
Number of parallelograms
0
50
100
150
200
250
300
350
400
450
500
Number of transversals
2
3
4
5
6
7

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