

Mathematics Portfolio
Standard Level Type I
Infinite Surds

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Apply $\lim_{n \rightarrow \infty}$

Here is a proved formula

$$a_{n+1} = \sqrt{1 + a_n}$$

$$a_n = \sqrt{1 + a_n}$$

$$a = \sqrt{1 + a}$$

$$a^2 = 1 + a$$

$$a^2 - a - 1 = 0$$

Use quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 + 4}}{2 \times 1} = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore a = \frac{1 + \sqrt{5}}{2} = 1.618033989 \text{ or } -0.6180339887$$

However, the root cannot be the negative number

So,

$$\therefore 1.618033989$$

Consider another infinite surd

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}} \text{ where the first term is } \sqrt{2 + \sqrt{2}}$$

Repeat the entire process above to find the exact value for this surd.

$$b_1 = \sqrt{2 + \sqrt{2}} = 1.847759065$$

$$b_2 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} = 1.961570561$$

$$b_3 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} = 1.990369453$$

$$b_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} = 1.997590912$$

$$b_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}} = 1.999397637$$

$$b_6 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}} = 1.999849404$$

$$b_7 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}}} = 1.999962351$$

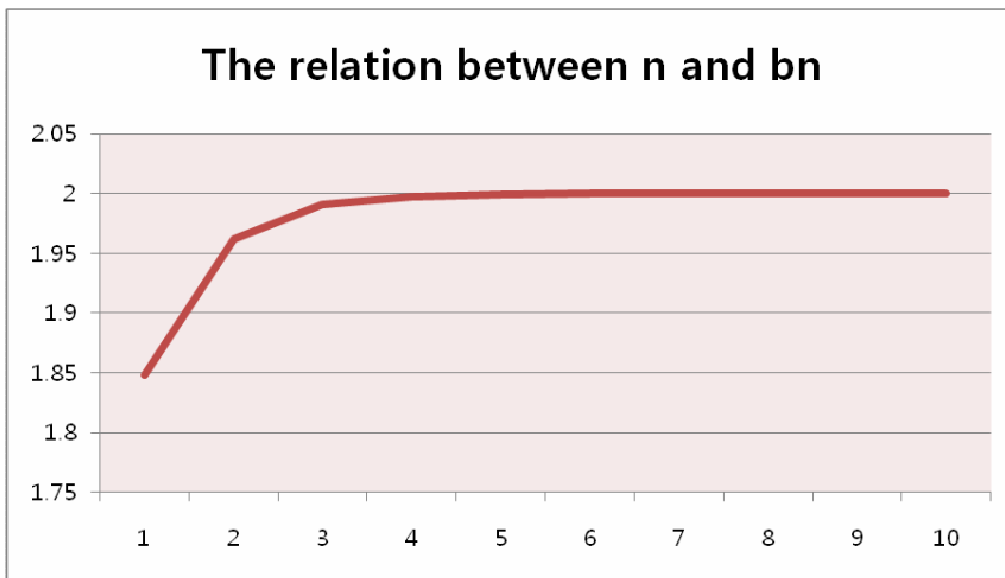
$$b_8 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}}}} = 1.999990588$$

$$b_9 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}}}}} = 1.999997647$$

$$b_{10} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}}}}}}} = 1.999999412$$

According to the result, you can aware that $b_2 = \sqrt{2 + b_1}$

Then analyzing the formula, $b_{n+1} = \sqrt{2 + b_n}$



This graph shows us that there is a huge change between 1 and 2. And it is constant from 4. b_n and $b_n - b_{n+1}$ is getting close to 0.

$$\therefore b_n - b_{n+1} = 0$$

Apply $\lim_{n \rightarrow \infty}$

Here is a proved formula

$$\begin{aligned} &= b_n = \sqrt{2 + b_n} \\ &= (b_n)^2 = 2 + b_n \\ &= b_n^2 - b_n - 2 = 0 \\ &= (b_n - 2)(b_n + 1) \\ &\therefore b_n = 2 \text{ or } b_n = -1 \end{aligned}$$

Again, the negative number is impossible in the root

\therefore the answer is -1

Now consider the general infinite surd $\sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k} \dots}}}$ where the first term is $\sqrt{k + \sqrt{k}}$. Find an expression for the exact value of this general infinite surd in terms of k.

$$a_1 = \sqrt{k + \sqrt{k}}$$

$$a_2 = \sqrt{k + a_1}$$

$$a_3 = \sqrt{k + a_2}$$

So, the general formula would be $a_{n+1} = \sqrt{k + a_n}$

Apply $\lim_{n \rightarrow \infty}$

Here is the solution

$$\begin{aligned} a_{n+1} &= \sqrt{k + a_n} \\ &= a_n = \sqrt{k + a_n} \\ &= (a_n^2) = (\sqrt{k + a_n})^2 \\ &= a_n^2 = k + a_n \\ &= a_n^2 - a_n - k = 0 \end{aligned}$$

Use quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 + 4k}}{2}$$

The infinite surd value for k=0 will be 0. the infinite surd value can be 0 if k=0, but it can't be exactly 1.

If you take k=0,000... (m zeros)...0001000...(m zeros)...0001, the infinite surd value will be 1,000...(m zeros)...0001. So, you can obtain values as close to 1 as you want, but it will never be 1.

Find some values of k that make the expression an integer. Find the general statement that represents all the values of k for which the expression is an integer

The value of an infinite surd is not always an integer

$$2=1 \times 2$$

$$6=2 \times 3$$

$$12=3 \times 4$$

$$20=4 \times 5$$

$$30=5 \times 6$$

These are proved, using this formula $k = n(n+1)$

$$5(5+1) = 30$$

5th term which is also described as $5 \times 6 = 30$

Test the validity of your general statement using other values of k.

First off, the formula is $k = n(n+1)$

Apply $\lim_{n \rightarrow \infty}$

With using quadratic equation,

$$\lim_{n \rightarrow \infty} = \frac{1 + \sqrt{1 + 4k}}{n}$$

$$8 \times 9 = 72 \text{ -----} \rightarrow y = \frac{1 + \sqrt{1 + 4 \times 72}}{2}$$

$$= 46$$

$$88 \times 89 = 7832 \text{ -----} \rightarrow y = \frac{1 + \sqrt{1 + 4 \times 7832}}{2}$$

$$= 89$$

$$222 \times 223 = 49506 \text{ -----} \rightarrow y = \frac{1 + \sqrt{1 + 4 \times 49506}}{2}$$

$$= 223$$

And so on..

It is able to notice that

$$\text{When } k \text{ is } 72, y = \frac{1 + \sqrt{1 + 4 \times 72}}{2} = 46$$

Discuss the scope and/or limitations of your general statement.

According to this, we've found that value of K and n always are

$$K \geq 0$$

$$n \geq 0$$